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A note on perfect dissections of an equilateral triangle

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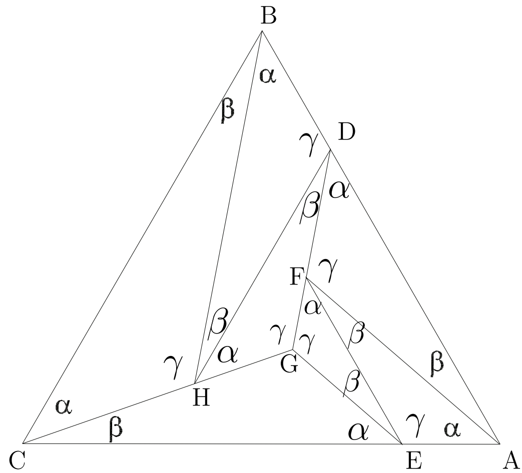
Abstract

A perfect dissection of a polygon P into a polygon P' is a decomposition of P into internally disjoint pairwise incongruent polygons all similar to P'. It is known that there is no perfect dissection of an equilateral triangle into smaller equilateral triangles. On the other hand, an equilateral triangle has trivial perfect dissections into any number n of right triangles where $n \geq 3$. We give an example of a perfect dissection of an equilateral triangle into 7 non-right triangles and prove that 7 is the smallest number of tiles in such dissection.

1 Introduction

Let P and P' be polygons in the Euclidean plane. A P'-dissection (tiling) of P is a decomposition of P into finitely many, internally disjoint polygons P'_1, ..., P'_n ($n \geq 2$) such that all of the P'_i are similar to P'. A dissection is perfect if the P'_i are pairwise incongruent. Polygons P'_i are called the tiles of the dissection.

Perfect dissections were first studied in the case of dividing a rectangle into squares. The first examples were found by Moroń [2] in 1925. In 1939 Sprague [3] found a perfect dissection of a square into 55 squares. The main contribution to this theory was made by Brooks et al. [1] in 1940, where the authors developed a beautiful and powerful method of transforming a square tiling into an equivalent electrical circuit. Tutte continued the study of perfect dissections and proved [4] in 1948 that there is no perfect dissection of an equilateral triangle into smaller equilateral triangles. On the other hand, one can easily divide an equilateral triangle into any number $n \geq 3$ of pairwise incongruent right triangles with angles $\pi/2$, $\pi/3$ and $\pi/6$. In this paper we prove that an equilateral triangle can also be perfectly divided into non-right triangles. We present an example of such a dissection into seven tiles (see Fig. 1) and prove that 7 is the smallest number of triangles in such a dissection.



$\gamma = 2\pi/3,\ \alpha + \beta = \pi/3,\ (\sin^2\alpha + 3/4)^2 \cdot \sin^3\beta = \sin^7\alpha$

Fig. 1. Perfect dissection of an equilateral triangle into non-right triangles.

2 Notation

In the paper T denotes an equilateral triangle and $\triangle$ denotes a non-right triangle which is a common shape of all tiles. We assume that T has corners A, B, C and $\triangle$ has angles α, β, γ. At times we mean by the word *dissection* a graph obtained as a result of a $\triangle$-dissection of T. Let V and E denote the set of its vertices and edges, respectively, and let v and e denote the number of its vertices and edges. We distinguish two sets among boundary vertices:

V_2—corners of T,

V_b—set of vertices lying on the boundary of T, but different from its corners.

Let v_2 and v_b denote the cardinalities of the above sets. Thus, $v_2 = 3$.

Similarly let us distinguish two sets among internal vertices:

V_4—set of vertices each of which is an internal point of a side of some tile,

V_3—set of vertices (lying in the interior of T) each of which is a corner of every tile it belongs to.

Let v_4 and v_3 denote the cardinalities of those sets. Let us recall the following definition. The *degree* of a vertex x, $\deg x$, is the number of edges incident to it, in other words the number of edges with x as an endpoint.

