AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **47** (2010), Pages 11–20

New classes of orthogonal designs and weighing matrices derived from near normal sequences

Christos Koukouvinos   Dimitris E. Simos

Department of Mathematics
National Technical University of Athens
Zografou 15773, Athens
Greece
ckoukouv@math.ntua.gr   dsimos@math.ntua.gr

Abstract

Directed sequences have been recently introduced and used for constructing new orthogonal designs. The construction is achieved by multiplying the length and type of suitable compatible sequences. In this paper we show that near normal sequences of length $n = 4m + 1$ can be used to construct four directed sequences of lengths $2m+1$, $2m+1$, $2m$, $2m$ and type $(4m+1, 4m+1) = (n, n)$ with zero NPAF. The above method leads to the construction of many large orthogonal designs. In addition, we obtain new infinite families of weighing matrices constructed by near normal sequences, such as $W(156+4k, 125)$, $W(160+4k, 144)$, $W(200+4k, 196)$ and $W(276 + 4k, 225)$ for all $k \geq 0$. These families resolve the existence and construction of over 30 weighing matrices which are listed as open in the second edition of the Handbook of Combinatorial Designs.

1 Introduction and preliminary results

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ $(s_i > 0)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$AA^T = (\sum_{i=1}^{u} s_i x_i^2) I_n.$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [3], where this was first defined, it was mentioned that

$$A^T A = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

and so our alternative description of A applies equally well to the columns of A. It was also shown in [3] that $u \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, b odd, $a = 4c + d$, $0 \leq d < 4$.

A weighing matrix $W = W(n, k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n; k)$. The number k is called the *weight* of W. If $k = n$, that is, all the entries of W are ± 1 and $WW^T = nI_n$, then W is called an Hadamard matrix of order n. In this case $n = 1, 2$ or $n \equiv 0 \pmod 4$.

Given the sequence $A = \{a_1, a_2, \ldots, a_n\}$ of length n the *non-periodic autocorrelation function* $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1, \tag{1}$$

and the *periodic autocorrelation function* $P_A(s)$ is defined, reducing $i + s$ modulo n, as

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1. \tag{2}$$

The following theorem which uses four circulant matrices in the Goethals-Seidel array is very useful in our construction for orthogonal designs.

Theorem 1 *[4, Theorem 4.49] Suppose there exist four circulant matrices A, B, C, D of order n satisfying*

$$AA^T + BB^T + CC^T + DD^T = fI_n.$$

Let R be the back diagonal matrix. Then the Goethals-Seidel array

$$GS = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^T R & -C^T R \\ -CR & -D^T R & A & B^T R \\ -DR & C^T R & -B^T R & A \end{pmatrix}$$

is a $W(4n, f)$ when A, B, C, D are $(0, 1, -1)$ matrices, and an orthogonal design $OD(4n; s_1, s_2, \ldots, s_u)$ on $x_1, x_2, \ldots, x_u$ when A, B, C, D have entries from $\{0, \pm x_1, \ldots, \pm x_u\}$ and $f = \sum_{j=1}^{u} (s_j x_j^2)$.

Corollary 1 *If there are four sequences A, B, C, D of length n with entries from the set $\{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$ with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals-Seidel array to form an $OD(4n; s_1, s_2, s_3, s_4)$. If there are sequences of length n with zero non-periodic autocorrelation function, then there are sequences with zero non-periodic autocorrelation function and the same list of weights as the sequences of length n, but of length $n + m$ for all $m \geq 0$.*

Notation. We use the following notation throughout this paper.

1. We use $\overline{a}$ to denote $-a$.
2. We use 0_n to denote a sequence of length n with every element zero.
3. We define the $NPAF$ (PAF) of a set of sequences as the sum of the corresponding $NPAF$ (PAF) of the individual sequences.
4. For given sequences $A = (a_1, a_2, \ldots, a_{m+1})$ and $C = (c_1, c_2, \ldots, c_m)$, the interleaved sequence A/C of A and C is defined as $A/C = (a_1, c_1, a_2, c_2, \ldots, a_m, c_m, a_{m+1})$.
5. We will say that some sequences of variables are *directed* if the sequences have zero autocorrelation function independently from the properties of the variables, i.e. they do not rely on commutativity to ensure zero autocorrelation. For example $\{a, b\}$ and $\{a, -b\}$ are two directed sequences while $\{a, b\}$ and $\{b, -a\}$ are not directed. Also $\{a, b\}$, $\{a, -b\}$, $\{c, d\}$ and $\{c, -d\}$ are four directed sequences while $\{a, b\}$, $\{b, -a\}$, $\{c, d\}$ and $\{c, -d\}$ are not directed.

