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Quasi-embeddings and intersections of latin squares of different orders

Peter Dukes*

Mathematics and Statistics
University of Victoria
Victoria, BC V8W 3R4
Canada
dukes@uvic.ca

Eric Mendelsohn†

Mathematics
University of Toronto
Toronto, ON M5S 3G3
Canada
mendelso@math.utoronto.ca

Abstract

We consider a common generalization of the embedding and intersection problems for latin squares. Specifically, we extend the definition of embedding to squares whose sides do not meet the necessary condition for embedding and extend the intersection problem to squares of different orders. Results are given for arbitrary latin squares, and those which are idempotent and idempotent symmetric. The latter topic is interpreted in terms of one-factorizations of the complete graph. Similar problems for Steiner triple systems, which can be regarded as totally symmetric idempotent latin squares, had been previously investigated by the authors, along with P. Danziger and T. Griggs.

1 Introduction

A *latin square* $\mathcal{L} = (L_{ij})$ of *order* (or *side*) n is an $n \times n$ matrix whose entries are from a set E, $|E| = n$, such that every row and column is a permutation of E. As usual, $\mathcal{L}$ is *symmetric* if $\mathcal{L} = \mathcal{L}^\top$.

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A *latin rectangle* is a $k \times n$ array, $k \leq n$, with entries from E such that every row is a permutation of E and every column has no repeated symbol. Later, we make use of the easy fact (see [5]) that every $k \times n$ latin rectangle can be completed to a latin square of side n.

More generally, a *partial latin square* of order n is an $n \times n$ array in which every cell is either blank, or contains a symbol from E, with the property that every symbol appears at most once in each row and column.

Let $\mathcal{L} = (L_{ij})$ and $\mathcal{M} = (M_{ij})$ be latin squares, each of order n, and with entry sets E and F, respectively. Then $\mathcal{L}$ and $\mathcal{M}$ are *orthogonal* if and only if the ordered pairs $\{(L_{ij}, M_{ij}) : 1 \leq i, j \leq n\}$ exhaust $E \times F$.

A *transversal* T of $\mathcal{L} = (L_{ij})$ is a collection of cells in L_{ij} so that every row, every column and every entry appears exactly once in T. It is well-known that the existence of an orthogonal mate $\mathcal{M}$ to $\mathcal{L}$ is equivalent to the existence of n pairwise disjoint transversals in $\mathcal{L}$.

Square $\mathcal{L}$ is called *idempotent* if the main diagonal of $\mathcal{L}$ is the transversal with $L_{ii} = i$ for each i. Idempotent latin squares exist for all orders except 2.

Suppose a latin square $\mathcal{L}$ of side n and entry set E has k disjoint transversals $\mathcal{T} = \{T_1, \ldots, T_k\}$. A *prolongation* of $\mathcal{L}$ *along* $\mathcal{T}$ is a latin square $\mathcal{L}^*$ of side $n+k$ and entry set $E \cup \{\infty_1, \ldots, \infty_k\}$, where if $L_{ij} = e \in E$ is on T_h, we have $L^*_{ij} = \infty_h$ and $L'_{n+h,j} = L'_{i,n+h} = e$. All entries of $\mathcal{L}$ not on some T_h are left unchanged in $\mathcal{L}^*$, and $L^*_{n+i,n+j} = Z_{ij}$, where $\mathcal{Z} = (Z_{ij})$ is a latin square of side k on entries $\{\infty_1, \ldots, \infty_k\}$. Below $\mathcal{L}^k$ represents $\mathcal{L}$ with each T_h replaced by ∞_h and $\mathcal{T}_R, \mathcal{T}_C$ represent 'ordered projections' of $\mathcal{T}$ onto rows and columns, respectively.

$$\mathcal{L}^* = \begin{array}{|c|c|} \hline \mathcal{L}^k & \mathcal{T}_C \\ \hline \mathcal{T}_R & \mathcal{Z} \\ \hline \end{array}$$

Let $\mathcal{L}' = (L'_{ij})$ be a latin square of order m. Let $I, J \subseteq \{1, \ldots, m\}$ with $|I| = |J| = n$. If the $n \times n$ submatrix $\mathcal{L}$ of $\mathcal{L}'$ consisting of the rows indexed by I and columns indexed by J is itself a latin square, then $\mathcal{L}$ is called a *latin subsquare* (or simply *subsquare*) of $\mathcal{L}'$. We also say that $\mathcal{L}$ is *embedded* in $\mathcal{L}'$.

