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# Minimum cost homomorphisms to locally semicomplete digraphs and quasi-transitive digraphs

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**Abstract**

For digraphs $G$ and $H$, a homomorphism of $G$ to $H$ is a mapping $f : V(G) \rightarrow V(H)$ such that $uv \in A(G)$ implies $f(u)f(v) \in A(H)$. If, moreover, each vertex $u \in V(G)$ is associated with costs $c_i(u)$, $i \in V(H)$, then the cost of a homomorphism $f$ is $\sum_{u \in V(G)} c_{f(u)}(u)$. For each fixed digraph $H$, the minimum cost homomorphism problem for $H$, denoted MinHOM($H$), can be formulated as follows: Given an input digraph $G$, together with costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, decide whether there exists a homomorphism of $G$ to $H$ and, if one exists, to find one of minimum cost. Minimum cost homomorphism problems encompass (or are related to) many well-studied optimization problems such as the minimum cost chromatic partition and repair analysis problems. We focus on the minimum cost homomorphism problem for locally semicomplete digraphs and quasi-transitive digraphs which are two well-known generalizations of tournaments. Using graph-theoretic characterization results for the two digraph classes, we obtain a full dichotomy classification of the complexity of minimum cost homomorphism problems for both classes.

## 1 Introduction

The minimum cost homomorphism problem was introduced, in the context of undirected graphs, in [12]. There, it was motivated by a real-world problem in defense logistics; in general, the problem appears to offer a natural and practical way to model many optimization problems. Special cases include the homomorphism problem, the list homomorphism problem [14, 17] and the optimum cost chromatic partition problem [13, 18, 19] (which itself has a number of well-studied special cases and applications [21, 22]).

For digraphs $G$ and $H$, a mapping $f : V(G) \rightarrow V(H)$ is a *homomorphism of $G$ to $H$* if $f(u)f(v)$ is an arc of $H$ whenever $uv$ is an arc of $G$. In the *homomorphism problem*, given a graph $H$, for an input graph $G$ we wish to decide whether there is a homomorphism of $G$ to $H$. In the *list homomorphism problem*, our input apart from $G$ consists of sets $L(u)$, $u \in V(G)$, of vertices of $H$, and we wish to decide whether there is a homomorphism $f$ of $G$ to $H$ such that $f(u) \in L(u)$ for each $u \in V(G)$. In the *minimum cost homomorphism problem* we fix $H$ as before, our inputs are a graph $G$ and costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$ of mapping $u$ to $i$, and we wish to check whether there exists a homomorphism of $G$ to $H$ and if it does exist, we wish to obtain one of minimum cost, where the cost of a homomorphism $f$ is $\sum_{u \in V(G)} c_{f(u)}(u)$. The homomorphism, list homomorphism, and minimum cost homomorphism problems are denoted by HOM($H$), ListHOM($H$) and MinHOM($H$), respectively. If the graph $H$ is *symmetric* (each $uv \in A(H)$ implies $vu \in A(H)$), we may view $H$ as an undirected graph. This way, we may view the problem MinHOM($H$) as also a problem for undirected graphs.

Our interest is in obtaining dichotomies: given a problem such as HOM($H$), we would like to find a class of digraphs $\mathcal{H}$ such that if $H \in \mathcal{H}$, then the problem is polynomial-time solvable and if $H \notin \mathcal{H}$, then the problem is NP-complete. For instance, in the case of undirected graphs it is well-known that HOM($H$) is polynomial-time solvable when $H$ is bipartite or has a loop, and NP-complete otherwise [16].

For undirected graphs $H$, a dichotomy classification for the problem MinHOM($H$) has been provided in [6]. (For ListHOM($H$), consult [4].) Since [6] interest has shifted to directed graphs. The first studies focused on loopless digraphs and dichotomies have been obtained for semicomplete digraphs [9] and semicomplete multipartite digraphs [10, 11]. A digraph is *semicomplete multipartite* if it can be obtained from a complete multipartite (undirected) graph by replacing every edge $\{x, y\}$ by arc $xy$, arc $yx$ or pair $xy, yx$ of arcs. A semicomplete multipartite digraph $D$ is *semicomplete* if every partite set of $D$ is a singleton.

More recently, [8] initiated the study of digraphs with loops allowed; and, in particular, of reflexive digraphs, where each vertex has a loop. While [7] gave a dichotomy for semicomplete digraphs with possible loops, [5] obtained a dichotomy for all reflexive digraphs.

A digraph $D$ is *locally semicomplete* if for every vertex $x$ of $D$, the in-neighbors of

$x$ induce a semicomplete digraph and the out-neighbors of $x$ also induce a semicomplete digraph. A digraph $D$ is *quasi-transitive* if, for every triple $x, y, z$ of distinct vertices of $D$ such that $xy$ and $yz$ are arcs of $D$, there is at least one arc between $x$ and $z$.

Semicomplete multipartite digraphs, locally semicomplete digraphs and quasi-transitive digraphs are the three most studied families of generalizations of tournaments [1]. Thus, it is a natural problem to obtain dichotomies for locally semicomplete digraphs and quasi-transitive digraphs and we solve this problem in the present paper. Like with semicomplete multipartite digraphs, structural properties of locally semicomplete digraphs and quasi-transitive digraphs play a key role in proving the dichotomies. Unlike for semicomplete multipartite digraphs, we also use structural properties of a family of undirected graphs.

In this paper we prove dichotomies for locally semicomplete digraphs and quasi-transitive digraphs; the dichotomies are formulated in Section 3 as Theorems 3.1 and 3.2. Before that in Section 2 we provide further terminology and notation and formulate a characterization of proper interval bigraphs that we use later. In Section 4 we prove the polynomial-time solvability parts of the two theorems. While the proof of the polynomial-time solvability part of Theorem 3.1 is relatively easy, this part of Theorem 3.2 is quite technical and lengthy. In Section 4 we prove the NP-hardness parts of the two theorems. There we use several known results and prove some new ones.

