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Maximal local edge-connectivity of diamond-free graphs

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Abstract

The edge-connectivity of a graph G can be defined as $\lambda(G) = \min\{\lambda_G(u,v) \,|\, u,v \in V(G)\}$, where $\lambda_G(u,v)$ is the local edge-connectivity of two vertices u and v in G. We call a graph G *maximally edge-connected* when $\lambda(G) = \delta(G)$ and *maximally local edge-connected* when $\lambda_G(u,v) = \min\{d(u), d(v)\}$ for all pairs u and v of distinct vertices in G.

In 2000, Fricke, Oellermann and Swart (unpublished manuscript) proved that a bipartite graph G of order $n(G)$ is maximally local edge-connected when $n(G) \leq 4\delta(G) - 1$. As an extension of this result, we will show in this work that it is sufficient for G to be diamond-free with $n(G) \leq 4\delta(G) - 1$ to guarantee the maximally local edge-connectivity.

1 Terminology and introduction

We consider finite graphs without loops and multiple edges. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set of all vertices adjacent to v, and $E_G(v) = E(v)$ is the set of all edges incident with v. The numbers $n(G) = |V(G)|$, $m(G) = |E(G)|$ and $d(v) = |N(v)|$ are called the *order*, the *size* of G and the *degree* of v, respectively. The *minimum degree* of a graph G is denoted by $\delta(G) = \delta$.

The *edge-connectivity* $\lambda(G)$ of a graph G is the smallest number of edges whose deletion disconnects the graph. The *local edge-connectivity* $\lambda_G(u,v) = \lambda(u,v)$ between two distinct vertices u and v of a graph G, is the maximum number of edge-disjoint u-v paths in G. It is a well-known consequence of Menger's theorem [13] that

$$\lambda(G) = \min\{\lambda_G(u,v) \,|\, u,v \in V(G)\}.$$

It is straightforward to verify that $\lambda(G) \leq \delta(G)$ and $\lambda(u,v) \leq \min\{d(u), d(v)\}$. We call a graph G *maximally edge-connected* when $\lambda(G) = \delta(G)$ and *maximally* *local edge-connected* when $\lambda(u,v) = \min\{d(u), d(v)\}$ for all pairs u and v of distinct vertices in G.

The graph obtained from a complete graph of order 4 by removing an arbitrary edge is called a *diamond*. A graph G is then called *diamond-free*, if it contains no diamond as a (not necessarily induced) subgraph.

Since $\lambda(G) \leq \delta(G)$, there is a special interest in a graph G with $\lambda(G) = \delta(G)$. Different authors have presented sufficient conditions for a graph to be maximally edge-connected, as, for example Dankelmann and Volkmann [2, 3, 4], Fàbrega and Fiol [5], Fiol [6], Hellwig and Volkmann [8, 10], Lin, Miller and Rodger [12], Moriarty and Christopher [14], Volkmann [16, 18], and Wang, Xu and Wang [19].

For more information on this topic, we refer the reader to the survey article by Hellwig and Volkmann [11]. However, closely related investigations for the local edge-connectivity have received little attention until recently. Fricke, Oellermann and Swart [7] studied the local edge-connectivity of p-partite graphs and graphs with bounded diameter. Hellwig and Volkmann [9] and Volkmann [17] gave sufficient conditions for the maximally local edge-connectivity of p-partite digraphs and graphs with bounded clique number.

In [15], Volkmann proved that bipartite graphs G with $n(G) \leq 4\delta(G) - 1$ are maximally edge-connected. Fricke, Oellermann and Swart [7] showed that this condition even guarantees the maximally local edge-connectivity of G. By looking at diamond-free graphs Dankelmann et al. [1] were able to generalize a similar result on the maximally local (vertex-)connectivity of bipartite graphs. In this work, we will give a generalization of the results of Volkmann and Fricke et al. by proving that it is sufficient for G to be diamond-free with $n(G) \leq 4\delta(G) - 1$ to imply maximally local edge-connectivity.