Theorem 1.1. ([5]) *Let $\mathcal{L}$ be any latin square of order n. There exists a latin square $\mathcal{L}'$ of order m such that $\mathcal{L}$ is a subsquare of $\mathcal{L}'$ if and only if $n = m$ or $2n \leq m$.*

Two latin squares of side n and with the same entry set have *intersection number* s if they agree in exactly s cells. The paper [7] gives a survey of results on the set of possible intersection numbers of two latin squares of side n. Determining this set for various types of latin squares is often loosely referred to as the 'intersection problem' for latin squares.

Theorem 1.2. ([7]) *Let $n \geq 5$. The set of all possible intersections numbers for two latin squares of order n is $[0, n^2] \setminus \{n^2 - 5, n^2 - 3, n^2 - 2, n^2 - 1\}$.*

In this paper, we consider a common generalization of embedding and the intersection problem for latin squares. Specifically, we extend the definition of embedding to squares whose sides do not meet the condition in Theorem 1.1, and extend the intersection problem to squares of different orders. We consider arbitrary latin squares (sections 2 and 3), and those which are idempotent (section 4), and idempotent symmetric (section 5). Results on the latter are interpreted in terms of one-factorizations of K_{2n}. The reader may wish to compare with [6] and [4], where similar problems are discussed for Steiner triple systems, which are equivalent to idempotent latin squares which are 'totally symmetric'.

2 Quasi-embeddings of latin squares

Let $\mathcal{L} = (L_{ij})$ and $\mathcal{L}' = (L'_{ij})$ be latin squares of sides $n \le m$ and entry sets $E \subseteq E'$, respectively. Define $\mathcal{L} \cap \mathcal{L}' = \{(i,j) : 1 \le i,j \le n \text{ and } L_{ij} = L'_{ij}\}$; that is, the set of cells with common entries in $\mathcal{L}$ and $\mathcal{L}'$. Define $I(n,m)$ to be the set of all possible values of $|\mathcal{L} \cap \mathcal{L}'|$, where $\mathcal{L}$ and $\mathcal{L}'$ are latin squares of sides n and m, respectively. Let $D(n,m)$ be the maximum element in $I(n,m)$. If $\mathcal{L}$ and $\mathcal{L}'$ achieve this maximum $D(n,m)$, we say that $\mathcal{L}$ is *quasi-embedded* in $\mathcal{L}'$. Of course, Theorem 1.1 states that quasi-embeddings coincide with embeddings when $n = m$ or $2n \le m$. Specifically, $D(n,m) = n^2$ and $I(n,n) \subseteq I(n,m) \subseteq [0,n^2]$ in these cases. Therefore, we focus on the case $m = n + k$, where $0 < k < n$.

Example 2.1. The latin squares below form a quasi-embedding and illustrate $D(2,3) = 3$. In fact, $I(2,3) = \{0,1,3\}$.

$$\boxed{\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}} \rightarrow \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}}$$

We now prove a general upper bound on $D(n,n+k)$, which we call the *counting bound.*

Lemma 2.2. *Suppose $k < n$ are positive integers. Then*

$$D(n,n+k) \le n^2 - k(n-k).$$

Proof: Suppose $\mathcal{L}$ is quasi-embedded in $\mathcal{L}'$, where the squares are of sides n and $n+k$ with entries E, E', respectively. In the $n \times (n+k)$ latin rectangle formed from the first n rows of $\mathcal{L}'$, at most k^2 entries from $E' \setminus E$ can appear in the last k columns. So at least $k(n-k)$ such entries appear in the first n columns. Therefore, $\mathcal{L}$ and $\mathcal{L}'$ disagree in these cells. □

Our goal in this section is to show equality in the counting bound. We begin with a construction combining prolongation with rectangle completion.