## 2 Further Terminology and Notation

In our terminology and notation, we follow [1]. From now on, all digraphs are loopless and do not have parallel arcs. A digraph $D$ is *transitive* if, for every pair of arcs $xy$ and $yz$ in $D$ such that $x \neq z$, the arc $xz$ is also in $D$. Sometimes, we will deal with *transitive oriented graphs*, i.e. transitive digraphs with no cycle of length two. Clearly, a transitive digraph is also quasi-transitive. Notice that a semicomplete digraph is both quasi-transitive and locally semicomplete.

An $(x, y)$-path in a digraph $D$ is a directed path from $x$ to $y$. A digraph $D$ is *strongly connected* (or, just, *strong*) if, for every pair $x, y$ of distinct vertices in $D$, there exist an $(x, y)$-path and a $(y, x)$-path. A *strong component* of a digraph $D$ is a maximal induced subgraph of $D$ which is strong. If $D_1, \ldots, D_t$ are the strong components of $D$, then clearly $V(D_1) \cup \ldots \cup V(D_t) = V(D)$ (recall that a digraph with only one vertex is strong). Moreover, we must have $V(D_i) \cap V(D_j) = \emptyset$ for every $i \neq j$ as otherwise all the vertices $V(D_i) \cup V(D_j)$ are reachable from each other, implying that the vertices of $V(D_i) \cup V(D_j)$ belong to the same strong component of $D$.

Let $D$ be any digraph. If $xy \in A(D)$, we say $x$ dominates $y$ or $y$ is dominated by $x$, and denote by $x \rightarrow y$. An arc $xy \in A(D)$ is *symmetric* if $yx \in A(D)$. For sets $X, Y \subset V(D)$, $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X, y \in Y$, but no vertex of $Y$ dominates a vertex in $X$. We denote by $B(D)$ the bipartite graph obtained from $D$

as follows. Each vertex $v$ of $D$ gives rise to two vertices of $B(D)$ - a *white* vertex $v'$ and a *black* vertex $v''$; each arc $vw$ of $D$ gives rise to an edge $v'w''$ of $B(D)$. The *converse* of $D$ is the digraph obtained from $D$ by reversing the directions of all arcs. A digraph $H$ is an *extension* of $D$ if $H$ can be obtained from $D$ by replacing every vertex $x$ of $D$ with a set $S_x$ of independent vertices such that if $xy \in A(D)$ then $uv \in A(H)$ for each $u \in S_x, v \in S_y$. A *tournament* is a semicomplete digraph which does not have any symmetric arc. An acyclic tournament on $p$ vertices is denoted by $TT_p$ and called a *transitive tournament.* The vertices of a transitive tournament $TT_p$ can be labeled $1, 2 \dots, p$ such that $ij \in A(TT_p)$ if and only if $1 \leq i < j \leq p$. By $TT_p^-$ $(p \geq 2)$, we denote $TT_p$ without the arc $1p$.

We say that a bipartite graph $H$ (with a fixed bipartition into white and black vertices) is a *proper interval bigraph* if there are two inclusion-free families of intervals $I_v$, for all white vertices $v$, and $J_w$ for all black vertices $w$, such that $vw \in E(H)$ if and only if $I_v$ intersects $J_w$. By this definition proper interval bigraphs are irreflexive and bipartite. A combinatorial characterization (in terms of forbidden induced subgraphs) of proper interval bigraphs is given in [15]: $H$ is a proper interval bigraph if and only if it does not contain an induced cycle $C_{2k}$, with $k \geq 3$, or an induced biclaw, binet, or bitent, as given in Figure 2.

A linear ordering $<$ of $V(H)$ is a *Min-Max ordering* if $i < j, s < r$ and $ir, js \in A(H)$ imply that $is \in A(H)$ and $jr \in A(H)$. For a bipartite graph $H$ (with a fixed bipartition into white and black vertices), it is easy to see that $<$ is a Min-Max ordering if and only if $<$ restricted to the white vertices, and $<$ restricted to the black vertices satisfy the condition of Min-Max orderings, i.e., $i < j$ for white vertices, and $s < r$ for black vertices, and $ir, js \in A(H)$, imply that $is \in A(H)$ and $jr \in A(H)$). A *bipartite Min-Max ordering* is an ordering $<$ specified just for white and for black vertices.

The following lemma exhibits that a proper interval bigraph always admits a bipartite Min-Max ordering.

**Lemma 2.1** *[6] A bipartite graph $G$ is a proper interval bigraph if and only if $G$ admits a bipartite Min-Max ordering.*

It is known that if $H$ admits a Min-Max ordering, then the problem MinHOM($H$) is polynomial time solvable [9], see also [3, 20]; however, there are digraphs with polynomial MinHOM($H$) which do not have Min-Max ordering [10].

## 3 Main Results

In this paper we prove the following two dichotomies:

**Theorem 3.1** *Let $H$ be a locally semicomplete digraph. MinHom($H$) is polynomial-time solvable if every connectivity component of $H$ is either acyclic or a directed cycle $\overrightarrow{C_k}$, $k \geq 2$. Otherwise, MinHom($H$) is NP-hard.*

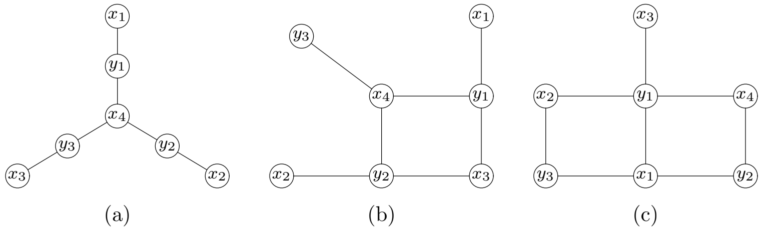


Figure 1: A biclaw (a), a binet (b) and a bitent (c).