2 Main Result

Theorem 2.1: *Let G be a diamond-free graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally local edge-connected.*

Proof: Assume G is not maximally local edge-connected. Therefore, we have two vertices $u, v \in V(G)$ with $r = \min\{d(u), d(v)\} - \delta \geq 0$ and an edge set S separating u and v with $|S| \leq \delta + r - 1$. Let U be the component of $G - S$ with $u \in V(U)$. Since $n \leq 4\delta - 1$ and by symmetry of u and v, without loss of generality, we may assume

$$n(U) \leq 2\delta - 1. \tag{1}$$

Furthermore, since $d(u) \geq \delta + r > |S|$ the vertex u must have at least one neighbour in $V(U)$ and, in addition, at least for one neighbour $u' \in V(U)$ of u, we have $E(u') \cap S = \emptyset$ (i.e. none of the edges incident with u' is in S). We distinguish two cases:

Case 1. u and u' have a common neighbour in $V(U)$.

Let $u'' \in N(u) \cap N(u') \cap V(U)$. Since G is diamond-free, u, u' and u'' can have no further common neighbours (pairwise). Let $W = (N(u) \cap V(U)) \setminus \{u', u''\}, W' = (N(u') \cap V(U))) \setminus \{u, u''\}$ and $W'' = (N(u'') \cap V(U)) \setminus \{u, u'\}$. Since G is diamond-free, $W \cap W' = \emptyset, W \cap W'' = \emptyset$ and $W' \cap W'' = \emptyset$. By $T = E(u) \cap S$ we refer to the edges of S incident with u, and let $T' = E(u') \cap S$ and $T'' = E(u'') \cap S$, respectively. Since no edge incident with u' is in S, we have

$$|W'| \geq \delta - 2. \tag{2}$$

Together with (1) this leads to

$$2\delta - 1 \geq n(U) \geq |W| + |W'| + |W''| + 3 \geq |W| + |W''| + \delta + 1.$$

Hence we have

$$|W| + |W''| \leq \delta - 2. \tag{3}$$

Obviously, we have $|T| + |W| \geq \delta + r - 2$ and $|T''| + |W''| \geq \delta - 2$. Thus, we deduce

$$2\delta + r - 4 \leq |T| + |T''| + |W| + |W''| \overset{(3)}{\leq} |T| + |T''| + \delta - 2,$$

which implies

$$|T| + |T''| \geq \delta + r - 2. \tag{4}$$

We now take a closer look at the vertices in W'. Assume there is a vertex $w \in W'$ with $E(w) \cap S = \emptyset$. Since G is diamond-free, w cannot be adjacent to u or u'', and have at most one neighbour in W'. Therefore, it follows that

$$2\delta - 1 \overset{(1)}{\geq} n(U) \geq |N(w) \setminus (W' \cup \{u'\})| + |W'| + |\{u, u', u''\}|$$

$$\overset{(2)}{\geq} \delta - 2 + \delta - 2 + 3 = 2\delta - 1.$$

So w must have exactly one neighbour $w' \in W'$ which cannot have further neighbours in U, and, of course, $\delta \geq 4$. Since G is diamond-free, w' is only adjacent to w and u', but cannot have neighbours in $(N(w) \setminus \{u', w'\}) \cup \{u, u''\} \cup (W' \setminus \{w\})$. Thus, w' must have at least $\delta - 2$ incident edges in S, i.e. $|E(w') \cap S| \geq \delta - 2$. Hence, every vertex in W' is either incident with at least one edge in S, or has exactly one neighbour in W' with at least 2 incident edges in S, and this neighbour cannot have further neighbours in W'. As a consequence, with $\delta \geq 4$ and $|T'| = |E(W') \cap S| = |\{E(x)|x \in W'\} \cap S|$ we obtain

$$|T'| \geq |W'| \geq \delta - 2. \tag{5}$$

By combining (5) with (4), we now deduce

$$|S| \geq |T| + |T'| + |T''| \geq \delta + r - 2 + \delta - 2 = \delta + r + (\delta - 4)$$

which is a contradiction to $|S| \leq \delta + r - 1$ for $\delta \geq 4$. In case $\delta = 3$ this deduction shows that all edges in S are incident with either u, u'' or w, where $w \in W'$, and since $|S| = r + 2$, $|T| + |T''| = r + 1$ and $|T'| = |W'| = 1$, the vertex w must have exactly one incident edge in S and one more neighbour $x \in V(U)$ besides u'. Since G is diamond-free, x can now only be adjacent to at most one of the vertices u, u' and u''. Hence, x must either have one more neighbour in U leading to $n(U) \geq 6 = 2\delta$, or an edge of S must be incident with x, a contradiction on the size of U or S.