Lemma 2.3. *A latin square $\mathcal{L}$ of side n with k disjoint transversals can be quasi-embedded in a latin square of side $n+k$ so that the counting bound is achieved.*

Proof: Given $\mathcal{L}$, construct the latin square $\mathcal{L}'$ of side $n+k$ as follows.

1. (rows 1 to k): Take the first k rows of $\mathcal{L}$, with a latin square of side k on entries $\{\infty_1, \dots, \infty_k\}$, appended to the right.

2. (rows $k+1$ to n): Take rows $k+1, \dots, n$ of a prolongation of $\mathcal{L}$ along k disjoint transversals.

3. (rows $n+1$ to $n+k$): Complete the $n \times (n+k)$ latin rectangle constructed above.

We leave it to the reader to verify that $|\mathcal{L} \cap \mathcal{L}'| = n^2 - k(n-k) = D(n, n+k)$. □

Variants of this construction are possible. We give only one example.

Example 2.4. Consider the latin square $\mathcal{L}$ of order 6 below. Replace the top five occurrences of '1' in $\mathcal{L}$ with new symbols $7, \dots, 11$. Let $\mathcal{M}$ be an idempotent latin square of order 5 on symbols $7, \dots, 11$. Prolong $\mathcal{M}$ along a transversal, and delete the last column of the result. Append this 6×5 array, and complete the resulting latin rectangle. This is a quasi-embedding, showing $D(6, 11) = 31$, since this achieves the counting bound in Lemma 2.2.

$\mathcal{L}$

1	2	3	4	5	6
6	1	2	3	4	5
5	6	1	2	3	4
4	5	6	1	2	3
3	4	5	6	1	2
2	3	4	5	6	1

$\rightarrow$

7	2	3	4	5	6	1	10	8	11	9
6	8	2	3	4	5	10	1	11	9	7
5	6	9	2	3	4	8	11	1	7	10
4	5	6	10	2	3	11	9	7	1	8
3	4	5	6	11	2	9	7	10	8	1
2	3	4	5	6	1	7	8	9	10	11

It is well-known (see [1]) that there exists a pair of orthogonal latin squares of every order $n \neq 2, 6$, as well as a latin square of side 6 with four disjoint transversals. The quasi-embeddings of Examples 2.1 and 2.4 settle the remaining cases not covered by Lemma 2.3.

Theorem 2.5. *For all* $0 < k < n$, $D(n, n+k) = n^2 - k(n-k)$. *In other words, all quasi-embeddings of latin squares achieve the counting bound.*

3 Intersections of latin squares

We now investigate the set $I(n, n+k)$ of possible intersection values in more detail. The following is a useful result of Ryser on completing partial latin squares.

Lemma 3.1. ([10]) *Suppose* P *is a partial latin square of order* n *in which a cell is filled if and only if it lies in the first* r *rows and* s *columns. If each symbol appears at least* $r+s-n$ *times in* P*, then* P *can be completed to a latin square of side* n.

There are various applications of Lemma 3.1 for determining $I(n, n+k)$.

First, we had previously mentioned that when $k \geq n$ an ordinary embedding of any latin square of order n into a latin square of order $n+k$ is possible. This handles most intersection values automatically. We have $I(n, n+k) \supseteq I(n, n)$, which by Theorem 1.2 covers all but a few values in $[0, n^2]$. The remaining intersections are easily obtained with Lemma 3.1.

Theorem 3.2. *If $k \geq n$, then $I(n, n+k) = [0, n^2]$.*

Proof: Take any latin square $\mathcal{L}$ of order n. For any $x \in [0, n^2]$, keep x entries in $\mathcal{L}$ and replace each other entry i by new symbol ∞_i. Regarded as a partial latin square of order $n+k$, this can be completed by Lemma 3.1. The resulting square $\mathcal{L}'$ has $|\mathcal{L} \cap \mathcal{L}'| = x$. □

Second, we generalize the idea in Lemma 2.3 to obtain an interval of large intersection values. Row permutations give a sparser collection of small intersections.