Directed sequences were introduced in [12] and used extensively in [2] to construct orthogonal designs. Base sequences and normal sequences were used in [2] to construct directed sequences. In this paper we show that near normal sequences can be used for the construction of directed sequences and then orthogonal designs. Moreover, we construct new infinite families of weighing matrices obtained by near normal sequences which resolve several open cases of weighing matrices.

2 Main results

In this section we use near normal sequences of length $n = 4m + 1$, i.e. $NNS(n)$, in order to construct four directed sequences of lengths $2m + 1$, $2m + 1$, $2m$, $2m$ and type (n, n) with zero NPAF. For the definitions of Base and near normal sequences, we refer to [7, 9]. Some sets of near normal sequences are given in [11] and [13]. A complete classification of near normal sequences $NNS(n)$, $n = 4m + 1$ for $1 \leq m \leq 11$, and partial results for $m = 12, 13, 14$ and 15, is given in [8].

Theorem 2 *Let $n = 4m + 1$. There are four directed sequences of lengths $2m + 1$, $2m + 1$, $2m$, $2m$ and type (n, n) with zero NPAF if and only if there are near normal sequences $NNS(n)$.*

Proof. A quadruple $(E, F; G, H)$ with $E = (1, X/O_{m-1})$ and $F = (Y/O_{m-1})$, are near normal sequences, if and only if the $(1, -1)$ sequences $A = (1, X/Y)$, $B = (1, X/-Y)$, $C = G + H$, $D = G - H$ of lengths $2m+1$, $2m+1$, $2m$, $2m$, respectively, are base sequences ([11, 13]). Applying Theorem 5 of [2] to the derived sequences we have that there are base sequences $BS(2m + 1, 2m)$ if and only if there are four directed sequences of lengths $2m+1$, $2m+1$, $2m$, $2m$ and type $(4m+1, 4m+1) = (n, n)$ with zero NPAF. □

The new directed sequences of lengths $2m+1, 2m+1, 2m, 2m$ and type $(4m+1, 4m+1)$ for $m = 1, \ldots, 15$, obtained by the known near normal sequences given in [8, 11, 13] and Theorem 2 can be found in [10].

Remark 1 The directed sequences given by Theorem 2 cannot be directly used in some array to construct orthogonal designs due to their unequal lengths. Their lengths can become equal by adding zeros to the end of the shorten sequences. Thus, the new sequences $\{A, B, [C, 0], [D, 0]\}$ will be of lengths $2m+1, 2m+1, 2m+1, 2m+1$, with zero NPAF and they can be used in the Goethals-Seidel array to give two-variable orthogonal designs of the form $OD(4(2m+1); 4m+1, 4m+1)$ (see Corollary 1).

The main advantage of directed sequences, their multiplication property, is that their variables can be replaced by sequences with zero NPAF to obtain longer sequences of different type, with zero NPAF, suitable for the construction of large orthogonal designs. This is illustrated in the Example 1. To construct orthogonal designs by eight circulant matrices we use the K-array (for details see [5, 6]).

Example 1 The sequences $E = (c, d)$ and $F = (c, -d)$ have zero NPAF and can be used for the construction of an orthogonal design $OD(4; 2, 2)$. Using the directed sequences

$$A = (a, a, b),\ B = (-a, b, b),\ C = (a, -a, 0),\ D = (-b, b, 0),$$

retrieved by [10] and applying Remark 1; we replace a by the sequence E, b by the sequence F and 0 by the sequence 0_2, and the following new sequences are obtained.

$$\begin{aligned} A' &= (c, d, c, d, c, -d), \qquad & B' &= (-c, -d, c, -d, c, -d), \\ C' &= (c, d, -c, -d, 0, 0), \qquad & D' &= (-c, d, c, -d, 0, 0). \end{aligned}$$

It is easy to check that $[A', 0_k]$, $[B', 0_k]$, $[C', 0_k]$ and $[D', 0_k]$ have zero NPAF and thus can be used in the Goethals-Seidel array to construct an orthogonal design $OD(4(6+k); 10, 10)$, for all $k \geq 0$.

