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A characterization of the set of lines either external to or secant to an ovoid in PG$(3, q)$

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Abstract

In this paper we prove that in PG$(3, q)$ a $\frac{q^4+q^2}{2}$−set of lines, having exactly $\frac{q^8-q^7-q^4+q^3}{8}$ pairs of skew lines, of type $(m, m+\frac{q^2+q}{2})$ with respect to stars of lines and of type $(m', m'+\frac{q^2+q}{2})$ with respect to ruled planes, is the set of lines either external to or secant to an ovoid.

1 Introduction and Motivations

We recall that the Klein quadric H_5 representing the lines in PG$(3, q)$ is the hyperbolic quadric of equation $l_{01}l_{23} - l_{02}l_{13} + l_{03}l_{12} = 0$ in PG$(5, q)$ where (l_{ij}), $i, j \in \{0, 1, 2, 3\}$, $i < j$, are the Plucker coordinates of a line ℓ in PG$(3, q)$; see [5]. So a point $L \in H_5$ represents in PG$(5, q)$ a line ℓ of PG$(3, q)$. Moreover a pencil of lines in PG$(3, q)$, i.e. all the lines through a point contained in the same plane, is represented in PG$(5, q)$ by a line contained in H_5. Thus, two lines in PG$(3, q)$ meeting in a point are represented by two *collinear* points of H_5, i.e. two points such that the line through them is completely contained in H_5.

In PG$(3, q)$ a maximal set of lines which pairwise meet in a point is either a star of lines, i.e. all the lines through a point, or a ruled plane, i.e. a plane considered as the set of its lines. Therefore, the Klein quadric H_5 has two systems of generating planes, called *greek* and *latin* planes for convenience (see [5]), which are maximal subspaces on H_5. A latin plane represents a star of lines and a greek plane represents a ruled plane in PG$(3, q)$. One of the most interesting problem in finite geometry is the combinatorial characterization of a remarkable set of lines as the point-set of H_5 having suitable incidence properties with respect to the subspaces of H_5. In PG$(3, q)$, $q > 2$, a set of $(q^2 + 1)$ points no three of which are collinear is called an *ovoid*.

In this paper we look for a common incidence condition in order to characterize both sets of secant lines and external lines of an ovoid in $\mathrm{PG}(3,q)$. It is easy to verify that the cardinality of the set of the external lines of an ovoid is equal to the cardinality of the set of the secant lines of an ovoid, and this number is $\frac{q^4+q^2}{2}$. Another common incidence property is the number of pairs of skew lines that is $\frac{q^8-q^7-q^4+q^3}{8}$. Let K denote a k-set of H_5, i.e. a set of k points of H_5. We recall that the *d-characters* of K, with respect to a family $\mathcal{F}$ of d-subspaces of H_5, are the numbers $t_i^d = t_i^d(K)$ of d-subspaces of $\mathcal{F}$ meeting K in exactly i points, $0 \le i \le \theta_d$, $\theta_d := q^d + q^{d-1} + \ldots + q + 1$, $d \in \{1,2\}$. A set K is said to be of *type* $(m_1, m_2, \ldots, m_s)$ with respect to a family $\mathcal{F}$ of d-subspaces of H_5, if any d-subspace of $\mathcal{F}$ contains either m_1, or $m_2, \ldots$, or m_s points of K, and every value occurs; see [7]. A set of type $(m_1, m_2, \ldots, m_s)$ is also called a *character set*; see [4]. Point k-sets on H_5 can be investigated in terms of their numbers of non zero characters; see [9]. The following results enter into this scheme of things.

