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Two forbidden subgraphs and the existence of a 2-factor in graphs

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Abstract

In 1996, Ota and Tokuda showed that a star-free graph with sufficiently high minimum degree admits a 2-factor. More recently it was shown that the minimum degree condition can be significantly reduced if one also requires that the graph is not only star-free but also of sufficiently high edge-connectivity. In this paper we reduce the minimum degree condition further for star-free graphs that also avoid WM_r, where WM_r is the graph $K_1 + rK_2$.

1 Introduction

In this paper, we consider only finite undirected graphs without loops and multiple edges. For basic terminology and notation not defined in this paper, we refer the reader to [2]. Let G, H_1, H_2 be three graphs. G is called H_1-free if H_1 is not an induced subgraph of G, and G is called $\{H_1, H_2\}$-free if neither H_1 nor H_2 is an induced subgraph of G. A graph isomorphic to $K_{1,r}$ for some r is called a *star*. A graph isomorphic to $K_1 + rK_2$ for some r is called a *windmill*, and is denoted by WM_r.

It is a natural problem to consider when a graph admits a 2-factor. In some sense this problem has been completely solved by Tutte[5]. For two disjoint subsets S and T of $V(G)$, we define $\mathcal{H}_G(S,T) = \{C \mid C \text{ is a component of } G-(S\cup T),\ e_G(V(C),T) \equiv 1 \pmod 2\}$ and $h_G(S,T) = |\mathcal{H}_G(S,T)|$.

Theorem 1 (Tutte [5]). *A graph G has a 2-factor if and only if*

$$\delta_G(S,T) = 2|S| + \sum_{x\in T}(d_{G-S}(x) - 2) - h_G(S,T) \geq 0$$

for any disjoint subsets S and T of $V(G)$.

The notion of a forbidden subgraph prespriction was introduced by Ota and Tokuda [4] where they proved the following result.

Theorem 2 (Ota and Tokuda [4]). *Let $n \geq 3$ be an integer and G be a $K_{1,n}$-free graph. If the minimum degree of G is at least $2n-2$, then G has a 2-factor.*

This result was shown in [4] to be best possible. The sharpness results demonstrated in [4] all contain bridges. Thus to generalize Theorem 2 we have either to impose edge-connectivity at least 2 or to alter the forbidden subgraph.

The following result shows that if we are only forbidding a single subgraph, then this subgraph must be a star.

Theorem 3. *Let k and d be positive integers. If every k-connected H-free graph with minimum degree at least d has a 2-factor, then H is a star.*

Proof. Let $r = \max\{k, d\}$. Let $G_1 = K_r + (r+1)K_1$ and $G_2 = K_{r,r+1}$, then both of G_1 and G_2 are k-connected graphs with minimum degree at least d, and neither G_1 nor G_2 has a 2-factor. Since H is an induced subgraph of G_1, H is a star or H contains a triangle. And since H is an induced subgraph of G_2, H is triangle-free, which implies H is a star. □

In [1] Aldred et al. proved the following.

Theorem 4 (Aldred et al. [1]). *Let k and n be integers with $k \geq 2$ and $n \geq 3$. Let G be a k-edge-connected $K_{1,n}$-free graph. If $(k,n) \neq (2,3)$ and the minimum degree of G is at least $n - 2 + \frac{n-1}{k-1}$, then G has a 2-factor.*

Thus the minimum degree condition on $K_{1,n}$-free graphs can be significantly reduced from the bound in Theorem 1 if one requires the edge-connectivity of the graph to be large.

Theorem 4 is also shown to be sharp. Thus, to improve on the lower bound for the minimum degree we consider a set of two forbidden subgraphs.

In considering a suitable pair of forbidden subgraphs we first introduce the following useful results.