We can obtain an $OD(8(6+k); 10, 10, 10, 10)$, for all $k \geq 0$, by using the sequences $A_1 = [A', 0_k]$, $A_2 = [B', 0_k]$, $A_3 = [C', 0_k]$, $A_4 = [D', 0_k]$, $A_5 = [E', 0_k]$, $A_6 = [F', 0_k]$, $A_7 = [G', 0_k]$, and $A_8 = [H', 0_k]$, where

$$\begin{aligned} E' &= (e, f, e, f, e, -f), \qquad & F' &= (-e, -f, e, -f, e, -f), \\ G' &= (e, f, -e, -f, 0, 0), \qquad & H' &= (-e, f, e, -f, 0, 0), \end{aligned}$$

and the corresponding circulant matrices satisfy

$$(A'E'^T - E'A'^T) + (B'F'^T - F'B'^T) + (C'G'^T - G'C'^T) + (D'H'^T - H'D'^T) = 0$$

□

Corollary 2 *Suppose that there exist two sequences E and F of length s and of type (u_1, u_2) with zero NPAF. Then there exist four sequences of length $(2m+1)\cdot s$ and of type $(u_1\cdot(4m+1), u_2\cdot(4m+1))$ with zero NPAF that can be used in the Goethals-Seidel array to construct an orthogonal design $OD(4\cdot((2m+1)\cdot s); u_1\cdot(4m+1), u_2\cdot(4m+1))$, for all $m = 1, \ldots, 15$. Furthermore, there exist eight sequences of length $(2m+1)\cdot s$ and type $(u_1\cdot(4m+1), u_1\cdot(4m+1), u_2\cdot(4m+1), u_2\cdot(4m+1))$ that can be used in the K-array to construct an orthogonal design $OD(8\cdot((2m+1)\cdot s); u_1\cdot(4m+1), u_1\cdot(4m+1), u_2\cdot(4m+1), u_2\cdot(4m+1))$, for all $m = 1, \ldots, 15$.*

Proof. There are four directed sequences A, B, C, D of lengths $2m+1$ (equal lengths; see Remark 1) and of type $(4m+1, 4m+1)$ constructed by Theorem 2. We replace the elements of these directed sequences by the given sequences and we obtain the result. To be more precise, replace the element a of the directed sequences by E, b by F and 0 by 0_s. The derived sequences are four sequences of length $(2m+1)\cdot s$ and of type $(u_1\cdot(4m+1), u_2\cdot(4m+1))$ with zero NPAF and can be used in Goethals-Seidel array to give the desirable orthogonal design $OD(4\cdot((2m+1)\cdot s); u_1\cdot(4m+1), u_2\cdot(4m+1))$.

To construct an orthogonal design $OD(8\cdot((2m+1)\cdot s); u_1\cdot(4m+1), u_1\cdot(4m+1), u_2\cdot(4m+1), u_2\cdot(4m+1))$ we need to find eight sequences $A', B', C', D', E', F', G', H'$ with zero NPAF for which the corresponding circulant matrices satisfy

$$(A'E'^T - E'A'^T) + (B'F'^T - F'B'^T) + (C'G'^T - G'C'^T) + (D'H'^T - H'D'^T) = 0$$

Construct A', B', C', D' as in the first stage of the proof. Copy these sequences, call the new sequences E', F', G', H' and replace their variables with new ones (i.e. replace e by g and f by h). It can be easily verified that these are the desirable eight sequences and can be used in K-array to construct an orthogonal design $OD(8\cdot((2m+1)\cdot s); u_1\cdot(4m+1), u_1\cdot(4m+1), u_2\cdot(4m+1), u_2\cdot(4m+1))$. □

Corollary 2 is valuable when large orthogonal designs are desirable. In Example 2 we shall use the known two-variable orthogonal designs and the constructed directed sequences to present some infinite families of large orthogonal designs.