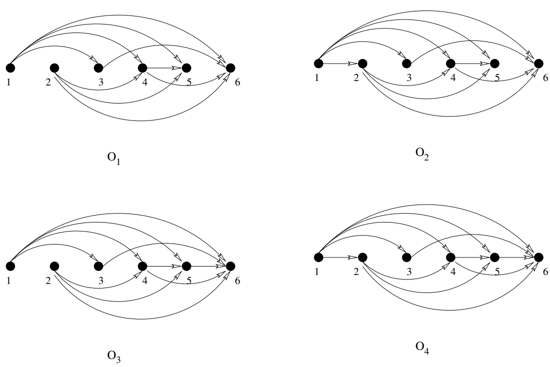


Figure 2: The obstructions $O_i$ with $i = 1, 2, 3, 4$

**Theorem 3.2** *Let $H$ be a quasi-transitive digraph. MinHom($H$) is polynomial-time solvable if every connectivity component $H'$ of $H$ is either $\overrightarrow{C_2}$ or an extension of $\overrightarrow{C_3}$ or acyclic, $B(H')$ is a proper interval bigraph and $H'$ does not contain $O_i$ with $i = 1, 2, 3, 4$ as an induced subgraph (the digraphs $O_i$ are defined as in Figure 2). Otherwise, MinHom($H$) is NP-hard.*

In fact, it is easy to see that it suffices to prove Theorems 3.1 and 3.2 only for connected digraphs $H$. Indeed, assume that Theorem 3.1 has been proved for each connected $H$ and we will show that Theorem 3.1 holds for a disconnected $H$ with connectivity components $H_1, \ldots, H_s$.

Let each $H_i$ is either acyclic or a directed cycle $\overrightarrow{C_k}$, $k \geq 2$. Let $D$ be an input digraph and assume that $D$ is connected. By the definition of a homomorphism, any homomorphism of $D$ to $H$ maps $D$ into only one component of $H$. Thus, the minimum cost homomorphism of $D$ to $H$ is the smallest minimum cost homomorphism to any $H_i$. Thus, MinHom($H$) is polynomial-time solvable. If $D$ is disconnected, we can solve the problem for each component of $D$ separately.

Now suppose that $H_1$ is not acyclic and it is not a directed cycle. We may force $D$ to be mapped to $H_1$ by setting the costs $c_i(x) = \infty$ for every $i \in V(H) \setminus V(H_1)$

and $x \in V(D)$. Thus, MinHom($H$) is NP-hard.

The rest of the paper is devoted to proving the two theorems for the case of connected $H$.

## 4 Polynomial cases

The most basic properties of strong components of a connected non-strong locally semicomplete digraph are given in the following result, due to Bang-Jensen [2].

**Theorem 4.1** *[2] Let $H$ be a connected locally semicomplete digraph that is not strong. Then the following holds for $H$.*

*(a) If $A$ and $B$ are distinct strong components of $H$ with at least one arc between them, then either $A \rightarrow B$ or $B \rightarrow A$.*

*(b) If $A$ and $B$ are strong components of $H$, such that $A \rightarrow B$, then $A$ and $B$ are semicomplete digraphs.*

*(c) The strong components of $H$ can be ordered in a unique way $H_1, H_2, \ldots, H_p$ such that there are no arcs from $H_j$ to $H_i$ for $j > i$, and $H_i$ dominates $H_{i+1}$ for $i = 1, 2, \ldots, p-1$.*

**Theorem 4.2** *Let $H$ be a connected locally semicomplete digraph. MinHOM($H$) is polynomial time solvable if $H$ is either acyclic or a directed cycle $\overrightarrow{C_k}$, $k \geq 2$.*

**Proof:** We already know that MinHOM($H$) is polynomial time solvable when $H$ is a directed cycle, see [9]. Assume that $H$ is a locally semicomplete digraph which is acyclic. Then $H$ is non-strong and every strong component of $H$ is a single vertex. Hence we know from Part (c) of Theorem 4.1 that the vertices of $H$ can be ordered in a unique way $1, 2, \ldots, p$ such that there are no arcs from $j$ to $i$ for $j > i$, and $i$ dominates $i+1$ for $i = 1, 2, \ldots, p-1$. We claim that this ordering is a Min-Max ordering and thus, MinHOM($H$) is polynomial time solvable.

Choose any two arcs $ir, js \in A(H)$ with $i < j$, $s < r$. Since all arcs are oriented forwardly with respect to the ordering, we have $i < j < s < r$. Also, there is a path $i, i+1, \ldots, j, \ldots, s, s+1, \ldots, r$ from $i$ to $r$ in $H$ due to the ordering property. Since vertex $i$ dominates both $i+1$ and $r$, there is an arc between $i+1$ and $r$, which must be oriented from $i+1$ to $r$. By induction, vertex $r$ is dominated by all vertices $i, \ldots, r-1$ on the path. This indicates that we have an arc $jr \in A(H)$. Following a similar argument, we conclude that there is an arc $is \in A(H)$. This proves that the ordering is a Min-Max ordering. ⋄

**Theorem 4.3** *Let $H$ be a connected quasi-transitive digraph. Then MinHOM($H$) is polynomial time solvable if either*

- *$H$ is $\overrightarrow{C_2}$ or $H$ is an extension of $\overrightarrow{C_3}$, or*
- *$H$ is acyclic, $B(H)$ is a proper interval bigraph and $H$ does not contain $O_i$ with $i = 1, 2, 3, 4$ as an induced subdigraph. (See Figure 2.)*

**Proof:** It has been proved in [9] that MinHOM($H$) is polynomial time solvable when $H$ is a directed cycle. The case for $H$ being $\overrightarrow{C_2}$ or $\overrightarrow{C_3}$ follows immediately.

Now assume that $H$ is acyclic. Then, it is straightforward to check that $H$ is exactly a transitive oriented graph $T$. We will show that a bipartite Min-Max ordering of $B(T)$ can be transformed to produce a Min-Max ordering of $T$. Recall that whenever $B(T)$ is a proper interval bigraph it has a bipartite Min-Max ordering due to Lemma 2.1.

Suppose $<$ is a bipartite Min-Max ordering of $B(T)$. A pair $x, y$ of vertices of $T$ is *proper for* $<$ if $x' < y'$ if and only if $x'' < y''$ in $B(T)$. We say a bipartite Min-Max ordering $<$ is *proper* if all pairs $x, y$ of $T$ are proper for $<$. If $<$ is a proper bipartite Min-Max ordering, then we can define a corresponding ordering $\prec$ on the vertices of $T$, where $x \prec y$ if and only if $x' < y'$ (which happens if and only if $x'' < y''$). It is easy to check that $\prec$ is now a Min-Max ordering of $T$.