Theorem 5 (Liu and Zhou [3]). *For any given positive integer g and κ with $g \geq 3$, there is a graph G with girth g and vertex connectivity κ.*

Corollary 6. *For any given positive integer g and κ with $g \geq 3$, there is a graph G with $|V(G)|$ odd, girth at least g and vertex connectivity at least κ, which contains an independent set of cardinality $\kappa + 1$.*

Proof. Let $r = \max\{g, 2(\kappa+1)\} + 1$. Then it follows from Theorem 5 that there exists a graph G with girth r and vertex connectivity κ. If $|V(G)|$ is odd, let $G' = G$, and if $|V(G)|$ is even, delete one vertex from G and let G' be the resulting graph. Let $C = u_1 u_2 \ldots u_{|V(C)|} u_1$ be the shortest cycle of G'. Then since C has no chord, $\{u_1, u_3, \ldots u_{2\kappa+1}\}$ is an independent set of cardinality $\kappa + 1$, and hence G is the required graph. □

Theorem 7. *Let k and d be positive integers. If every k-connected $\{H_1, H_2\}$-free graph with minimum degree at least d has a 2-factor, then H_1 or H_2 is a star.*

Proof. Let $r = \max\{k, d, 2\}$. Assume neither H_1 nor H_2 is a star. Let $G_1 = K_r + (r+1)K_1$ and $G_2 = K_{r,r+1}$, then both G_1 and G_2 are k-connected graphs with minimum degree r, and neither G_1 nor G_2 has a 2-factor. Now H_i is an induced subgraph of G_1 for $i = 1$ or 2. Without loss of generality, we may assume that $i = 1$. Then since H_1 is not a star, H_1 contains a triangle. On the other hand, H_j is an induced subgraph of G_2 for $j = 1$ or 2. Since H_j is not a star, the girth of H_j is 4, which implies $j = 2$.

Now we construct a k-connected graph G_3 with minimum degree at least d and girth at least 5 which has no 2-factor. Let $S = \{y_i \mid 1 \leq i \leq r\}$, $T = \{x_{i,j} \mid 1 \leq i, j \leq r\}$. Moreover, let $\mathcal{C} = \{C_{i,j,l} \mid 1 \leq i, j, l \leq r\}$ be a set which consists of r-connected graphs with odd order, girth at least 5, and each of which contains an independent set of cardinality $r+1$. (The existence of such graphs is guaranteed by Corollary 6.) For i, j and l with $1 \leq i, j, l \leq r$, let $\{w^m_{i,j,l} \mid 1 \leq m \leq r+1\}$ be an independent set in $C_{i,j,l}$. Now we define G_3 as

$$\begin{aligned} V(G_3) &= S \cup T \cup \left(\bigcup_{C \in \mathcal{C}} V(C) \right) \text{ and} \\ E(G_3) &= \{y_i x_{i,j} \mid 1 \leq i, j \leq r\} \\ &\quad \cup \{x_{i,j} w^1_{i,j,l} \mid 1 \leq i, j, l \leq r\} \\ &\quad \cup \{w^{m+1}_{i,j,l} y_m \mid 1 \leq i, j, l, m \leq r\} \\ &\quad \cup \left(\bigcup_{1 \leq i,j,l \leq r} E(C_{i,j,l}) \right) \end{aligned}$$

Then G_3 is a k-connected graph with minimum degree at least $r \geq d$ and girth at least 5. Moreover, since $\mathcal{H}_{G_3}(S,T) = \mathcal{C}$ and $r \geq 2$, we have $\delta_{G_3}(S,T) = 2|S| + \sum_{x \in T}(d_{G_3 - S}(x) - 2) - h_{G_3}(S,T) = 2r + \sum_{x \in T}(r-2) - |\mathcal{C}| = 2r(1-r) < 0$, and hence G_3 has no 2-factor. Now neither H_1 nor H_2 is an induced subgraph of G_3, a contradiction. □

By Theorem 7 we must use a star for one of the forbidden subgraphs. To find a second non-redundant forbidden subgraph we consider the sharpness examples for Theorem 4 demonstrating $K_{1,n}$-free k-edge-connected graphs with minimum degree $n-2+\lceil\frac{n-1}{k-1}\rceil-1$. Each such graph has multiple occurrences of induced subgraphs isomorphic to windmills. Thus we consider a windmill as our second induced subgraph and investigate the existence of a 2-factor in a k-edge-connected $\{K_{1,n}, WM_r\}$-free graph. Since $K_{1,n}$-free graph is also WM_n-free, WM_r becomes redundant if $r \geq n$. Therefore, without loss of generality, we may assume $r \leq n-1$. Then we obtain the following theorem.