Set $N = \{(3,1,4), (6,2,8), (6,5,5), (10,10,10), (10,4,16), (14,13,13), (18,5,20), (20,8,32), (24,17,17), (26,26,26), (30,10,40), (30,25,25), (40,34,34), (42,13,52)\}$.

Example 2 The following infinite families of orthogonal designs exist.

- $OD(4\cdot(2m+1+k); 4m+1, 4m+1)$ and $OD(8\cdot(2m+1+k); 4m+1, 4m+1, 4m+1, 4m+1)$ for all $m = 1, \ldots, 15$ and all $k \geq 0$. (Use the four directed sequences constructed in Theorem 2).
- $OD(4\cdot(2^t\cdot(2m+1)+k); 2^t\cdot(4m+1), 2^t\cdot(4m+1))$ and $OD(8\cdot(2^t\cdot(2m+1)+k); 2^t\cdot(4m+1), 2^t\cdot(4m+1), 2^t\cdot(4m+1), 2^t\cdot(4m+1))$ for all $t \geq 0$, all $m = 1, \ldots, 15$ and all $k \geq 0$. (Use the four directed sequences constructed in Theorem 2 and the two sequences of length 2 and type $(2,2)$, with zero NPAF, given in [12]).

- $OD(4\cdot(s\cdot(2m+1)+k); u_1\cdot(4m+1), u_2\cdot(4m+1))$ and $OD(8\cdot(s\cdot(2m+1)+k); u_1\cdot(4m+1), u_1\cdot(4m+1), u_2\cdot(4m+1), u_2\cdot(4m+1))$ for all triples $(s, u_1, u_2) \in N$, all $m = 1, \ldots, 15$ and all $k \geq 0$. (Use the four directed sequences constructed in Theorem 2 and the two sequences of length s and type (u_1, u_2), with zero NPAF, given in [12]).

□

Theorem 3 *If there exist two directed sequences of lengths s and type (u_1, u_2) with zero NPAF and two sequences of lengths t and type (u_3, u_4) with zero NPAF then there exist an orthogonal design $OD(4 \cdot s \cdot t \cdot (2m+1); u_1 \cdot u_3 \cdot (4m+1), u_2 \cdot u_4 \cdot (4m+1))$ for all $m = 1, \ldots, 15$.*

Proof. There are four directed sequences of length $2m+1$ and type $(4m+1, 4m+1)$ constructed in Theorem 2, for $m = 1, \ldots, 15$. If we replace the two variables of these sequences by the two directed sequences of lengths s and type (u_1, u_2) we obtain four directed sequences of length $s \cdot (2m+1)$ and type $(u_1 \cdot (4m+1), u_2 \cdot (4m+1))$ with zero NPAF. Then the variables of the derived directed sequences are replaced by the two sequences of lengths t and type (u_3, u_4) with zero NPAF. The last derived sequences give the result. □

Set $M = \{(2,2,2), (6,5,5), (10,10,10), (14,13,13), (24,17,17), (26,26,26), (30,25,25), (40,34,34)\}$.

Corollary 3 *Let $x_1 \geq 0, x_2 \geq 0, \ldots, x_8 \geq 0, k \geq 0$ be integer numbers. There exists an orthogonal design $OD(4(\cdot t \cdot 2^{x_1} \cdot 6^{x_2} \cdot 10^{x_3} \cdot 14^{x_4} \cdot 24^{x_5} \cdot 26^{x_6} \cdot 30^{x_7} \cdot 40^{x_8} \cdot (2m+1) + k); u_1 \cdot 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8} \cdot (4m+1), u_2 \cdot 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8} \cdot (4m+1))$ for all $(t, u_1, u_2) \in N$ and $m = 1, \ldots, 15$.*