Lemma 3.3. *If $0 < k < n$ and $(n, k) \neq (2, 1), (6, 5)$, then $I(n, n+k)$ contains all integers of the form $(f+g)(n-k)+h$, where $h \in [0, fk]$, $0 \leq f \leq k$, $g \geq 0$, and $f+g \in [0, n-2] \cup \{n\}$.*

Proof: Consider a latin square $\mathcal{L}$ of side n and having k disjoint transversals, $T_1, \ldots, T_k$. We build a partial latin square $\mathcal{L}^{\#}$ of side $n+k$, filling all and only those cells in the first n rows and columns. In the first f rows, change any $fk-h$ of the entries on the transversals, placing ∞_i on cells in T_i. In rows $f+1$ through n, change all cells on the transversals in this way. Each symbol, both old and new, appears at least $2n-(n+k) = n-k$ times in $\mathcal{L}^{\#}$. By Lemma 3.1, $\mathcal{L}^{\#}$ admits a completion to a latin square $\mathcal{L}'$ of side $n+k$. We have $|\mathcal{L} \cap \mathcal{L}'| = n(n-k)+h$.

For smaller intersection values, apply row permutations to the original square $\mathcal{L}$. We fix the top f rows of $\mathcal{L}$, maintaining intersection $f(n-k)+h$ with $\mathcal{L}'$ in those rows. Apply any permutation with exactly g fixed points to the remaining $n-f$ rows. The fixed rows contribute $g(n-k)$ to the intersection with $\mathcal{L}'$ and the non-fixed rows contribute zero. So $(f+g)(n-k)+h \in I(n, n+k)$. □

In particular, chosing $f = k$ and $g = n-k$ shows that $I(n, n+k) \supseteq [n^2-nk, n^2-nk+k^2]$, provided $(n, k) \neq (2, 1)$ or $(6, 5)$. It should be remarked that although the earlier construction in Lemma 2.3 is simply a special case of this, we chose to include a separate constructive proof that was not dependent on a completion via Ryser's Theorem.

Finally, an interval of small intersections is possible with the use of transversals which restrict to a subsquare.

Lemma 3.4. *Suppose $2 \leq t \leq n-2$ and $t \leq k$. If there exists a latin square $\mathcal{L}$ of side n with both k disjoint transversals and a $t \times t$ subsquare, then $I(n, n+k) \supseteq I(t, t)$.*

Proof: Assume the asserted $t \times t$ subsquare in $\mathcal{L}$ is in the first t rows and columns. In rows $t+1$ through n, change all $(n-t)k$ transversal entries to new symbols in the usual way, and complete to a latin square $\mathcal{L}'$ of side $n+k$ by Ryser's Theorem. We now modify $\mathcal{L}$. Apply a permutation to rows $1, \ldots, t$ of $\mathcal{L}$, leaving no fixed row, and a second permutation to rows $t+1$ through n of $\mathcal{L}$, again leaving no fixed row.

Call this resulting square $\mathcal{L}^0$. We have $|\mathcal{L}^0 \cap \mathcal{L}'| = 0$. But each of $\mathcal{L}^0$ and $\mathcal{L}'$ contain a $t \times t$ subsquare on the same entries. These can be replaced by any pair of squares of order t, realizing any intersection number in $I(t,t)$. □

The hypothesis of Lemma 3.4 is guaranteed when there exists orthogonal latin squares of order n having orthogonal subsquares of order t. By a result of Heinrich, [8], these exist for $t \le n/3$, $t \neq 2, 6$, $n \neq 6$. Any $k \in [t, n]$ is permitted. In conjunction with Lemma 3.3, we can now settle the intersection problem for 'most' values of n and k.

Theorem 3.5. *Let $n \ge 135$. For $\sqrt{2n} - 1 \le k < n$, we have $I(n, n+k) = [0, n^2 - nk + k^2]$.*