Theorem 8. *Let $n \geq 3$, $k \geq 2$, $1 \leq r \leq n-1$ and G be a k-edge-connected $\{K_{1,n}, WM_r\}$-free graph with minimum degree at least $n-1+\frac{r-1}{k-1}$. Then G has a 2-factor.*

2 Proof of Theorem 8

We generally follow [1] for the terminology and notation used in the proof of Theorem 8. Let G be a graph and let S and T be disjoint subsets of $V(G)$. We denote $\bigcup_{v\in T}(N_G(v)\cap S)$ by $N_S(T)$. The number of edges joining S and T is denoted by $e_G(S,T)$. We often identify a subgraph H of G with its vertex set $V(H)$. For example, $e_G(V(H),T)$ is often denoted by $e_G(H,T)$. Moreover, for a vertex x, we sometimes denote $\{x\}$ by x when there is no fear of confusion.

Let G be a graph which has no 2-factor. If a pair of disjoint subsets (S,T) of $V(G)$ is chosen so that $|S|+|T|$ is minimum among those satisfying $\delta_G(S,T)<0$, then we call it *a minimal barrier of* G (Note that the existence of a minimal barrier is guaranteed by Theorem 1.) We use the following lemmas in the proof of Theorem 8.

Lemma 1 (Aldred et al. [1]). *Let G be a graph which has no 2-factor and let (S,T) be a minimal barrier of G. Then $|S|<|T|$.*

Lemma 2 (Aldred et al. [1]). *Let G be a graph which has no 2-factor and let (S,T) be a minimal barrier of G. Then T is independent, and $d_{G-S}(x)=|\{C\in\mathcal{H}_G(S,T) \mid e_G(x,C)>0\}|$ for every $x\in T$.*

Note that if (S,T) is a minimal barrier of a graph without a 2-factor, then we have $T\neq\emptyset$ by Lemma 1.

Proof of Theorem 8. If $n=3$ and $r=1$, then $\delta(G)\geq n-1=2$. Since $r=1$, G is triangle-free, and since G is also $K_{1,3}$-free, there is no vertex of degree three. This implies that G is a 2-regular graph, and the theorem holds since G itself is a cycle. So assume $(n,r)\neq(3,1)$, and by way of contradiction, suppose that there is no 2-factor in G. Take a minimal barrier (S,T). Let $U=V(G)\setminus(S\cup T)$ and

$\mathcal{U} = \mathcal{H}_G(S,T)$. Let

$$\begin{aligned}
\mathcal{U}_1 &= \{C \in \mathcal{U} \mid e_G(T,C) = 1\},\\
\mathcal{U}_{\geq 3} &= \{C \in \mathcal{U} \mid e_G(T,C) \geq 3\},\\
U_1 &= \bigcup_{C \in \mathcal{U}_1} V(C) \text{ and}\\
U_{\geq 3} &= \bigcup_{C \in \mathcal{U}_{\geq 3}} V(C).
\end{aligned}$$

Note that by the definition of $h_G(S,T)$, $h_G(S,T) = |\mathcal{U}_1| + |\mathcal{U}_{\geq 3}|$.

For every $C \in \mathcal{U}_1$, $N_C(T)$ consists of precisely one vertex, say w_C. Note that $N_C(S) \neq \emptyset$ for each $C \in \mathcal{U}_1$ since G is 2-edge-connected. Now we define

$$\begin{aligned}
\mathcal{U}_1^1 &= \{C \in \mathcal{U}_1 \mid N_C(S) = \{w_C\}\}\\
\mathcal{U}_1^2 &= \mathcal{U}_1 \setminus \mathcal{U}_1^1.
\end{aligned}$$

Then for every $C \in \mathcal{U}_1^2$, it follows that $N_C(S) \setminus \{w_C\} \neq \emptyset$. Let v_C be a vertex in $N_C(S) \setminus \{w_C\}$, and let y_C be a vertex in $N_S(v_C)$.