Proof. There are directed sequences of length s and type (t_1, t_2) with zero NPAF for all $(s, t_1, t_2) \in M$ (see [12]). For any integers $x_1 \geq 0, x_2 \geq 0, \ldots, x_8 \geq 0$ one can construct directed sequences with zero NPAF which will be of length $2^{x_1} \cdot 6^{x_2} \cdot 10^{x_3} \cdot 14^{x_4} \cdot 24^{x_5} \cdot 26^{x_6} \cdot 30^{x_7} \cdot 40^{x_8}$ and type $(2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8}, 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8})$. These sequences are plugged in the four directed sequences of Theorem 2 to give the desirable directed sequences. Then, for $(t, u_1, u_2) \in N$, we use the sequences with zero NPAF from [12] padded with k zeros at the end and the result follows. □

Example 3 For $x_2 = 1$, and $x_5 = 1$, whereas $x_i = 0, i = 1, 3, 4, 6, 7, 8$ we obtain two directed sequences of length 144 and type $(85, 85)$. Using the four directed sequences of length 9 and type $(17, 17)$ (derived from $NNS(17)$; see Theorem 2 and Remark 1) we obtain four directed sequences of lengths 1296 and of type $(1445, 1445)$ with zero NPAF. There are two sequences of length 3 and type $(1, 4)$ with zero NPAF. Thus, we can construct four sequences of length 3888 and of type $(1445, 5780)$ that can be used in the Goethals-Seidel array to give an $OD(15552 + 4k; 1445, 5780)$, for all $k \geq 0$.

Corollary 4 *Let* $x_1 \geq 0, x_2 \geq 0, \ldots, x_8 \geq 0, k \geq 0$ *be integer numbers. There exists an orthogonal design* $OD(8(\cdot t \cdot 2^{x_1} \cdot 6^{x_2} \cdot 10^{x_3} \cdot 14^{x_4} \cdot 24^{x_5} \cdot 26^{x_6} \cdot 30^{x_7} \cdot 40^{x_8} \cdot (2m+1) + k); u_1 \cdot 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8} \cdot (4m+1), u_2 \cdot 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8} \cdot (4m+1), u_1 \cdot 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8} \cdot (4m+1), u_2 \cdot 2^{x_1} \cdot 5^{x_2} \cdot 10^{x_3} \cdot 13^{x_4} \cdot 17^{x_5} \cdot 26^{x_6} \cdot 25^{x_7} \cdot 34^{x_8} \cdot (4m+1))$ *for all* $(t, u_1, u_2) \in N$ *and* $m = 1, \ldots, 15$.

Proof. Construct the zero NPAF sequences as in Corollary 3. Then, copy these sequences into new ones and rename their variables. The derived sequences are the desirable eight sequences that can be used in the K-array to obtain the result. □

As is obvious, the suggested multiplication constructions for orthogonal designs and sequences with zero autocorrelation will be useful for the construction of large orthogonal designs. Moreover, it is believed that these methods might be helpful in the theoretical study of orthogonal designs and their asymptotic properties.

3 New weighing matrices from near normal sequences

Orthogonal designs with fewer variables can be obtained by Equating and Killing variables [4] from the orthogonal designs constructed from directed sequences and produce weighing matrices, since an orthogonal design $OD(n; k)$ is equivalent to a weighing matrix $W = W(n, k)$. In a similar way, orthogonal designs derived by directed sequences can produce infinite families of weighing matrices.

3.1 An infinite family of $W(156 + 4k, 125)$

Corollary 5 *There exists an infinite family of orthogonal designs* $OD(156 + 4k; 25, 100)$. *Furthermore, there exists an infinite family of weighing matrices* $W(156 + 4k, 125)$*, for all* $k \geq 0$.