Assume, on the other hand, that the bipartite Min-Max ordering $<$ on $B(T)$ is not proper. That is, there are vertices $x', y'$ such that $x' < y'$ and $y'' < x''$. Suppose that for every pair of vertices $c''$ and $d''$ such that $d'' < c''$ and $x'd'', y'c'' \in E(B(T))$, we have both $x'c''$ and $y'd''$ in $E(B(T))$. Then we can exchange the positions of $x'$ and $y'$ in $<$ while preserving the Min-Max property. Furthermore, it can be checked that this exchange strictly increases the number of proper pairs in $H$: if a proper pair turns into an improper pair or vice versa by this exchange, then one of the two vertices must be $x$ or $y$. Clearly the improper pair consisting of $x$ and $y$ is turned into a new proper pair. Suppose that vertex $w$ constitutes a pair with $x$ or $y$ which is possibly affected by the exchange. Observe that we have $x' < w' < y'$ or $y'' < w'' < x''$. When $w$ lies between $x$ and $y$ in both partite sets in $B(T)$, the improper pairs $(w, x)$, $(w, y)$ are transformed to proper pairs by the exchange of $x'$ and $y'$. When $x' < w' < y'$ and $w''$ is not between $x''$ and $y''$, there is a newly created proper pair and improper pair respectively, which compensate the effect of each other in the number of proper pairs in $H$. Similarly, there is no change in the number of proper pairs of the form $(w, x)$ or $(w, y)$ when $y'' < w'' < x''$ and $w'$ is not between $x'$ and $y'$. Hence, the exchange increases the number of proper pairs at least by one.

Analogously, we can exchange the positions of $x''$ and $y''$ in $<$ if for every pair of vertices $a'$ and $b'$ such that $b' < a'$ and $a'x'', b'y'' \in E(B(T))$, we have both $a'y''$ and $b'x''$ in $E(B(T))$. This exchange does not affect the Min-Max ordering of $B(T)$ and strictly increases the number of proper pairs as well.

In the remaining part, we will show that we can always exchange the positions of $x'$ and $y'$ or the positions of $x''$ and $y''$ in $<$ whenever we have an improper pair $x$, $y$ and $<$ is a Min-Max ordering of $B(T)$.

Suppose, to the contrary, that we performed the above exchange for every improper pair as far as possible and still the Min-Max ordering is not proper. Then, there must be an improper pair $x$ and $y$ with $x' < y'$, $y'' < x''$ in $<$ which satisfies the following condition: 1) there exist vertices $c''$ and $d''$, $d'' < c''$ such that $x'd'', y'c'' \in E(B(T))$ and at least one of $y'd''$ and $x'c''$ is missing in $B(T)$. 2) there exist vertices $a'$ and $b'$, $b' < a'$ such that $b'y'', a'x'' \in E(B(T))$ and at least one of $b'x''$ and $a'y''$ is missing in $B(T)$.

Notice that $a$, $d$ and $x$ are distinct vertices in $T$ since otherwise, the edges $a'x''$ and $x'd''$ induce $\overrightarrow{C_2}$ or a loop in $T$. With the same argument $b$,$c$ and $y$ are distinct vertices in $T$. On the other hand, by transitivity of $T$, the edges $a'x''$ and $x'd''$ imply the existence of edge $a'd''$ in $E(B(T))$. Similarly, there is an edge $b'c''$ in $E(B(T))$. Note that we do not have $x'x''$ and $y'y''$ in $E(B(T))$ as $T$ is irreflexive.

We will consider cases according to the positions of $a', b', c'', d''$ in the ordering $<$. We remark the two edges $b'y''$ and $y'c''$ cannot cross each other. That is, they either satisfy $b' < y'$ and $y'' < c''$, or $y' < b'$ and $c'' < y''$, since otherwise there has to be an edge $y'y''$ by the Min-Max property, which is a contradiction. Similarly, the two edges $a'x''$ and $x'd''$ cannot cross each other, since otherwise there has to be an edge $x'x''$ by the Min-Max property, which is a contradiction. Hence we have either $x' < a'$ and $d'' < x''$, or $a' < x'$ and $x'' < d''$.

When $y' < b'$ and $c'' < y''$, the positions of all vertices are determined immediately so that we have $x' < y' < b' < a'$ and $d'' < c'' < y'' < x''$. On the other hand, when $b' < y'$ and $y'' < c''$ we can place the edges $a'x''$ and $x'd''$ in two ways, namely to satisfy either $x' < a'$ and $d'' < x''$, or $a' < x'$ and $x'' < d''$ due to the argument in the above paragraph. In the latter case, however, the positions of all vertices are determined as well and this is just a converse of the case when $y' < b'$ and $c'' < y''$. Therefore we may assume that $x' < a'$ and $d'' < x''$ whenever $b' < y'$ and $y'' < c''$.

**CASE 1** $b' < y'$ and $y'' < c''$ ($x' < a'$ and $d'' < x''$).

We will consider are following cases. We show that in every case we have a contradiction.

**Case 1-1** $y' < a'$ and $d'' < y''$.

The two edges $a'd'', y'c'' \in E(B(H))$ imply the existence of $y'd'' \in E(B(T))$ by the Min-Max property. The edge $y'd''$, however, together with $b'y'' \in E(B(H))$ enforce the edge $y'y'' \in E(B(H))$, which is a contradiction.

**Case 1-2** $y' \leq a'$ and $y'' \leq d''(< x'')$.

Case 1-2-1: $b' < x'$. We know that $a'd'' \in E(B(T))$. We can easily see that $y'd'' \in E(B(T))$ since $<$ is a Min-Max ordering . ( Note that $y'c'', a'd'' \in E(B(T))$). By the taking of two vertices $c'', d''$, the existence of $y'd'' \in E(B(T))$ enforces $x'c'' \notin E(B(T))$. On the other hand, however, we must have the edge $x'c'' \in E(B(H))$ due to edges $b'c'', x'd'' \in E(B(H))$ and the Min-Max property, a contradiction.

Case 1-2-2: $x' \leq b' < y'$. If $x' = b'$ or $y'' = d''$ then $x'y'' \in E(B(T))$ since $b'y'' \in E(B(T))$ and $x'd'' \in E(B(T))$. If $x' < b'$ and $y'' < d''$ it is easy to see that we have $x'y'' \in E(B(T))$ by the Min-Max property. (Note that $b'y'', x'd'' \in E(B(T))$). With $a'x'', x'y'' \in E(B(H))$, the transitivity of $T$ implies $a'y'' \in E(B(T))$. However, this is a contradiction since we have $y'y'' \notin E(B(H))$ by the Min-Max property and

$y'c'', a'y'' \in E(B(H))$.