For every $x \in T$, we define the following sets:

$$\begin{aligned}
\mathcal{U}_1^1(x) &= \{C \in \mathcal{U}_1^1 \mid e_G(x,C) = 1\};\\
\mathcal{U}_1^2(x) &= \{C \in \mathcal{U}_1^2 \mid e_G(x,C) = 1\};\\
S(x) &= \bigcup_{C \in \mathcal{U}_1^1(x)} N_S(w_C);\\
E_1(x) &= \{w_C y \mid C \in \mathcal{U}_1^1(x),\ y \in N_S(w_C)\};\\
E_2(x) &= \{v_C y_C \mid C \in \mathcal{U}_1^2(x)\};\\
E_3(x) &= \{xy \mid y \in S(x) \cap N_G(x)\};\\
E_4(x) &= \{xy \mid y \in (S \cap N_G(x)) \setminus S(x)\};\\
D &= \bigcup_{i=1}^{4} \left\{ \bigcup_{x \in T} E_i(x) \right\};\\
F &= \bigcup_{x \in T} E_3(x).
\end{aligned}$$

(See Figure 1.) Note that each component of $\mathcal{U}_1$ contains exactly one vertex which is incident with an edge of D. Moreover, since G is k-edge-connected, each component of $\mathcal{U}_1^1$ must incident to $k-1$ or more edges of E_1. Since $E_1(x) \cap E_1(x') = \emptyset$ for every $x, x' \in T$ with $x \neq x'$,

$$\frac{|E_1(x)|}{k-1} \geq |\mathcal{U}_1^1(x)| \tag{1}$$

holds.

Claim 1. $|D \setminus F| \leq (n-1)|S|$.

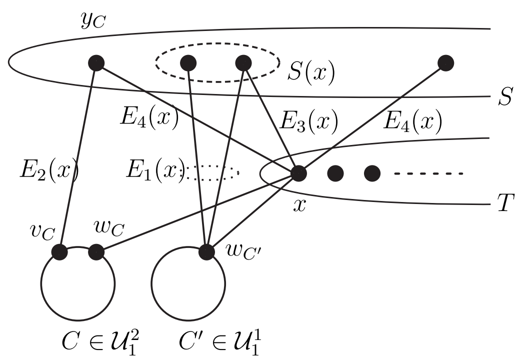


Figure 1:

Proof. Assume $|D \setminus F| > (n-1)|S|$; then there exists $y \in S$ which is incident with n edges of $D \setminus F$, say $yz_1, yz_2, \ldots, yz_n$.

Since G is $K_{1,n}$-free, $z_i z_j \in E(G)$ for some i and j. By the construction of D, $z_i, z_j \in T \cup U_1$. If both of z_i and z_j are in U_1, then they belong to distinct components of $\mathcal{U}_1$ by the definition of E_1 and E_2, and hence they cannot be adjacent. Thus $\{z_i, z_j\} \cap T \neq \emptyset$. Without loss of generality, we may assume $z_i \in T$. By Lemma 2, T is independent, and hence $z_j \in U_1$. Let C be the component that contains z_j, then $C \in \mathcal{U}_1^1(z_i) \cup \mathcal{U}_1^2(z_i)$. If $C \in \mathcal{U}_1^2(z_i)$, then $z_j = v_C$. However, it follows from the definition of v_C that $z_i z_j \notin E(G)$, a contradiction. Consequently $C \in \mathcal{U}_1^1(z_i)$, $z_j = w_C$, and by the definition of E_3, $yz_i \in E_3(z_i)$. This implies $yz_i \in F$, contradicting the fact that $yz_i \in D \setminus F$. □

By the definition we have $e(T, U_{\geq 3}) \geq 3|\mathcal{U}_{\geq 3}|$ and $e(T, U_1) = |\mathcal{U}_1|$. Hence $h_G(S,T) = |\mathcal{U}_1| + |\mathcal{U}_{\geq 3}| \leq e(T, U_1) + \frac{1}{3}e(T, U_{\geq 3})$. Since (S,T) is a minimal barrier, it follows from Lemma 2 that

$$\begin{aligned}
0 &> \delta_G(S,T) \\
&= 2|S| + \sum_{x \in T}(d_{G-S}(x) - 2) - h_G(S,T) \\
&= 2|S| - 2|T| + \sum_{x \in T} d_{G-S}(x) - h_G(S,T) \\
&= 2|S| - 2|T| + e_G(T, U_1) + e_G(T, U_{\geq 3}) - h_G(S,T) \\
&\geq 2|S| - 2|T| + \frac{2}{3} e_G(T, \mathcal{U}_{\geq 3}).
\end{aligned}$$

Hence

$$e_G(T,\mathcal{U}_{\geq 3}) < 3(|T| - |S|).$$

Now it follows from Lemma 1 and Claim 1 that

$$\begin{aligned}|D \setminus F| + e_G(T,\mathcal{U}_{\geq 3}) &< (n-1)|S| + 3(|T| - |S|) \\ &\leq (n-1)|S| + \max\{n-1, 3\}(|T| - |S|).\end{aligned}$$