3 Preparatory results

Recall the following known way of counting vertices in a dissection. Summing the angles of n tiles we obtain

$$n \cdot \pi = \pi + v_b \cdot \pi + v_3 \cdot 2\pi + v_4 \cdot \pi.$$

Thus,

$$v_b + 2v_3 + v_4 = n - 1. \tag{1}$$

Using Euler's formula $e = n + v - 1$ and formula (1) we obtain

$$e = 2n + 1 - v_3. \tag{2}$$

Proposition 1 *Every $\triangle$-dissection of T satisfies the following conditions:*

1. *if $x \in V_2$ then* $\deg x \geq 2$,
2. *if $x \in V_3$ then* $\deg x \geq 3$,
3. *if $x \in V_4$ or $x \in V_b$ then* $\deg x \geq 4$.

Proof. Properties 1 and 2 are obvious. To prove property 3, note that the sum of any two angles (not necessary different) of a non-right triangle is always different from π. ■

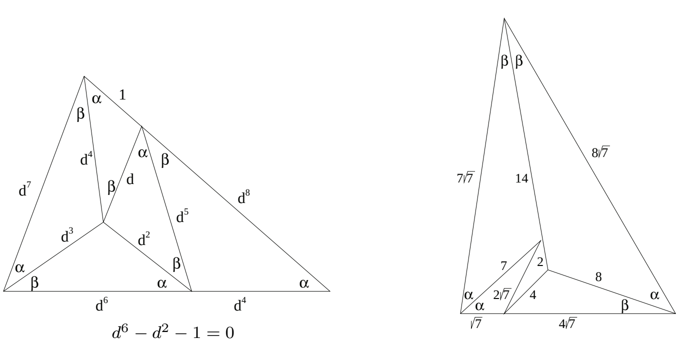


Fig. 2. The only triangles that have a perfect $\triangle$-dissection into 5 tiles.

Proposition 2 *Consider a perfect* $\triangle$*-dissection of* T *with* $V_3 \neq \emptyset$*. Let* $x \in V_3$*. If* $\deg x = 3$ *then every corner of* T *has degree greater than or equal to* 3*.*

Proof. Suppose on the contrary that one of the corners of T, say A, has degree equal to 2. Hence one of the angles of $\triangle$ is equal to $\pi/3$. Note that because T is divided into triangles the sum of any two angles at x is greater than or equal to π. Since these angles are angles of $\triangle$, they are equal. Thus all three angles at x are equal, hence are equal to $2\pi/3$. Therefore the angles of $\triangle$ are $\pi/3$, $2\pi/3$ and 0, a contradiction. ∎

Proposition 3 *In every perfect* $\triangle$*-dissection of* T*,* $v_3 \geq 1$*.*

Proof. Suppose on the contrary that $v_3 = 0$. Using (1), (2) and Proposition 1, we obtain

$$4n+2 = 2e = \sum_{x\in V_2} \deg x + \sum_{x \in V_b \cup V_4} \deg x \geq$$

$$6 + 4(v_b + v_4) = 4n+2.$$

Thus for all $x \in V_2$, $\deg x = 2$ and for all $x \in V_b \cup V_4$, $\deg x = 4$. Hence one of the angles of $\triangle$, say α, is equal to $\pi/3$. Moreover, because $\triangle$ is non-equilateral (by Tutte's result), $\beta \neq \pi/3$ and $\gamma \neq \pi/3$. Since each angle of T is equal to $\pi/3 = \alpha$, there exists a vertex $y \in V_b \cup V_4$ that compensates the global number of angles α in the dissection. Thus, there exists a vertex $y \in V_b \cup V_4$ such that no angle at it equals α. Hence, $p\beta + q\gamma = \pi$ with $p + q = 3$. We may assume that $p \geq q$. Hence, $2\beta + \gamma = \pi$ or $3\beta = \pi$. However, in both cases $\triangle$ is equilateral. Thus, the dissection cannot be perfect. ∎