Proof: We prove that under the given hypotheses, the values in Lemmas 3.3 and 3.4 cover the stated interval, and this is sufficient by Lemma 2.2. At the upper end, the intervals $(f+g)(n-k) + [0, fk]$ together cover the interval $[f(n-k), n^2 - nk + k^2]$ for integers $f \ge (2(n-k)-1)/k$. Observe that the lower bound on k is equivalent to $k^2 \ge 2(n-k) - 1$. Thus any legal intersection value at least

$$(n-k)\left\lceil \frac{2(n-k)-1}{k} \right\rceil \le \frac{(n-k)(2n-k-2)}{k}$$

can be achieved. On the other hand, the lower bound on n certainly assures by Lemma 3.4 that any intersection value less than

$$\left\lfloor \frac{n}{3} \right\rfloor^2 - 5 \ge \left(\frac{n-2}{3}\right)^2 - 5$$

is possible. It remains to check that

$$\frac{(n-k)(2n-k-2)}{k} \le \left(\frac{n-2}{3}\right)^2 - 5. \qquad (*)$$

For a fixed n, the left side of (*) is decreasing in k. So it suffices to consider the lower bound on k; that is, put $k = z$ and $n = (z+1)^2/2$. After some simplification, (*) becomes

$$z^5 - 14z^4 - 20z^3 - 12z^2 - 189z + 18 \ge 0.$$

The polynomial on the left has largest positive root $z \approx 15.4 < \sqrt{270} - 1$. So the inequality holds for $n \ge 135$. □

We remark that smaller values of n can be handled at the expense of more delicate calculations. For instance, if $n = 32$, any value of $k \ge 10$ is settled by a combination of the results in this section. Small intersections from 10×10 subsquares extend from 0 up to $10^2 - 6 = 94$, and starting f from 3 yields a run of intersection sizes from 72 to the counting bound.

Complete determination of $I(n, n+k)$ seems tricky, at least with elementary techniques. A general procedure for obtaining any given intersection $x < n^2 - nk$

is straightforward: replace k disjoint transversals of $\mathcal{L}$ by new symbols $\infty_1, \dots, \infty_k$, retain x old symbols on the remaining $n^2 - nk$ cells, and ensure that the other $n^2 - nk - x$ entries are assigned an old symbol not equal to the corresponding entry in $\mathcal{L}$. This partial latin square admits a completion by Lemma 3.1, provided each old symbol is used at least $n - k$ times. Keeping intersection at x is usually easy, but we presently see no systematic method.

Nonetheless, the following statement seems like a safe guess.

Conjecture 3.6. *$I(n, n+k) = [0, n^2 - nk + k^2]$ for all but finitely many pairs n, k with $1 \le k \le n$.*

The most difficult case appears to be small k and intersection number slightly less than $n^2 - nk$. We present an example illustrating the tightness here.

Example 3.7. Put $k = 1$ and suppose $\mathcal{L}$, $\mathcal{L}'$ are squares of orders $n, n+1$, respectively, with $|\mathcal{L} \cap \mathcal{L}'| = n^2 - n - 1$. Let the symbol set of $\mathcal{L}$ be $\{1, \dots, n\}$ and suppose ∞ is the new symbol for $\mathcal{L}'$. Either ∞ appears $n - 1$ or n times in the upper-left $n \times n$ subsquare of $\mathcal{L}'$. In the latter case, for instance, those cells in $\mathcal{L}$ which contain ∞ in $\mathcal{L}'$ cannot form a transversal. (Otherwise, $|\mathcal{L} \cap \mathcal{L}'| = n^2 - n$.) So some symbol in $\mathcal{L}$, say 1, is replaced by ∞ twice, some other symbol, say 2, is not replaced by ∞, and all other symbols are replaced by ∞ exactly once. This requires $\mathcal{L}$ and $\mathcal{L}'$ complete the structures shown below.

$$\overset{\mathcal{L}}{\boxed{\begin{array}{cccccc} 1 & 3 & & & & \\ 2 & 1 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & \ddots & \\ & & & & & n \end{array}}} \quad \rightarrow \quad \overset{\mathcal{L}'}{\boxed{\begin{array}{ccccccc} \infty & 3 & & & & & 1 \\ 1 & \infty & & & & & 2 \\ & & \infty & & & & 3 \\ & & & \infty & & & 4 \\ & & & & \ddots & & \vdots \\ & & & & & \infty & n \\ 2 & 1 & 3 & 4 & \cdots & n & \infty \end{array}}}$$

Completing the left partial latin square is not even possible for $n = 3$ or 4. It is easy to see that a completion does exist for $n = 5$, giving $19 \in I(5, 6)$.