Hence, if $n \geq 4$, we have

$$|D \setminus F| + e_G(T,\mathcal{U}_{\geq 3}) < (n-1)|S| + (n-1)(|T| - |S|) = (n-1)|T|, \tag{2}$$

and if $n = 3$, we have

$$\begin{aligned}|D \setminus F| + e_G(T,\mathcal{U}_{\geq 3}) &< (n-1)|S| + 3(|T| - |S|) \\ &= 2|S| + 3(|T| - |S|) \;=\; 3|T| - |S|.\end{aligned} \tag{3}$$

Claim 2. $|F| \leq (r-1)|S|$.

Proof. If $F \neq \emptyset$, then there exists a triangle which contains an edge of F. Hence if $r = 1$, it follows that $F = \emptyset$, and the claim holds. So assume $r \geq 2$, and assume to the contrary that $|F| > (r-1)|S|$. Then there exists $y \in S$ which is incident with r edges of F, say $yx_1, yx_2, \ldots, yx_r$. By the definition of F, for every i with $1 \leq i \leq r$, there exists $C_i \in \mathcal{U}_1^1(x_i)$ such that $w_{C_i} \in N_G(y) \cap N_G(x_i)$. Now, by Lemma 2, for every i, j with $i \neq j$, we have $x_ix_j \notin E(G)$, and by the definition of $\mathcal{U}_1^1$ we have $x_iw_{C_j}, x_jw_{C_i} \notin E(G)$. Moreover, since C_i and C_j are in distinct components of $G - (S \cup T)$, we may conclude that $w_{C_i}w_{C_j} \notin E(G)$. Therefore, $\{y, x_1, x_2, \ldots, x_r, w_{c_1}, w_{c_2}, \ldots, w_{c_r}\}$ induces WM_r in G, a contradiction. □

Claim 3. $|F| \leq \sum_{x \in T} |E_1(x)|$.

Proof. For every $xy \in F$ with $x \in T$ and $y \in S$, there exists $C \in \mathcal{U}_1^1$ such that $w_C \in N_G(x) \cap N_G(y)$. Let $f(xy) = w_C$ and let $g(xy) = yf(xy)$. Then, if $x, x' \in T$ and $x \neq x'$, $g(xy) \neq g(x'y)$ holds since $f(xy) \neq f(x'y)$, and if $y \neq y'$ for $y, y' \in S$, obviously $g(xy) \neq g(xy')$. Hence g is an injection from F to $\bigcup_{x \in T} E_1(x)$. Since $E_1(x) \cap E_1(x') = \emptyset$ for every $x, x' \in T$ with $x \neq x'$, the claim holds. □

By Lemma 2, (1) and Claims 2 and 3,

$$\begin{aligned}
&|D \setminus F| + e_G(T, \mathcal{U}_{\geq 3}) \\
&\quad = \sum_{x \in T} (|E_1(x)| + |E_2(x)| + |E_3(x)| + |E_4(x)| - |E_3(x)|) + e_G(T, \mathcal{U}_{\geq 3}) \\
&\quad = \sum_{x \in T} \big(|E_1(x)| + |E_2(x)| + d_G(x) - |\mathcal{U}_1^1(x)| - |\mathcal{U}_1^2(x)| - e_G(x, \mathcal{U}_{\geq 3}) - |E_3(x)|\big) \\
&\qquad + e_G(T, \mathcal{U}_{\geq 3}) \\
&\quad = \sum_{x \in T} \big(|E_1(x)| + d_G(x) - |\mathcal{U}_1^1(x)| - |E_3(x)|\big) - e_G(T, \mathcal{U}_{\geq 3}) + e_G(T, \mathcal{U}_{\geq 3}) \\
&\quad = \sum_{x \in T} \big(|E_1(x)| + d_G(x) - |\mathcal{U}_1^1(x)|\big) - |F| \\
&\quad \geq \sum_{x \in T} \left(|E_1(x)| + d_G(x) - \frac{|E_1(x)|}{k-1}\right) - |F| \\
&\quad = \sum_{x \in T} \left(\frac{k-2}{k-1}|E_1(x)| + d_G(x)\right) - |F| \\
&\quad \geq \frac{k-2}{k-1} \sum_{x \in T} |E_1(x)| + \delta(G) \cdot |T| - |F| \\
&\quad \geq -\frac{1}{k-1}|F| + \delta(G) \cdot |T| \\
&\quad \geq -\frac{r-1}{k-1}|S| + \delta(G) \cdot |T|.
\end{aligned}$$