Result 1 ([3]) Let $\mathcal{L}$ be a family of lines in $\mathrm{PG}(3,q)$, q odd, for which the following hold:

(i) $\mathcal{L}$ is of type $(0, \frac{q-1}{2}, \frac{q+1}{2}, q)$ with respect to pencils of lines;
(ii) $\mathcal{L}$ is of type $(\frac{q^2-q}{2}, q^2)$ with respect to stars of lines;
(iii) $\mathcal{L}$ is of type $(0, \frac{q^2+q}{2})$ with respect to ruled planes;
(iv) any plane containing some line of $\mathcal{L}$ contains no pencil all of whose lines do not belong to L.

Then $\mathcal{L}$ is the family of the secant lines of an elliptic quadric in $\mathrm{PG}(3,q)$.

Result 2 ([1]) Let $\mathcal{L}$ be a family of lines in $\mathrm{PG}(3,q)$, q even, $q > 2$, such that:

(i) $\mathcal{L}$ is of type $(0, \frac{q}{2}, q)$ with respect to pencils of lines;
(ii) $\mathcal{L}$ is of type (n, q^2), $0 < n < q^2$, with respect to stars of lines.

Under these assumptions, $n = \frac{q^2-q}{2}$ and $\mathcal{L}$ is the set of secants of an ovoid in $\mathrm{PG}(3,q)$.

Result 3 ([2]) Let $\mathcal{L}$ be a family of lines in $\mathrm{PG}(3,q)$, satisfying the following properties.

(i) $\mathcal{L}$ is of type $(\frac{q^2-q}{2}, q^2)$ with respect to stars of lines.
(ii) $\mathcal{L}$ is of type $(0, \frac{q^2+q}{2})$ with respect to ruled planes.

Then $\mathcal{L}$ is the family of the secant lines of an ovoid in $\mathrm{PG}(3,q)$.

Result 4 ([2]) Let $\mathcal{L}$ be a family of lines in $\mathrm{PG}(3,q)$, satisfying the following properties.

(i) $\mathcal{L}$ is of type $(0, \frac{q^2+q}{2})$ with respect to stars of lines.

(ii) $\mathcal{L}$ is of type $(\frac{q^2-q}{2}, q^2)$ with respect to ruled planes.

Then $\mathcal{L}$ is the family of the external lines of an ovoid in $\mathrm{PG}(3, q)$.

In this paper we give a characterization of the set of points of H_5 which represents the set of lines either external to or secant to an ovoid in $\mathrm{PG}(3, q)$ as two character set of H_5 with respect to the two families of maximal subspaces of H_5. In particular we prove the following.

Theorem *In $PG(3, q)$, a $\frac{q^4+q^2}{2}$−set $\mathcal{L}$ of lines having exactly $\frac{q^8-q^7-q^4+q^3}{8}$ pairs of skew lines such that*

(I) $\mathcal{L}$ is of type $(m, m + \frac{q^2+q}{2})$ with respect to stars of lines,

(II) $\mathcal{L}$ is of type $(m', m' + \frac{q^2+q}{2})$ with respect to ruled planes,

is a set of lines either external to or secant to an ovoid.

2 The proof of the Theorem

Suppose that K is a k-set of type (m, n) with respect to latin (greek) planes of H_5. Let α denote a latin (greek) plane. By counting in double way the total number of latin (greek) planes, of incident point-planes pairs (P, α) with $P \in K \cap \alpha$, and triples (P, Q, α) with $P, Q \in K \cap \alpha$, we have what are referred to as the *standard equations* on the integers t_m and t_n; see [7],

$$\begin{aligned} t_m + t_n &= q^3 + q^2 + q + 1 \\ mt_m + nt_n &= k(q + 1) \\ m(m-1)t_m + n(n-1)t_n &= k(k-1) - 2\tau, \end{aligned}$$

where τ denotes the number of pairs of non collinear points. Thus, a two character set with respect to latin (greek) planes depends by six parameters k, τ, m, n, t_m and t_n and a complete classification seems to be extremely difficult, see [4], [6], [8] and [10]. For $k = \frac{1}{2}(q^4 + q^2)$ and $\tau = \frac{1}{8}(q^8 - q^7 - q^4 + q^3)$, the system of linear equations becomes

$$\begin{aligned} t_m + t_n &= q^3 + q^2 + q + 1 \\ mt_m + nt_n &= \frac{q^2}{2}(q^2 + 1)(q + 1) \\ m(m-1)t_m + n(n-1)t_n &= \frac{q^2}{2}(q^2 + 1)(\frac{q^2}{2}(q^2 + 1) - 1) - \frac{q^3}{4}(q^4 - 1)(q - 1). \end{aligned}$$