4 Main result

Theorem 1 *An equilateral triangle has a perfect dissection into* 7 *non-right triangles with angles* α, β, γ *satisfying* $\gamma = 2\pi/3$*,* $\alpha + \beta = \pi/3$ *and*

$$(\sin^2 \alpha + 3/4)^2 \cdot \sin^3 \beta = \sin^7 \alpha \tag{3}$$

(see Fig. 1*).*

Proof. First note that one can arrange three similar triangles with any angles α, β, γ satisfying $\alpha + \beta = \pi/3$ in such a way that they form a quadrangle with opposite angles equal to $\pi/3$ and $2\pi/3$; see quadrangle $BCDG$ in Fig. 1. Consequently, by an appropriate arrangement of such a dissected quadrangle and its similar copy one can obtain a pentagon with angles $4\pi/3, 2\pi/3+\beta, \alpha, \pi/3, \pi/3$, where angles $2\pi/3+\beta$ and α are adjacent to the angle $4\pi/3$; see pentagon $BCGEA$ in Fig. 1. Now, by drawing a line connecting the vertices at the angles $2\pi/3+\beta$ and α (vertices E and C in Fig. 1), one obtains a figure dissected into 7 triangles, 6 of which are similar. We shall prove that if α, β satisfy (3), then the resulting dissection is a perfect dissection of an equilateral triangle into 7 similar triangles. We use the notation of Fig. 1. It is enough to prove that in the triangle CEG, $\angle E = \alpha$ and $\angle C = \beta$, and that all the tiles are pairwise incongruent. Without a loss of generality we assume that $|EF| = 1$.

Then, by the law of sines, $|EG| = \frac{\sin\alpha}{\sin\gamma} = \frac{2\sin\alpha}{\sqrt{3}}$, $|FG| = \frac{\sin\beta}{\sin\gamma} = \frac{2\sin\beta}{\sqrt{3}}$ and $|AF| = \frac{\sin\gamma}{\sin\alpha} = \frac{\sqrt{3}}{2\sin\alpha}$. Furthermore,

$$\begin{aligned}
|DF| &= \frac{\sin\beta}{\sin\alpha}|AF| = \frac{\sqrt{3}\sin\beta}{2\sin^2\alpha},\\
|DH| &= \frac{\sin\gamma}{\sin\alpha}\left(|GF| + |DF|\right) = \frac{\sin\beta}{\sin^3\alpha}(\sin^2\alpha + 3/4) = \sqrt{\frac{\sin\beta}{\sin\alpha}}, \text{by (3)},\\
|GH| &= \frac{\sin\beta}{\sin\gamma}|DH| = \frac{2\sqrt{\sin\beta\sin\alpha}}{\sqrt{3}},\\
|BH| &= \frac{\sin\gamma}{\sin\alpha}|DH| = \frac{\sqrt{3}}{2\sqrt{\sin\beta\sin\alpha}},\\
|CH| &= \frac{\sin\beta}{\sin\alpha}|BH| = \frac{\sqrt{3\sin\beta}}{2\sin\alpha\sqrt{\sin\alpha}}.
\end{aligned}$$

Now, in triangle CEG

$$\frac{|EG|}{\sin C} = \frac{|CH| + |GH|}{\sin E}.$$

Hence

$$\begin{aligned}
\frac{\sin E}{\sin C} &= \left(\frac{\sqrt{3\sin\beta}}{2\sin\alpha\sqrt{\sin\alpha}} + \frac{2\sqrt{\sin\beta\sin\alpha}}{\sqrt{3}}\right) \div \frac{2\sin\alpha}{\sqrt{3}} = \frac{3\sqrt{\sin\beta}}{4\sin^2\alpha\sqrt{\sin\alpha}} + \frac{\sqrt{\sin\beta}}{\sqrt{\sin\alpha}}\\
&= \frac{\sqrt{\sin\beta}(3/4 + \sin^2\alpha)}{\sin^2\alpha\sqrt{\sin\alpha}} = \frac{\sin\alpha}{\sin\beta}, \text{ by (3)}.
\end{aligned}$$

Moreover, $\sin\alpha = \sin(\pi/3 - \beta) = \frac{\sqrt{3}}{2}\cos\beta - \frac{1}{2}\sin\beta$ and $\sin E = \frac{\sqrt{3}}{2}\cos C - \frac{1}{2}\sin C$. Thus $\cot C = \cot\beta$. Therefore $C = \beta$ and $E = \alpha$. Hence all tiles are similar. Checking that the tiles are pairwise incongruent is left to the reader. ■

Theorem 2 *There is no perfect dissection of T into fewer than 7 non-right triangles.*

Proof. To simplify the discussion, let us distinguish the following situation:

(**) there is a segment, different from any side of T, that connects one of the vertices of T with the side opposite to it.