Proof. Use the two sequences of length 3 and type $(1, 4)$ with zero NPAF from [12]. There are four directed sequences of lengths 13 (equal lengths; see Remark 1) and of type $(25, 25)$ constructed by Theorem 2 (see [10]). By applying Corollary 2 we obtain an infinite family of orthogonal designs $OD(156 + 4k; 25, 100)$. By equating the variables in the previous family and replacing them with 1 we produce four sequences of length 39 with zero NPAF, which padded with k zeros give rise to an infinite family of weighing matrices $W(156 + 4k, 125)$, for all $k \geq 0$:

```
++-++--0---+-0-++-+0+++-+0+++--0-++-+0+
--++0+++-+0+--++0+--++0+++--0-++-+0++0+
+0+++---+++-++-+0+-0-++---+--+--+-0-000
++--0--0--0-+0+++---++0++0+-0-+0+--+000
```

□

3.2 An infinite family of $W(160+4k, 144)$

Corollary 6 *There exists an infinite family of orthogonal designs* $OD(160+4k; 72, 72)$. *Furthermore, there exists an infinite family of weighing matrices* $W(160+4k, 144)$*, for all* $k \geq 0$.

Proof. Use the two sequences of length 8 and type $(8, 8)$ with zero NPAF from [12]. There are four directed sequences of lengths 5 (equal lengths; see Remark 1) and of type $(9, 9)$ constructed by Theorem 2 (see [10]). By applying Corollary 2 we obtain an infinite family of orthogonal designs $OD(160+4k; 72, 72)$. By equating the variables in the previous family and replacing them with 1 we produce four sequences of length 40 with zero NPAF, which padded with k zeros give rise to an infinite family of weighing matrices $W(160+4k, 144)$, for all $k \geq 0$:

```
++-++++-++-++++-+---+-++++-++++--+++-+--
++-++++-+---+-++--+----++---+-+++---+-++
++-++++--+++-+--+---+-++--+----+00000000
-+++-+----+----+++-++++-+---+-++00000000
```

□

3.3 An infinite family of $W(200+4k, 196)$

Corollary 7 *There exists an infinite family of orthogonal designs* $OD(200+4k; 98, 98)$. *Furthermore, there exists an infinite family of weighing matrices* $W(200+4k, 196)$*, for all* $k \geq 0$.

Proof. Use the two sequences of length 2 and type $(2, 2)$ with zero NPAF from [12]. There are four directed sequences of lengths 25 (equal lengths; see Remark 1) and of type $(49, 49)$ constructed by Theorem 2 (see [10]). By applying Corollary 2 we obtain an infinite family of orthogonal designs $OD(200+4k; 98, 98)$. By equating the variables in the previous family and replacing them with 1 we produce four sequences of length 50 with zero NPAF, which padded with k zeros give rise to an infinite family of weighing matrices $W(200+4k, 196)$, for all $k \geq 0$:

```
++-++++-++-+--+----++++---+----++++-+++---+-+++-+-
----+----+--+---+-++-+---+--+-++-+---++++---+-+++-
++++-+---+--+---++-++--+-+-+-+--++-++++-+++---++00
+--+--+---+-+++--++++++++---+++--+---+++-++++-+-00
```

□

3.4 An infinite family of $W(276+4k, 225)$

Corollary 8 *There exists an infinite family of orthogonal designs* $OD(276+4k; 45, 180)$. *Furthermore, there exists an infinite family of weighing matrices* $W(276+4k, 225)$*, for all* $k \geq 0$.

Proof. Use the two sequences of length 3 and type $(1, 4)$ with zero NPAF from [12]. There are four directed sequences of lengths 23 (equal lengths; see Remark 1) and of type $(45, 45)$ constructed by Theorem 2 (see [10]). By applying Corollary 2 we obtain an infinite family of orthogonal designs $OD(276 + 4k; 45, 180)$. By equating the variables in the previous family and replacing them with 1 we produce four sequences of length 69 with zero NPAF, which padded with k zeros give rise to an infinite family of weighing matrices $W(276 + 4k, 225)$, for all $k \geq 0$:

```
++--0-++--0---++0+--+-0-++-+0+--+-0---+-0-++-+0+++-+0+++-+0+++-+0++0+
--+--++0+--++0+--++0+--++0+++--0-++--0---++0+++--0---+-0-++-+0+++-+0+
++-+0+++--0---+--+++---+++-+0++0++0+-0-++-++---+++-++-+0+--++0+++-000
+0+--+-0---++0++0+-0-+0++0+++---+--+--++0+-0-+0+-0--0-++-+0+--++0+000
```

□

In the following table, the first column gives the new family of weighing matrices while the second column gives the range of the weighing matrices listed. The third column gives the weight of the family, while in the fourth column we give explicitly the order n of each weighing matrix. We use the following convention: if the order appears without parentheses in the record corresponding to w, then a $W(n, w)$ is known to exist (see [1]); if n appears in parentheses, then its existence was previously unresolved and listed as open in the second edition of the Handbook of Combinatorial Designs. Finally, the last column gives the total number of cases resolved by the respective families of weighing matrices listed.

