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Orthogonal double covers of complete bipartite graphs

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Abstract

Let $\mathcal{H} = \{A_1, \ldots, A_n, B_1, \ldots, B_n\}$ be a collection of $2n$ subgraphs of the complete bipartite graph $K_{n,n}$. The collection $\mathcal{H}$ is called an *orthogonal double cover* (ODC) of $K_{n,n}$ if each edge of $K_{n,n}$ occurs in exactly two of the graphs in $\mathcal{H}$; $E(A_i) \cap E(A_j) = \phi = E(B_i) \cap E(B_j)$ for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and for any $i, j \in \{1, \ldots, n\}$, $|E(A_i) \cap E(B_j)| = 1$. If $A_i \cong G \cong B_i$ for all $i \in \{1, \ldots, n\}$, then $\mathcal{H}$ is called an ODC *of* $K_{n,n}$ *by* G. In this paper, we establish a product construction for ODCs of complete bipartite graphs.

1 Introduction

Throughout this paper, all graphs are finite, simple and undirected. Let $V(G)$ and $E(G)$ denote, respectively, the vertex set and the edge set of a graph G. For $A \subseteq V(G)$, $G[A]$ denotes the subgraph induced by A in G. Let $K_{r,r}$ be the complete bipartite graph with part sizes both r. We refer to the book [1] for graph theory notation and terminology not described in this paper.

Let H be an arbitrary simple graph with $|V(H)|$ vertices and let $\mathcal{H} = \{H_1, H_2, \ldots, H_{|V(H)|}\}$ be a collection of $|V(H)|$ subgraphs of H. The collection $\mathcal{H}$ is called an *orthogonal double cover* (briefly, ODC) of H if there exists a bijective mapping $\phi : V(H) \to \mathcal{H}$ such that:

(i) Every edge of H is contained in exactly two of the graphs in $\mathcal{H}$.

(ii) For every choice of distinct vertices u, w of H,

$$|\, E(\phi(u)) \cap E(\phi(w))\,| = \begin{cases} 1 & \text{if } uw \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

If $H_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{H}$ is called an ODC of H by G.

For results on ODCs of graphs, see the survey article [2]. It can be easily observed that an ODC of $K_{m,m}$ by mK_2 is equivalent to a pair of orthogonal latin squares of order m, where mK_2 denotes m disjoint copies of K_2. The aim of this paper is to generalize an earlier result of design theory to ODCs of complete bipartite graphs.

For simple graphs G and H, the *tensor product*, $G \times H$, of G and H, is the graph with vertex set $V(G) \times V(H)$ and $E(G \times H) = \{\,(u, v)(x, y) : ux \in E(G) \text{ and } vy \in E(H)\}$. If the simple graphs G and H are bipartite with bipartitions (X, Y) and (U, V), respectively, then the induced subgraphs $(G \times H)[(X \times U) \cup (Y \times V)]$ and $(G \times H)[(X \times V) \cup (Y \times U)]$ are called the *weak-tensor products* of G and H. We denote the weak-tensor product $(G \times H)[(X \times U) \cup (Y \times V)]$ by $G \times_{\frac{1}{2}} H$. Clearly, $K_{m,m} \times_{\frac{1}{2}} K_{n,n} \cong K_{mn,mn}$ and $mK_2 \times_{\frac{1}{2}} nK_2 \cong mnK_2$.

2 Result

Theorem 2.1. *If there is an ODC of $K_{m,m}$ by G and if there is an ODC of $K_{n,n}$ by H, then there is an ODC of $K_{mn,mn}$ by $G \times_{\frac{1}{2}} H$.*

Proof. Let $\mathcal{G}^1 \cup \mathcal{G}^2$, where $\mathcal{G}^k = \{G_1^k, G_2^k, \ldots, G_m^k\}$, $k \in \{1, 2\}$, be an ODC of $K_{m,m}$ by G on $V(K_{m,m}) = X \cup Y$, where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$ is the bipartition of $K_{m,m}$, with the mapping $\phi(x_i) = G_i^1$ and $\phi(y_i) = G_i^2$, for $i \in \{1, 2, \ldots, m\}$ and let $\mathcal{H}^1 \cup \mathcal{H}^2$, where $\mathcal{H}^k = \{H_1^k, H_2^k, \ldots, H_n^k\}$, $k \in \{1, 2\}$, be an ODC of $K_{n,n}$ by H on $V(K_{n,n}) = U \cup V$, where $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ is the bipartition of $K_{n,n}$, with the mapping $\psi(u_i) = H_i^1$ and $\psi(v_i) = H_i^2$, for $i \in \{1, 2, \ldots, n\}$.