In [5] it was proved that no triangle has a perfect dissection into fewer than 5 non-right triangles and only two triangles, both non-equilateral (see Fig. 2), have a perfect dissection into 5 non-right triangles. Therefore, it remains to show that an equilateral triangle cannot be dissected into 6 pairwise incongruent triangles, all similar to a non-right triangle $\triangle$. Suppose on the contrary that T is dissected in a perfect way into 6 triangles similar to $\triangle$. Note that we can assume that (**) does not occur. Indeed, otherwise T is divided into two triangles. Hence one of these triangles is divided into 2, 3, 4 or 5 tiles. This is not possible, because there is no perfect dissection of any triangle into fewer than 5 non right triangles. Moreover, one can see that the only perfect dissections into 5 non-right triangles (Fig. 2) cannot be a part of a perfect dissection of T into 6 non-right tiles.

We may further assume that at least two sides of T are divided by vertices from V_b (hence $v_b \geq 2$), because otherwise two sides of T are sides of two congruent tiles. Therefore, by formula (1) and Proposition 3 we have to analyze only two cases.

1. $v_3 = 1, v_b = 3, v_4 = 0$,
2. $v_3 = 1, v_b = 2, v_4 = 1$.

Assume case 1. Let $x \in V_3$. Note that if each corner of T has degree greater than or equal to 3, then (since (**) does not occur) each corner of T is connected by an edge with x. Hence three edges incident to x divide T into three triangles, two of them containing vertices from V_b. Since every vertex from V_b has degree greater than or equal to 4, these two triangles are divided into at least three tiles each. Hence the number of tiles in the dissection is at least 7, a contradiction. The above situation occurs if $\deg x = 3$; see Proposition 2. Hence we may assume that $\deg x \geq 4$ and one of the corners of T, say A, has degree equal to 2. Thus one of the angles of $\triangle$, say α, is equal to $\pi/3$. Furthermore, x is connected to another corner of T, say B. Because (**) does not occur, $\deg B = 3$. Hence angle B of T is divided into two smaller angles, both different from α. Since $\alpha + \beta + \gamma = \pi$ these two angles are equal to each other and hence both equal $\pi/6$. Thus $\triangle$ is a right triangle, a contradiction. Hence case 1 is not possible.

Assume case 2. Let $x \in V_3$ and $y \in V_4$. Suppose that one of the corners of T, say A, has degree equal to 2. Note that because (**) does not occur, every corner of T with degree equal to 2 is adjacent to two vertices from V_b. Since $v_b \leq 2$, $\deg B \geq 3$ and $\deg C \geq 3$. Moreover, analogously as in case 1, we show that $\deg B \neq 3$ and $\deg C \neq 3$. Hence $\sum_{v \in V_2} \deg v \geq 10$. By formula (2), $e = 12$. Thus

$$24 = 2e = \sum_{v \in V} \deg v = \sum_{v \in V_2} \deg v + \sum_{v \in V_b} \deg v + \deg x + \deg y \geq 10 + 8 + 3 + 4 = 25,$$

a contradiction. Therefore every corner of T has degree greater than or equal to 3 and hence is connected to x or y. If x is connected to y, then three or four edges divide T into three regions, two of them containing a vertex from V_b. Since every vertex from V_b has degree greater than or equal to 4, these two regions are divided into at least three tiles each. Hence the number of tiles in the dissection is at least 7, a contradiction. Therefore x and y are not connected. Hence y lies on the segment connecting two vertices different from x. Since (**) does not occur, y lies on the segment connecting two vertices from V_b. This segment divides T into a triangle and a quadrangle. Because there is no perfect dissection of any triangle into fewer than 5 non-right tiles, the quadrangle is divided into 5 tiles. Thus a corner of T has degree equal to 2, which is a contradiction with previous observations. Therefore case 2 is not possible. This completes the proof of the theorem. ■

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