Table 1: New families of weighing matrices derived from near normal sequences.

Family	Range	Weight	Order					Total
$W(156 + 4k, 125)$	$0 \leq k \leq 4$	125	156	160	(164)	168	(172)	2
$W(160 + 4k, 144)$	$0 \leq k \leq 4$	144	160	(164)	168	(172)	(176)	3
$W(200 + 4k, 196)$	$0 \leq k \leq 29$	196	200	(204)	208	(212)	(216)	20
			(220)	(224)	228	(232)	(236)	
			240	(244)	(248)	(252)	256	
			(260)	(264)	(268)	272	(276)	
			(280)	(284)	288	(292)	(296)	
			300	304	(308)	312	(316)	
$W(276 + 4k, 225)$	$0 \leq k \leq 9$	225	(276)	(280)	(284)	288	(292)	7
			(296)	300	304	(308)	(312)	

Acknowledgements

This research was financially supported from the General Secretariat of Research and Technology by a grant PENED 03ED740. The research of the second author was also supported by a scholarship awarded by the Research Committee of National Technical University of Athens.

References

[1] R. Craigen and H. Kharaghani, Orthogonal designs, in *Handbook of Combinatorial Designs*, 2nd ed. (Eds. C.J. Colbourn and J.H. Dinitz), Chapman and Hall/CRC Press, Boca Raton, Fla., 2006, 280–295.

[2] S. Georgiou and C. Koukouvinos, On sequences with zero autocorrelation and orthogonal designs, *J. Combin. Theory Ser. A* **94** (2001), 15–33.

[3] A.V. Geramita, J.M. Geramita and J. Seberry Wallis, Orthogonal designs, *Linear and Multilinear Algebra* **3** (1976), 281–306.

[4] A.V. Geramita and J. Seberry, *Orthogonal designs: Quadratic forms and Hadamard matrices*, Marcel Dekker, New York-Basel, 1979.

[5] W.H. Holzmann and H. Kharaghani, On the Plotkin arrays, *Australas. J. Combin.* **22** (2000), 287–299.

[6] H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Des.* **8** (2000), 127–130.

[7] H. Kharaghani and C. Koukouvinos, Complementary, Base and Turyn Sequences, in *Handbook of Combinatorial Designs*, 2nd ed. (Eds. C.J. Colbourn and J.H. Dinitz), Chapman and Hall/CRC Press, Boca Raton, Fla., 2006, pp. 317–321.

[8] I.S. Kotsireas, C. Koukouvinos and D.E. Simos, Inequivalent Hadamard matrices from near normal sequences, *J. Combin. Math. Combin. Comput.* (to appear).

[9] C. Koukouvinos, Sequences with Zero Autocorrelation, in *The CRC Handbook of Combinatorial Designs*, (Eds. C.J. Colbourn and J.H. Dinitz), Chapman and Hall/CRC Press, 1996, Part IV, Ch. 42, 452–456.

[10] C. Koukouvinos, online appendix, `http://www.math.ntua.gr/~ckoukouv`.

[11] C. Koukouvinos, S. Kounias, J. Seberry, C.H. Yang and J. Yang, Multiplication of Sequences with Zero Autocorrelation, *Australas. J. Combin.* **10** (1994), 5–15.

[12] C. Koukouvinos and J. Seberry, New weighing matrices and orthogonal designs constructed using two sequences with zero autocorrelation function — a review, *J. Statist. Plann. Inf.* **81** (1999), 153–182.

[13] C.H. Yang, On Composition of Four-Symbol δ-Codes and Hadamard Matrices, *Proc. Amer. Math. Soc.* **107** (1989), 763–776

(Received 4 Feb 2009; revised 12 Nov 2009)