From the first two equations, we get

$$\begin{aligned} t_m &= \frac{(q^2+1)(q+1)}{n-m}(n-\frac{q^2}{2}) \\ t_n &= \frac{(q^2+1)(q+1)}{n-m}(\frac{q^2}{2}-m). \end{aligned}$$

By substituting into the third equation we get

$$(q^2-2m)(2n-q^2)=q^3.$$

Since $n=m+\frac{q^2+q}{2}$, the equation becomes $m(q^2-q-2m)=0$. Therefore, either $m=0$ or $m=(q^2-q)/2$. We get that either $m=0$ and $n=(q^2+q)/2$, or $m=(q^2-q)/2$ and $n=q^2$. Thus K is either of type $(0,\frac{q^2+q}{2})$ or of type $(\frac{q^2-q}{2},q^2)$ with respect to latin (greek) planes of H_5. We prove that:

Proposition I *The Klein quadric H_5 contains no $\frac{q^4+q^2}{2}$-set of type $(0,\frac{q^2+q}{2})$ with respect to both latin and greek planes.*

Proof. Let H denote the set theoretic union of the lines in K. Let α be a plane in $\mathrm{PG}(3,q)$. Assume that α contains no line in K. Clearly, any line of K meets α in one point. Moreover the star of lines through $P\in H$ shares $\frac{q^2+q}{2}$ lines with K. Therefore, $|\alpha\cap H|\frac{q^2+q}{2}=\frac{q^4+q^2}{2}$. Thus $|\alpha\cap H|=q^2-q+2-\frac{2}{q+1}$, which is not an integer, a contradiction.

Proposition II *The Klein quadric H_5 contains no $\frac{q^4+q^2}{2}$-set of type $(\frac{q^2-q}{2},q^2)$ with respect to both latin and greek planes.*

Proof. Let α be a plane in $\mathrm{PG}(3,q)$. Assume that α contains q^2 lines of K. Any line of $K-\alpha$ meets α in one point. Let x and y denote the number of points of α which are centres of $\frac{q^2-q}{2}$−stars and q^2−stars, respectively. By counting, in two different ways, the total number of points of α, the numbers of pairs (P,ℓ), where $P\in\alpha\cap\ell$, $\ell\in K$, we get

$$\begin{aligned} x+y &= q^2+q+1, \\ \frac{q^2-q}{2}x+q^2y &= q^2(q+1)+\frac{q^4+q^2}{2}-q^2, \end{aligned}$$

which has no integer solution, a contradiction.

From Propositions I and II, K is either of type $(0,\frac{q^2+q}{2})$ with respect to stars of lines and of type $(\frac{q^2-q}{2},q^2)$ with respect to ruled planes, or of type $(\frac{q^2-q}{2},q^2)$ with respect to stars of lines and of type $(0,\frac{q^2+q}{2})$ with respect to ruled planes. In view of Results 3 and 4, K is either the set of the external lines or is the set of the secant lines of an ovoid in $\mathrm{PG}(3,q)$. Thus the Theorem is completely proved.

3 Conclusion

In this paper we have given a characterization of the point-subset of the Klein quadric H_5 which represents the set of lines either external to or secant to an ovoid in PG$(3, q)$ as two character set of H_5 with respect to its maximal subspaces. The arguments leading to these results are combinatorial arguments based largely on the integrality of the parameters at stake.

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