If $n \geq 4$, it follows from Lemma 1 that

$$\begin{aligned}
|D \setminus F| + e_G(T, \mathcal{U}_{\geq 3}) &\geq -\frac{r-1}{k-1}|S| + \left(n - 1 + \frac{r-1}{k-1}\right)|T| \\
&= (n-1)|T| + \frac{r-1}{k-1}(|T| - |S|) \\
&\geq (n-1)|T|,
\end{aligned}$$

which contradicts (2). If $n = 3$, then we may consider only the case $r = 2$. Now $\delta(G) \geq \lceil n - 1 + \frac{r-1}{k-1} \rceil = 3$. Hence

$$\begin{aligned}
|D \setminus F| + e_G(T, \mathcal{U}_{\geq 3}) &\geq -\frac{r-1}{k-1}|S| + \delta(G) \cdot |T| \\
&\geq 3|T| - |S|,
\end{aligned}$$

which contradicts (3). This completes the proof of Theorem 8. □

3 Sharpness

In this section, we discuss the sharpness of Theorem 8. First, we look at the bound on the minimum degree. If $n - 1 + \frac{r-1}{k-1} \geq n - 2 + \frac{n-1}{k-1}$, then Theorem 4 trivially implies

Theorem 8. Thus, Theorem 8 is meaningful only if $(k,n) = (2,3)$ or $n-1+\frac{r-1}{k-1} < n-2+\frac{n-1}{k-1}$. The latter inequality yields $r < n-k+1$. Since r,n,k are all natural numbers, we have $(k,n) = (2,3)$ or $r \le n-k$.

We claim that the minimum degree condition of Theorem 8 is sharp if $(k,n) = (2,3)$ or $3 \le r \le n-k$. The sharpness for $r \le 2$ remains open. First consider the case $(k,n) = (2,3)$. Now $\delta = r \le n-1 = 2$. There is no need to consider the case $r = 1$, because $k \ge 2$ implies $\delta \ge 2$. So assume $r = 2$. Let H_i be a sufficiently large complete graph for $i = 1$ and 2, and let u_i, v_i and w_i be distinct vertices in H_i. Let G be a graph such that $V(G) = V(H_1) \cup V(H_2) \cup \{u_3, v_3, w_3\}$ and $E(G) = E(H_1) \cup E(H_2) \cup \{u_1u_3, u_2u_3, v_1v_3, v_2v_3, w_1w_3, w_2w_3\}$. Then G is a 2-edge connected $\{K_{1,3}, WM_2\}$-free graph with minimum degree 2, and G has no 2-factor.

Next consider the case $(k,n) \ne (2,3)$. Then $r \le n-k$. Since $r \ge 1$, we have $k \le n-1$. Let β be an integer such that $n-1+\left\lceil\frac{r-1}{k-1}\right\rceil - 1 = n-1+\frac{r-\beta}{k-1}$. (Note that $2 \le \beta \le k$.) Since $n-1+\frac{r-\beta}{k-1} = n-1+\left\lceil\frac{r-1}{k-1}\right\rceil - 1 < n-2+\frac{n-1}{k-1}$, we have $r-\beta+k-1 < n-1$. We define $I = \{(i,j) \mid 1 \le i \le n-1,\ 1 \le j \le r-\beta+k\}$ and $L = \left\{l \;\middle|\; 1 \le l \le \frac{r-\beta}{k-1}\right\}$. Let

$$S = \{y_{i,j} \mid (i,j) \in I\} \text{ and } T = \{x_{i,j} \mid (i,j) \in I\} \cup \{\tilde{x}\}.$$