Quite likely, a more sophisticated use of permutations and list-colourings may help to drop the unnatural hypotheses on n and k. However, in contrast with Theorem 3.5, Example 2.1 shows that $I(n, n+k)$ need not be an interval, even for $k > 0$.

4 Quasi-embeddings of idempotent latin squares

Let us define $D_i(n, m)$ in a similar way to $D(n, m)$, except where the maximum intersection is taken over pairs of *idempotent* latin squares of orders n and m. We assume $n, m \neq 2$ and $n < m < 2n$. The following result is analogous to Lemma 2.2.

Lemma 4.1. *Suppose $k < n$ are positive integers. Then*

$$D_i(n, n+k) \le n^2 - k(n - k + 1).$$

Proof: Suppose $\mathcal{L}$ is quasi-embedded in $\mathcal{L}'$, where the squares are idempotent of sides n and $n+k$ with entries E, E', respectively. As $\mathcal{L}'$ is idempotent, each entry of $E' \setminus E$ appears on the diagonal of $\mathcal{L}'$ in the last k rows. Therefore, at most $k^2 - k$ entries from $E' \setminus E$ can appear in the first n rows and last k columns of $\mathcal{L}'$. This forces at least $nk - (k^2 - k)$ disagreements between $\mathcal{L}$ and $\mathcal{L}'$. □

A construction similar to that in Lemma 2.3 establishes equality in this bound for small values of k.

Theorem 4.2. *Let k be a positive integer, $k \neq 3$, and suppose $n \geq 3k$. Then $D_i(n, n+k) = n^2 - k(n-k+1)$.*

Proof: We first claim that, under the given hypotheses, there exists an idempotent latin square $\mathcal{N}$ of side $n-k+1$ having $2k-1$ disjoint off-diagonal transversals. Any square of order $n-k+1$ having an orthogonal mate furnishes such an $\mathcal{N}$. Note $2 = n-k+1 \geq 2k+1$ is a contradiction. And $6 = n-k+1 \geq 2k+1$ forces $k \leq 2$, in which case we may take $\mathcal{N}$ of side 6 with 4 transversals, as in the previous section. Prolong $\mathcal{N}$ along $k-1$ of its transversals, using an idempotent square $\mathcal{Z}$ on symbols $n-k+2, \ldots, n$. The condition $k \neq 3$ permits this. Let $\mathcal{L}$ be this resulting square of order n. We now construct $\mathcal{L}'$ of side $n+k$ agreeing with $\mathcal{L}$ in the required number of cells.

1. (rows 1 to $n-k+1$): Replace the remaining k off-diagonal transversals T_i, $i = 1, \ldots, k$ from $\mathcal{N}$ with new symbol ∞_i. To the right, place the transversal entries from each T_i in column $n+i$.
2. (rows $n-k+2$ to n): Leave unchanged these rows of $\mathcal{L}$. To the right, append $k-1$ rows of any latin square of order k on entries $\{\infty_1, \ldots, \infty_k\}$ in columns $n+1$ to $n+k$, so that ∞_i is missing from column $n+i$.
3. (rows $n+1$ to $n+k$): Complete the $n \times (n+k)$ latin rectangle constructed above, ordering rows so that ∞_i now appears in column $n+i$ of row $n+i$.

Note that the completion as above is possible, since each of the last k rows must contain exactly one of the new points ∞_i within the last k columns. Furthermore, there are exactly $k(n-k+1)$ disagreements between $\mathcal{L}$ and $\mathcal{L}'$ arising from the replacements in step 1. So we have $|\mathcal{L} \cap \mathcal{L}'| = n^2 - nk + k^2 - k$. □

Example 4.3. We illustrate the construction in Theorem 4.2 for $n = 6, k = 2$. Begin with a back-circulant latin square $\mathcal{N}$ of side 5, and prolong along one off-diagonal transversal to get $\mathcal{L}$.

$\mathcal{L}$

1	6	2	5	3	4
4	2	6	3	1	5
2	5	3	6	4	1
5	3	1	4	6	2
6	1	4	2	5	3
3	4	5	1	2	6

$\rightarrow$

1	6	∞_1	∞_2	3	4	2	5
4	2	6	∞_1	∞_2	5	3	1
∞_2	5	3	6	∞_1	1	4	2
∞_1	∞_2	1	4	6	2	5	3
6	∞_1	∞_2	2	5	3	1	4
3	4	5	1	2	6	∞_2	∞_1

In a completion of this rectangle, entries ∞_1 and ∞_2 must each appear once in column 6. So the main diagonal of $\mathcal{L}'$ is forced to complete as $1, \ldots, 6, \infty_1, \infty_2$.