Let $W = V(K_{mn,mn})$ and let the partite sets of $K_{mn,mn}$ be

$$\{(x_i, u_j) : i \in \{1, 2, \ldots, m\},\ j \in \{1, 2, \ldots, n\}\}$$

and

$$\{(y_i, v_j) : i \in \{1, 2, \ldots, m\},\ j \in \{1, 2, \ldots, n\}\}.$$

Consider the set $\mathcal{F}^k = \{(G_i^k \times H_j^k)[W] : i \in \{1, 2, \ldots, m\},\ j \in \{1, 2, \ldots, n\}\}$, $k \in \{1, 2\}$, of subgraphs of $K_{mn,mn}$. Define the mapping $f : W \to \mathcal{F}^1 \cup \mathcal{F}^2$ by

$f((x_i, u_j)) = (G_i^1 \times H_j^1)[W]$ and $f((y_i, v_j)) = (G_i^2 \times H_j^2)[W]$. Clearly, $(G_i^k \times H_j^k)[W] \cong G \times_{\frac{1}{2}} H$, for $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, $k \in \{1, 2\}$.

Claim 1. Every edge of $K_{mn,mn}$ occurs in exactly two graphs of $\mathcal{F}^1 \cup \mathcal{F}^2$.

Consider an arbitrary edge $(x_i, u_j)(y_k, v_\ell)$ of $K_{mn,mn}$. Since $\mathcal{G}^1 \cup \mathcal{G}^2$ and $\mathcal{H}^1 \cup \mathcal{H}^2$ are ODCs of $K_{m,m}$ and $K_{n,n}$, respectively, the edges $x_i y_k$ and $u_j v_\ell$ are, respectively, in exactly two graphs of $\mathcal{G}^1 \cup \mathcal{G}^2$ and $\mathcal{H}^1 \cup \mathcal{H}^2$. Let the two graphs containing $x_i y_k$ be G_p^1 and G_q^2 and that of $u_j v_\ell$ be H_r^1 and H_s^2. Then the two graphs containing the edge $(x_i, u_j)(y_k, v_\ell)$ are $(G_p^1 \times H_r^1)[W]$ and $(G_q^2 \times H_s^2)[W]$.

Claim 2. Let $k \in \{1, 2\}$. Any two graphs in $\mathcal{F}^k$ have no edge in common.

The two graphs $(G_p^k \times H_r^k)[W]$ and $(G_q^k \times H_s^k)[W]$ have no edge in common, because $E(G_p^k) \cap E(G_q^k) = \emptyset$ and $E(H_r^k) \cap E(H_s^k) = \emptyset$.

Claim 3. A graph in $\mathcal{F}^1$ and a graph in $\mathcal{F}^2$ have exactly one edge in common.

The two graphs $(G_p^1 \times H_r^1)[W]$ and $(G_q^2 \times H_s^2)[W]$ have exactly one edge in common, since $\left|E(G_p^1) \cap E(G_q^2)\right| = 1$ and $|E(H_r^1) \cap E(H_s^2)| = 1$.

By Claims 1, 2 and 3, $\mathcal{F}^1 \cup \mathcal{F}^2$ is an ODC of $K_{mn,mn}$ by $G \times_{\frac{1}{2}} H$. This completes the proof. ■

If $G = mK_2$ and $H = nK_2$, then Theorem 2.1 is:

Corollary 2.1. *If there is an ODC of $K_{m,m}$ by mK_2 and if there is an ODC of $K_{n,n}$ by nK_2, then there is an ODC of $K_{mn,mn}$ by mnK_2.*

Equivalently, we have the following:

Corollary 2.2. *If A_1 and A_2 are orthogonal latin squares of order m and B_1 and B_2 are orthogonal latin squares of order n, then the direct products $A_1 \times B_1$ and $A_2 \times B_2$ are orthogonal latin squares of order mn.*

The above corollaries are well-known results in design theory.

El-Shanawany and Gronau have proved the following: "Let $m \neq 2, 6$ be a positive integer, and assume that there exist ODCs $\mathcal{G}_i$ of $K_{n,n}$ by G_i for $i \in \{0, 1, \ldots, m-1\}$. Then there exists an ODC of $K_{mn,mn}$ by $G_0 \cup G_1 \cup \ldots \cup G_{m-1}$" (see Theorem 5.11 in [2]). In the special case when all the G_i's are isomorphic to H, the above statement is: "Let $m \neq 2, 6$ be a positive integer, and assume that there exists an ODC $\mathcal{G}$ of $K_{n,n}$ by H. Then there exists an ODC of $K_{mn,mn}$ by mH." If $G = mK_2$, then Theorem 2.1 is: "Let $m \neq 2, 6$ be a positive integer. If there is an ODC of $K_{n,n}$ by H, then there is an ODC of $K_{mn,mn}$ by $mK_2 \times_{\frac{1}{2}} H$." Since $mK_2 \times_{\frac{1}{2}} H \cong mH$, this is the special case of El-Shanawany and Gronau.

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