**Case 1-3** $(x' <)a' \leq y'$ and $y'' \leq d''(< x'')$.

Case 1-3-1: $x'' < c''$. We will show that we cannot avoid having the edge $x'c'' \in E(B(T))$. Once this is the case, the two edges $x'c''$ and $a'x''$ imply the existence of edge $x'x'' \in E(B(T))$, which is a contradiction.

When $x' \leq b'$ we again easily observe that $x'y'' \in E(B(T))$ and thus, $x'c'' \in E(B(T))$ for $T$ is transitive and $x'y'', y'c'' \in E(B(T))$. On the other hand, when $b' < x'$ we have $x'c'' \in E(B(T))$ again by the Min-Max property and the two edges $b'c'', x'd'' \in E(B(T))$.

Case 1-3-2: $c'' \leq x''$. We again easily observe that $y'x'' \in E(B(T))$ by the Min-Max property and the two edges $y'c'', a'x'' \in E(B(T))$.

When $x' \leq b'$, the Min-Max property implies $x'y'' \in E(B(T))$. Since $T$ does not contain $\overrightarrow{C_2}$ as an induced subgraph, this is a contradiction.

When $b' < x'$. It is again implied that $b'x'' \in E(B(T))$ as $T$ is transitive and $b'y'', y'x'' \in E(B(T))$. The two edges $b'x'', x'd'' \in E(B(H))$ enforce the existence of $x'x'' \in E(B(T))$ by the Min-Max property, which is a contradiction.

**Case 1-4** $(x' <)a' \leq y'$ and $d'' < y''$.

We will show that we cannot avoid having the edge $b'x'' \in E(B(T))$. Once this is the case, by the taking of two vertices $a', b'$, the existence of $b'x'' \in E(B(T))$ enforces $a'y'' \notin E(B(T))$. On the other hand, however, we must have the edge $a'y'' \in E(B(H))$ due to edges $a'd'', b'y'' \in E(B(H))$ and the Min-Max property, a contradiction.

When $x'' = c''$, we trivially have $b'x'' \in E(B(T))$. When $x'' < c''$, the Min-Max property and the two edges $b'c'', a'x'' \in E(B(T))$ imply $b'x'' \in E(B(T))$. When $x'' > c''$, the Min-Max property and the two edges $a'x'', y'c'' \in E(B(T))$ imply $y'x'' \in E(B(T))$. For $b'y'', y'x'' \in E(B(T))$, we again have $b'x'' \in E(B(T))$ by the transitivity of $T$. This completes the argument.

**CASE 2** $y' < b'$ and $c'' < y''$.

We now prove that $T$ has one of $O_i$ with $i = 1, 2, 3, 4$ as an induced subgraph. Remember that $x' < y' < b' < a'$ and $d'' < c'' < y'' < x''$. Since $T$ is transitive we have $a'd'', b'c'' \in E(B(T))$. Since $<$ is a bipartite Min-Max ordering,

$$\{a'x'', a'y'', a'c'', a'd'', b'y'', b'c'', b'd'', y'c'', y'd'', x'd''\} \subset E(B(T)).$$

Now by the taking of $a, b$ and $c, d$ we have $b'x'', x'c'' \notin E(B(T))$; hence $y'x'', x'y'' \notin E(B(T))$ as $<$ is a bipartite Min-Max ordering. It is easy to see from the set of edges existing in $B(T)$ that $a, b, x, y, c, d$ are distinct vertices in $T$ . Let us define $T' = T[\{a, b, x, y, c, d\}]$. Since $T'$ is acyclic we do not have symmetric arcs in $T'$.

From $E(B(T'))$, we have $\{ax, ay, ac, ad, by, bc, bd, xd, yc, yd\} \subset A(T')$ and $xy, yx, bx, xc \notin A(T')$. We can easily see that $xb \notin A(T')$, since otherwise from $xb, by \in A(T')$ and the transitivity of $T'$ we must have $xy \in A(T')$, a contradiction. With the same argument we will see that $ba, cx, dc \notin A(T')$. Therefore we can only add a subset of $S = \{ab, cd\}$ to the previous arc subset of $T'$ mentioned above each of which makes $T'$ to be isomorphic to one of $O_i$ with $i = 1, 2, 3, 4$ with the mapping $g$ where $g(a) = 1, g(b) = 2, g(x) = 3, g(y) = 4, g(c) = 5, g(d) = 6$. ⋄

## 5 NP-Hardness

We begin this section with a few simple observations. The first one is easily proved by setting up a natural polynomial time reduction from MinHOM($B(H)$) to MinHOM($H$) [8].

**Proposition 5.1** *[8] If MinHOM(B(H)) is NP-hard, then MinHOM($H$) is also NP-hard.* ⋄

The next observation is folklore, and is proved by obvious reduction, cf. [7].

**Proposition 5.2** *Let $H'$ be an induced subgraph of the digraph $H$. If MinHOM($H'$) is NP-hard, then MinHOM($H$) is NP-hard.* ⋄

The following lemmas are the NP-hardness part of the main results in [9, 10, 8].

**Lemma 5.3** *[9] Let $H$ be a semicomplete digraph containing a cycle and let $H \notin \{\overrightarrow{C}_2, \overrightarrow{C}_3\}$. Then MinHOM($H$) is NP-hard.*

**Lemma 5.4** *[10] Let $H$ be a semicomplete $k$-partite digraph, $k \geq 3$ which is not an extension of $TT_k, \overrightarrow{C}_3$ or $TT_p^-$ $(p \geq 4)$. Then MinHOM($H$) is NP-hard.*

**Lemma 5.5** *[8] Let $H'$ be a digraph obtained from $\overrightarrow{C}_k = 12 \ldots k1, k \geq 2$, by adding an extra vertex $k+1$ dominated by at least two vertices of the cycle. Let $H$ be $H'$ or its converse. Then MinHOM($H$) is NP-hard.*

We need two more lemmas for our classification. We will use the well-known fact that the maximum independent set problem, i.e., the problem of finding the maximum number of vertices with no edges between them in a given graph, is NP-hard.