Moreover, let

$$\mathcal{C} = \{C^l_{i,j} \mid (i,j) \in I,\ l \in L\} \cup \{C^{\frac{r-\beta}{k-1}+1}_{i,r-\beta+k} \mid 1 \le i \le n-1\} \text{ and } \tilde{\mathcal{C}} = \{\tilde{C}^l \mid l \in L\}$$

be two sets each of which consists of sufficiently large complete graphs. From each $C^l_{i,j} \in \mathcal{C}$ and $\tilde{C}^l \in \tilde{\mathcal{C}}$, we choose one vertex $w^l_{i,j}$ and $\tilde{w}^l$, respectively. Now we consider $x_{n,j} = x_{1,j}$ and $y_{0,j} = y_{n-1,j}$ for every j, $x_{i,j} = x_{i+1,j-(r-\beta+k)}$ for $j \ge r-\beta+k+1$ and $y_{i,j} = y_{i-1,j+(r-\beta+k)}$ for $j \le 0$. Let G be a graph defined by

$$V(G) = S \cup T \cup \left(\bigcup_{C \in \mathcal{C} \cup \tilde{\mathcal{C}}} V(C)\right) \text{ and}$$

$$\begin{aligned}
E(G) \\
= \;& (\{y_{i,j}x_{i,j'} \mid (i,j) \in I,\ j \le j' \le j+n-2\} \setminus \{y_{i,r-\beta+k}, x_{i,r-\beta+k} \mid 1 \le i \le n-1\}) \\
& \cup \{y_{i,r-\beta+k}\tilde{x} \mid 1 \le i \le n-1\} \\
& \cup \{x_{i,j}w^l_{i,j} \mid (i,j) \in I,\ l \in L\} \cup \{x_{i,r-\beta+k}w^{\frac{r-\beta}{k-1}+1}_{i,r-\beta+k} \mid 1 \le i \le n-1\} \\
& \cup \{\tilde{x}\tilde{w}^l \mid l \in L\} \\
& \cup \{w^l_{i,j}y_{i,j'} \mid (i,j) \in I,\ l \in L,\ j-l(k-1) \le j' \le j-(l-1)(k-1)-1\} \\
& \cup \{w^{\frac{r-\beta}{k-1}+1}_{i,r-\beta+k}y_{i,j'} \mid 1 \le i \le n-1,\ 1 \le j' \le k-1\} \\
& \cup \{\tilde{w}^l y_{i,r-\beta+k} \mid l \in L,\ n-l(k-1) \le i \le n-1-(l-1)(k-1)\} \\
& \cup \left(\bigcup_{C \in \mathcal{C} \cup \tilde{\mathcal{C}}} E(C)\right).
\end{aligned}$$

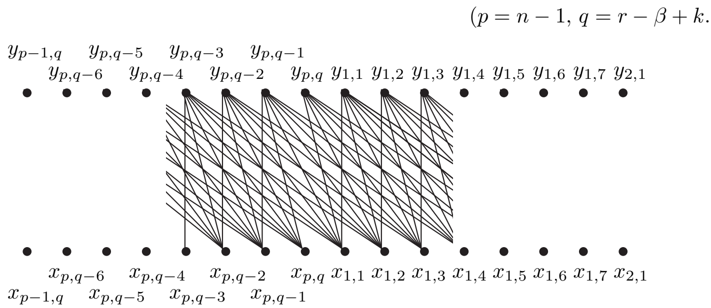


Figure 2: $G[S \cup T]$ in case of $n = 8$, $r = 6$ and $k = 3$.

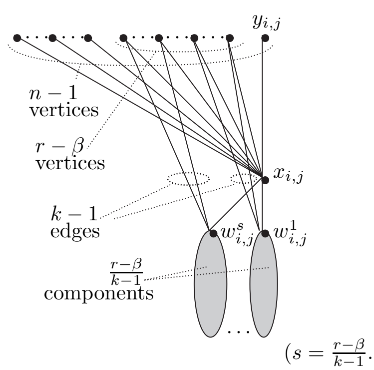
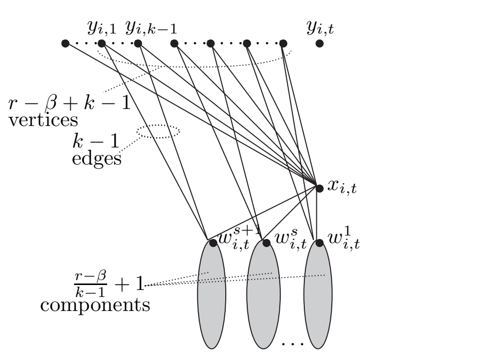


Figure 3: Around $x_{i,j}$ with $j \neq r - \beta + k$.