Case 2. u and u' have no common neighbour in $V(U)$.
Again, we define $W = (N(u) \cap V(U)) \setminus \{u'\}$ and $W' = (N(u') \cap V(U)) \setminus \{u\}$. Now $W \cap W' = \emptyset$ and

$$|W'| \geq \delta - 1. \tag{6}$$

Let $T = E(u) \cap S$. Since $|W| \geq \delta + r - 1 - |T|$, we conclude

$$2\delta - 1 \overset{(1)}{\geq} n(U) \geq |W| + |W'| + 2 \geq \delta + r - 1 - |T| + \delta - 1 + 2 = 2\delta + r - |T|$$

and, therefore,

$$|T| \geq r + 1. \tag{7}$$

Here (6) and (7) together with $|S| \leq \delta + r - 1$ lead to the conclusion that there must be a vertex $u'' \in W'$ such that no edge in S is incident with u''. Now u'' can have at most one neighbour in W', hence we have $|W''| \geq \delta - 2$ where $W'' = N(u'') \setminus (W' \cup \{u, u'\})$, which leads us to

$$2\delta - 1 \overset{(1)}{\geq} n(U) \geq |W'| + |W''| + 2 \geq \delta - 1 + \delta - 2 + 2 = 2\delta - 1.$$

We conclude that u'' must have a neighbour $w' \in W'$, and since G is diamond-free, w' must be incident with at least $\delta - 2$ edges in S, i.e. $|T'| \geq \delta - 2$ where $T' = E(w') \cap S$. Furthermore, we must have $W \subseteq W''$, and

$$\delta + r - 1 \geq |S| \geq |T| + |T'| \overset{(7)}{\geq} r + 1 + \delta - 2 = \delta + r - 1.$$

Then it follows that $|T| = r + 1$ and thus $|W| = \delta - 2$ and $W = W''$. Now, an arbitrary vertex $w \in W (= W'')$ cannot be adjacent to u' or w', and no edge in S is incident with w. Furthermore, w cannot have a neighbour in W, otherwise we would have a diamond in U together with u and u''. Thus, $W \cap (N(w) \setminus \{u, u''\} = \emptyset$, leading us to

$$2\delta - 1 \overset{(1)}{\geq} n(U) \geq |W| + |N(w) \setminus \{u, u''\}| + |\{u, u', u'', w'\}| \geq \delta - 2 + \delta - 2 + 4 = 2\delta,$$

a contradiction. □

As a direct consequence of Theorem 2.1 we obtain the following results.

Corollary 2.2: *Let G be a diamond-free graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally edge-connected.*

Corollary 2.3 (Volkmann [15]) *Let G be a bipartite graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally edge-connected.*

Corollary 2.4 (Fricke, Oellermann, Swart [7]) *Let G be a bipartite graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally local edge-connected.*

To see that Theorem 2.1 and Corollary 2.2 are sharp in the sense that for every integer p there exists a diamond-free graph G with $\delta(G) = p$ and $n(G) = 4\delta(G)$, which is not maximally edge-connected and, therefore, not maximally local edge-connected, we consider the following example.

Example 2.5: Let G be the graph obtained from two complete bipartite graphs $K_{p,p}$ ($p \geq 2$) by adding one arbitrary edge between them. Of course, G is diamond-free with edge-connectivity $\lambda(G) = 1$, while $\delta(G) = p$ and $n(G) = 4\delta(G)$. Therefore, G is not maximally edge-connected and not maximally local edge-connected.

To see that Theorem 2.1 does not hold for $\delta(G) = 2$, we consider the following graph.

Example 2.6: Let G be the graph obtained from two 3-cycles by adding one arbitrary edge between them. Then $\delta(G) = 2$, but $\lambda(G) = 1$. Thus, G is not maximally edge-connected and not maximally local edge-connected, but we have $n(G) = 6 \leq 7 = 4\delta(G) - 1$.

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