**Lemma 5.6** *Let $H'_1$ be a digraph obtained from $\overrightarrow{C}_k = 12 \ldots k1, k \geq 2$, by adding an extra vertex $k+1$ such that $i \rightarrow k+1$ and $k+1 \rightarrow i+1$, where $i, i+1$ are two consecutive vertices in $\overrightarrow{C}_k$. Let $H_1$ be $H'_1$ or its converse. Then MinHOM($H_1$) is NP-hard. (See Figure 3.)*

**Proof:** Without loss of generality, we may assume that $V(H_1) = \{1, \ldots, k+1\}$, $123 \ldots k$ is a cycle of length $k$, and the vertex $k+1$ is dominated by $k$ and dominates 1.

We will construct a polynomial time reduction from the maximum independent set problem to MinHOM($H_1$). Let $G$ be an arbitrary undirected graph. We replace every edge $uv \in E(G)$ by the digraph $D_{uv}$ defined as follows:

$V(D_{uv}) = \{c_1, c_2, \ldots, c_{k(k+1)}\} \cup \{x, y, u', u, v', v\}$
$A(D_{uv}) = \{c_i c_{i+1} : 1 \le i \le k(k+1)\} \cup \{c_{2k}u', u'u, c_{k(k+1)-1}v', v'v\} \cup \{xy, xc_1, yc_1\}$
where addition is taken modulo $k(k+1)$.

Observe that in any homomorphism $f$ of $D_{uv}$ to $H_1$, we must have $f(c_1) = 1$. Once we assign the first $k$ vertices $c_1, \ldots c_k$ color $1 \ldots k$, the vertex $c_{k+1}$ is assigned with either color 1 or color $k+1$. If we opt for color 1, then through the whole remaining vertices $c_{k+1}, \ldots, c_{k(k+1)}$ we must assign these vertices with colors along the $k$-cycle $12 \ldots k$ in $H_1$. Else if we opt for color $k+1$, then we must assign the whole remaining vertices with colors along the $(k+1)$-cycle $12 \ldots k+1$ in $H_1$. To see this, suppose to the contrary that we assign the vertices $c_1, \ldots, c_{k(k+1)}$ in $H$ with colors along the $k$-cycle $s$ times and with colors along the $(k+1)$-cycle $t$ times, where $0 < t < k$. Then, we have the following equation $k(k+1) = sk + t(k+1)$ which implies $(k+1)(k-t) = sk$.

Since the least common denominator of $k$ and $k+1$ is $k(k+1)$, we arrive at a contradiction. Hence, $(f(c_1), \ldots, f(c_{k(k+1)}))$ coincides with one of the following sequences:

$(1, 2, \ldots, k, \ldots, 1, \ldots, k)$ (the sequence $1, 2, \ldots, k$ appears $k+1$ times) or

$(1, 2, \ldots, k, k+1, \ldots, 1, \ldots, k+1)$ (the sequence $1, 2, \ldots, k+1$ appears $k$ times).

If the first sequence is the actual one, then we have $f(c_{2k}) = k$, $f(u') \in \{1, k+1\}$, $f(u) \in \{1, 2\}$, $f(c_{k(k+1)-1}) = k-1$, $f(v') = k$ and $f(v) \in \{1, k+1\}$. If the second one is the actual one, then we have $f(c_{2k}) = k-1$, $f(u') = k$, $f(u) \in \{1, k+1\}$, $f(c_{k(k+1)-1}) = k$, $f(v') \in \{1, k+1\}$ and $f(v) \in \{1, 2\}$. In both cases, we can assign both of $u$ and $v$ color 1. Furthermore by choosing the right sequence, we can color one of $u$ and $v$ with color 2 and the other with color 1. However we cannot assign color 2 to both $u$ and $v$ in a homomorphism.

Let $D$ be the digraph obtained by replacing every edge $uv \in E(G)$ by $D_{uv}$. Here $D_{uv}$ is placed in an arbitrary direction. Note that $|V(D)| = |V(G)| + |E(G)| \cdot (k(k+1)+4)$ and this reduction can be done in polynomial time.

Let all costs $c_i(t) = 0$ for $t \in V(D)$, $i \in V(H)$ apart from $c_1(x) = 1$ and $c_{k+1}(x) = |V(G)|$ for all $x \in V(G)$. Let $f$ be a homomorphism of $D$ to $H$ and let $S = \{u \in V(G) : f(u) = 2\}$. Then, $S$ is an independent set in $G$ since we cannot assign color 2 to both $u$ and $v$ in $V(G)$ whenever there is an edge between them. Observe that a minimum cost homomorphism will assign as many vertices of $V(G)$ color 2.

Conversely, suppose we have an independent set $I$ of $G$. Then we can build a homomorphism $f$ of $D$ to $H_1$ such that $f(u) = 2$ for all $u \in I$ and $f(u) = 1$ for all $u \in G(V) \setminus I$. Note that all the other vertices from $D_{uv}$, $uv \in E(G)$ can be assigned with an appropriate color from $H_1$.

Hence, a minimum cost homomorphism $f$ of $D$ to $H_1$ yields a maximum independent set of $G$ and vice versa, which completes the proof. ◇

**Lemma 5.7** *Let $H_2'$ be a digraph obtained from $\overrightarrow{C_k} = 12 \ldots k1, k \ge 3$, by adding an*

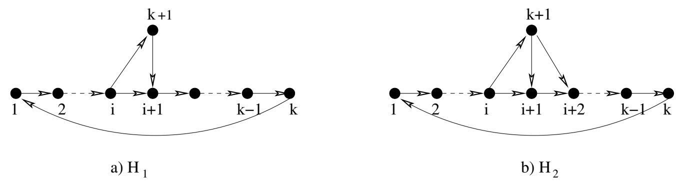


Figure 3: $H_1$ and $H_2$.