($s = \frac{r-\beta}{k-1}$, $t = r - \beta + k$.)

Figure 4: Around $x_{i,r-\beta+k}$.

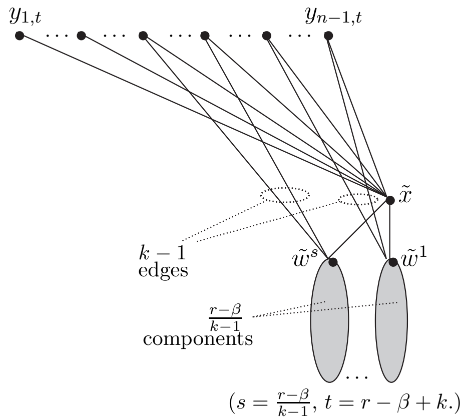


Figure 5: Around $\tilde{x}$.

(See Figures 2–5.) Then, $G' = G[S \cup T \setminus \{\tilde{x}\}]$ contains the edge set $\{y_{i,j}x_{i,j+\gamma} \mid y_{i,j} \in S,\ 1 \le \gamma \le n-2\}$, and hence G' is $(n-2)$-edge-connected. Note that $\tilde{x}$ is adjacent to $n-1 \ge k$ vertices of S, and each $w^l_{i,j}$ or $\tilde{w}^l$ is adjacent to at least k vertices of $V(G) \setminus V(C^j_i)$. This implies that G is $(n-1)$-edge-connected, that is, k-edge-connected. Next, we check that G is a $\{K_{1,n},\ WM_r\}$-free graph with minimum degree $n-1+\lceil \frac{r-1}{k-1} \rceil - 1$.

For every $y \in S$, y is adjacent to $n-1$ vertices in T and $r-\beta$ or $r-\beta+1$ vertices in $U = V(G) \setminus (S \cup T)$. Hence $d_G(y) \ge n-1+r-\beta \ge n-1+\frac{r-\beta}{k-1} = n-1+\lceil \frac{r-1}{k-1} \rceil - 1$. Now $w^l_{i,j} \in N_G(y) \cap U$ if and only if $x_{i,j} \in N_G(y) \cap N_G(w^l_{i,j})$, and $\tilde{w}^l \in N_G(y) \cap U$ if and only if $\tilde{x} \in N_G(y) \cap N_G(\tilde{w}^l)$. Hence there exists no $K_{1,n}$ with center y in G, and if there exists $WM_{r'}$ with center y for some r', then $r' \le r-\beta+1 \le r-1$.

For every $x \in T$, x is adjacent to $n-1$ or $n-2$ vertices of S, and $\frac{r-\beta}{k-1}$ or $\frac{r-\beta}{k-1}+1$ vertices of U, respectively. Hence $d_G(x) = n-1+\frac{r-\beta}{k-1} = n-1+\lceil \frac{r-1}{k-1} \rceil - 1$. If $w^l_{i,j} \in N_G(x) \cap U$, then there exists some j' such that $y_{i,j'} \in N_G(x) \cap \{w^l_{i,j}\}$, and if $\tilde{w}^l \in N_G(\tilde{x}) \cap U$, then there exists some i' such that $y_{i',r-\beta+k} \in N_G(\tilde{x}) \cap \tilde{w}^l$. Hence there exists no $K_{1,n}$ with center x or $\tilde{x}$ in G, and if there exists $WM_{r'}$ with center x or $\tilde{x}$ for some r', then $r' \le \frac{r-\beta}{k-1}+1 \le r-\beta+1 \le r-1$.

Let $w \in U$; then w has sufficiently large degree. If $w \in N_G(S \cup T)$, then w has exactly one neighbor in T, say x, and w has $k-1$ neighbors in S each of which is adjacent to x. Note that S is independent in G. Hence, if there exists $K_{1,n'}$ with center w for some n', then $n' \le k \le n-1$, and if there exists $WM_{r'}$ with center w for some r', then $r' \le 2 \le r-1$.

Therefore, G is a $\{K_{1,n},\ WM_r\}$-free graph with minimum degree $n-1+\lceil \frac{r-1}{k-1} \rceil - 1$. Now $\delta_G(S,T) = 2|S| - 2|T| < 0$, and hence G has no 2-factor.

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