*extra vertex $k+1$ such that $i \to k+1$ and $k+1 \to i+1, i+2$, where $i, i+1, i+2$ are three consecutive vertices in $\overrightarrow{C_k}$. Let $H_2$ be $H_2'$ or its converse. Then MinHOM($H_2$) is NP-hard. (See Figure 3.)*

**Proof:** Without loss of generality, we may assume that $V(H_2) = \{1, \ldots, k+1\}$, $123 \ldots k$ is a cycle of length $k$, and the vertex $k+1$ is dominated by $k$ and dominates 1 and 2.

We will construct a polynomial time reduction from the maximum independent set problem to MinHOM($H_2$). Let $G$ be an arbitrary undirected graph. Modify the digraph $D_{uv}$ defined in the proof of Lemma 5.6 by adding the arc $(y, c_2)$ and replace every edge $uv \in E(G)$ by the new digraph $D_{uv}$.

Observe that in any homomorphism $f$ of $D_{uv}$ to $H_2$, we should have $f(c_1) = 1$. By the same argument discussed in the proof of Lemma 5.6, the vertices of $k(k+1)$-cycle in $D_{uv}$ must be mapped to either the $k$-cycles, $12 \ldots k$ and $(k+1)2 \ldots k$, or to the $(k+1)$-cycle $12 \ldots k+1$ in $H_2$. If the vertices of $k(k+1)$-cycle in $D_{uv}$ are mapped to the $k$-cycles in $H_2$, then we have $f(c_{2k}) = k$, $f(u') \in \{1, k+1\}$, $f(u) \in \{1, 2\}$, $f(c_{k(k+1)-1}) = k-1$, $f(v') = k$ and $f(v) \in \{1, k+1\}$. If the vertices of $k(k+1)$-cycle in $D_{uv}$ are mapped to the $(k+1)$-cycle in $H_2$, then we have $f(c_{2k}) = k-1$, $f(u') = k$, $f(u) \in \{1, k+1\}$, $f(c_{k(k+1)-1}) = k$, $f(v') \in \{1, k+1\}$ and $f(v) \in \{1, 2\}$. In both cases, we can assign both of $u$ and $v$ color 1. Furthermore by choosing the right sequence, we can color one of $u$ and $v$ with color 2 and the other with color 1. However we cannot assign color 2 to both $u$ and $v$ in a homomorphism.

Now it is easy to check that the same argument as that in the proof of Lemma 5.6 applies, completing the proof. ◇

Let $\mathcal{I}$ denote the following decision problem: given a graph $X$ and an integer $k$, decide whether or not $X$ contains an independent set of $k$ vertices. We denote by $\mathcal{I}_3$ the restriction of $\mathcal{I}$ to graphs with a given three-coloring. In the following lemma, we give a polynomial time reductions from $\mathcal{I}_3$. The following lemma shows that MinHOM($O_i$) is NP-hard for $i = 1, 2, 3, 4$.

**Proposition 5.8** *[6] The problem $\mathcal{I}$ is NP-complete, even when restricted to three-colorable graphs (with a given three-coloring).* ◇

**Lemma 5.9** *Let $H'$ be an arbitrary digraph over vertex set $\{1, 2, 3, 4, 5, 6\}$ such that*

$$\{13, 14, 15, 16, 24, 25, 26, 36, 45, 46\} \subseteq A(H'),$$

$$A(H') \subseteq \{12, 13, 14, 15, 16, 24, 25, 26, 36, 45, 46, 56\}.$$

*Let $H$ be $H'$ or its converse. Then MinHOM($H$) is NP-hard. (See Figure 2.)*

**Proof:** Let $X$ be a graph whose vertices are partitioned into independent sets $U, V, W$, and let $k$ be a given integer. We construct an instance of MinHOM($H$) as follows: the digraph $G$ is obtained from $X$ by replacing each edge $uv$ of $X$ with $u \in U, v \in V$ by an arc $uv$, replacing each edge $vw$ of $X$ with $v \in V, w \in W$ by an arc $vw$, and replace each edge $uw$ of $X$ with $u \in U, w \in W$ by an arc $um_{uw}, n_{uw}m_{uw}, n_{uw}w$, where $m_{uw}, n_{uw}$ are new vertices. Define a cost function $c_2(u) = 0$, $c_1(u) = 1$, $c_3(v) = 0$, $c_4(v) = 1$, $c_5(w) = 0$, $c_6(w) = 1$, $c_3(m_{uw}) = c_3(n_{uw}) = -|V(X)|$, $c_i(m_{uw}) = c_i(n_{uw}) = |V(X)|$ for $i \neq 3$. Apart from these, set all cost to $|V(X)|$.

We now claim that $X$ has an independent set of size $k$ if and only if $G$ admits a homomorphism to $H$ of cost $|V(X)| - k$. Let $I$ be an independent set in $G$. We can define a mapping $f : V(G) \to V(H)$ as follows:

- $f(u) = 2$ for $u \in U \cap I$, $f(u) = 1$ for $u \in U - I$
- $f(v) = 3$ for $v \in V \cap I$, $f(v) = 4$ for $v \in V - I$
- $f(w) = 5$ for $w \in W \cap I$, $f(w) = 6$ for $w \in W - I$.

  When $uw \in E(X)$:

- If $f(u) = 2, f(w) = 6$ then set $f(m_{uw}) = 6, f(n_{uw}) = 3$.
- If $f(u) = 1$ and $f(w) \in \{5, 6\}$ then set $f(m_{uw}) = 3, f(n_{uw}) = 1$,

One can verify that $f$ is a homomorphism from $G$ to $H$, with cost $|V(X)| - k$.

Let $f$ be a homomorphism of $G$ to $H$ of cost $|V(X)| - k$. If $k \leq 0$ then we are trivially done so assume that $k > 0$. Note that we cannot assign color 3 to both $n_{uw}$ and $m_{uw}$ simultaneously due to the arc $n_{uw}m_{uw}$. Hence, the fact that the cost of homomorphism $f$ is $|V(X)| - k$, $k > 0$ implies that all the vertices in $V(X)$ are assigned so that their individual costs are either zero or one, and for each edge $uw \in E(X)$ the costs of assigning $m_{uw}$ and $n_{uw}$ to vertices of $V(H)$ sum up to zero.

Let $I = \{u \in V(X) \mid c_{f(u)}(u) = 0\}$ and note that $|I| = k$. It can be seen that $I$ is an independent set in $G$, as if $uw \in E(G)$, where $u \in I \cap U$ and $w \in I \cap W$ then $f(u) = 2$ and $f(w) = 5$, which implies that $f(m_{uw}) \neq 3$ and $f(n_{uw}) \neq 3$ contrary to $f$ being a homomorphism of cost $|V(X)| - k$. $\diamond$

**Lemma 5.10** *Let $H$ be a connected locally semicomplete digraph which is neither acyclic nor a directed cycle. Then MinHOM($H$) is NP-hard.*

**Proof:** Since $H$ is neither acyclic nor a directed cycle, it has an induced cycle $\overrightarrow{C_k} = 12 \ldots k1, k \geq 2$ and a vertex $k+1$ outside this cycle. For $H$ is connected, the vertex $k+1$ is adjacent with at least one of the $\overrightarrow{C_k}$ vertices.

If $\overrightarrow{C_k} = \overrightarrow{C_2}$ and vertex $k+1$ is adjacent with 1, then $k+1$ is adjacent with vertex 2 as well. By Lemma 5.3 and Lemma 5.2, MinHOM($H$) is NP-hard in this case.

Therefore, we assume that $H$ does not have any symmetric arc hereinafter. Observe that the vertex $k+1$ cannot be adjacent with more than four vertices of $\overrightarrow{C_k}$, since otherwise $k+1$ either dominates or is dominated by at least three vertices on $\overrightarrow{C_k}$, which is a contradiction by the existence of a chord between two $\overrightarrow{C_k}$ vertices. With the same argument, vertices which dominate or are dominated by $k+1$ are consecutive on the cycle $\overrightarrow{C_k}$ and the number of these vertices are at most two.

Now without loss of generality, assume that $k+1$ is dominated by 1 and is not dominated by $k$. Since both $k+1$ and 2 are outneighbors of vertex 1, there is an arc between $k+1$ and 2. Consider the following cases.

Case 1. $k+1 \rightarrow 2$: The vertex $k+1$ either dominates 3 or is nonadjacent with 3. Since $k+1$ is dominated by 1, $k+1$ cannot be dominated by 3.

Case 1-1. $k+1 \rightarrow 3$: The digraph $H[\{1, 2, \ldots, k+1\}]$ is isomorphic to $H_2$. Hence, MinHOM($H$) is NP-hard by Lemma 5.7 and Lemma 5.2. Observe that there is no arc between $k+1$ and the vertices of $\overrightarrow{C_k}$ other than $1, 2$ and 3.

Case 1-2. $k+1$ is nonadjacent with 3: There is no arc between $k+1$ and the vertices of $\overrightarrow{C_k}$ other than 1 and 2, thus MinHOM($H$) is NP-hard by Lemma 5.6 and Lemma 5.2.

Case 2. $2 \rightarrow k+1$: Since $k+1$ and 3 are outneighbors of vertex 2, there is an arc between $k+1$ and 3. Moreover, $k+1$ is dominated by two vertices 1 and 2, which implies that $k+1 \rightarrow 3$. Now the vertex $k+1$ either dominate 4 or is nonadjacent with 4.

Case 2-1. $k+1 \rightarrow 4$: The digraph $H[\{1, 3, \ldots, k+1\}]$ is isomorphic to $H_1$, thus MinHOM($H$) is NP-hard by Lemma 5.6 and Lemma 5.2.

Case 2-2. $k+1$ is nonadjacent with 4: Observe that there is no arc between $k+1$ and the vertices of $\overrightarrow{C_k}$ other than $1, 2$ and 3. Hence, the digraph $H[\{1, 2, \ldots, k+1\}]$ is isomorphic to $H_2$. MinHOM($H$) is NP-hard by Lemma 5.7 and Lemma 5.2. $\diamond$

**Lemma 5.11** *Let $H$ be a connected quasi-transitive digraph which is neither acyclic nor $\overrightarrow{C_2}$ nor an extension of $\overrightarrow{C_3}$. Then MinHOM(H) is NP-hard.*

**Proof:** We can easily observe that $H$ has an induced cycle $\overrightarrow{C_k} = 12 \ldots k1, k \geq 2$. If it has an induced cycle $\overrightarrow{C_2}$, then there is a vertex $k+1$ outside this cycle which is adjacent with one of the vertices in $\overrightarrow{C_2}$. Furthermore, the quasi-transitivity of $H$ enforces $k+1$ to be adjacent with both vertices in this cycle, and the cycle $\overrightarrow{C_2}$ together with $k+1$ induce a semicomplete digraph. By Lemma 5.3 and Lemma 5.2, MinHOM($H$) is NP-hard in this case. Therefore, we assume that $H$ does not have any symmetric arc hereinafter.

Note that $H$ cannot have an induced cycle $\overrightarrow{C_k} = 12 \ldots k1$ of length greater than 3. Otherwise, by quasi-transitivity of $H$ a chord appears in the cycle, a contradiction. Hence we may consider only $\overrightarrow{C_3}$ as an induced cycle of $H$. Choose a maximal induced subdigraph $H'$ of $H$ which is an extension of $\overrightarrow{C_3}$ with partite sets $X_1, X_2$ and $X_3$. Clearly such subdigraph $H'$ exists.

By assumption $H' \neq H$ and we have a vertex $x$ which is adjacent with at least one vertex of $H'$. Without loss of generality, suppose that $x \to 1$, for some $1 \in X_1$. Since $H$ is quasi-transitive, vertex $x$ should be adjacent with every vertex of $X_2$. There are two possibilities.

Case 1. $x \to 2$ for some $2 \in X_2$. Then MinHOM($H$) is NP-hard by Lemmas 5.3, 5.5 and 5.2.

Case 2. $X_2 \to x$. Then there is an arc between $x$ and each vertex of $X_1$ by quasi-transitivity. If $1' \to x$ for some $1' \in X_1$, then MinHOM($H$) is NP-hard by Lemmas 5.3, 5.5 and 5.2. Else if $x \to X_1$, there is a vertex $3 \in X_3$ which is adjacent with $x$ since otherwise, $H' \cup \{x\}$ is an extension of $\overrightarrow{C_3}$, a contradiction to the maximality assumption. Again MinHOM($H$) is NP-hard by Lemmas 5.3 and 5.2. ◇

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