The remaining values of k remain open. We remark that the construction in Lemma 2.3 can be applied in the context of idempotent latin squares, but requires a careful completion of the latin rectangle in step 3. Unfortunately, there is no nice analog of Ryser's Theorem for idempotent squares. See section 5 for an infinite family of quasi-embeddings of idempotent latin squares (from n to $n+k$) with $k \approx n/2$.

We have not considered the intersection problem for idempotent latin squares of different sizes. The intersection problem for idempotent latin squares of the same size has been solved by Fu and Fu. See [7] for details and references. The main result is given below.

Theorem 4.4. ([7]) *Let $n \geq 5$. The set of all possible intersections numbers for two idempotent latin squares of order n is*

$$[n+1, n^2] \setminus \{n^2-5, n^2-3, n^2-2, n^2-1\}.$$

A combination of this result with the methods in [4] should prove fruitful in future work.

5 Quasi-embeddings of idempotent symmetric latin squares

A latin square $\mathcal{L} = (L_{ij})$ which is both idempotent and symmetric must have odd side $2n-1$, and is equivalent to a *one-factorization* of the complete graph K_{2n}. Specifically, suppose $E = \{1, \ldots, 2n-1\}$ is the entry set of $\mathcal{L}$. For each $e \in E$, define

$$F_e = \{\{\infty, e\}\} \cup \{\{i, j\} : i \neq j \text{ and } L_{ij} = e\}.$$

Then since $\mathcal{L}$ is an idempotent symmetric latin square, $\mathcal{F} = \{F_1, \ldots, F_{2n-1}\}$ is a set of edge-disjoint one-factors of K_{2n} on vertices $\{1, \ldots, 2n-1\} \cup \{\infty\}$; that is, $\mathcal{F}$ is a one-factorization of K_{2n}.

As before, take positive integers $n \leq m$. Let $\mathcal{L}$ and $\mathcal{L}'$ be idempotent symmetric latin squares of sides $2n-1$ and $2m-1$, and with respective entry sets $\{1, \ldots, 2n-1\}$, $\{1, \ldots, 2m-1\}$. If $\mathcal{F}, \mathcal{F}'$ are the one-factorizations corresponding to $\mathcal{L}, \mathcal{L}'$, respectively, then

$$\textstyle\sum_{i=1}^{2n-1} |F_i \cap F_i'| = \frac{1}{2}(|\mathcal{L} \cap \mathcal{L}'| + 2n - 1).$$

Each side above equals $\binom{2n}{2}$ if and only if $\mathcal{F}$ admits a standard embedding into $\mathcal{F}'$ as one-factorizations. For this reason, we feel that $|\mathcal{L} \cap \mathcal{L}'|$ is also perhaps the most natural measure of intersection of corresponding one-factorizations of K_{2n}. The minimum possible intersection is nonempty and occurs when the two one-factorizations of K_{2n} are *orthogonal* and form a *Room square*, [11]. Despite this natural definition from the point of view of one-factorizations of K_{2n}, the squares $\mathcal{L}$ and $\mathcal{L}'$ depend not only on $\mathcal{F}$ and $\mathcal{F}'$, but also on the choice of a specified vertex as ∞.

Other authors have considered intersections of one-factorizations of K_{2n} in this way. See [9], for example.

For $n \le m$, define $D'(2n-1, 2m-1) = \max \frac{1}{2}(|\mathcal{L} \cap \mathcal{L}'| + 2n - 1)$, where the maximum is taken over all pairs $\mathcal{L}$ and $\mathcal{L}'$ of idempotent symmetric latin squares of sides $2n-1$ and $2m-1$, and entry sets $E \subseteq E'$. As before, if $\mathcal{L}$ and $\mathcal{L}'$ achieve this maximum, we say that $\mathcal{L}$ is *quasi-embedded* in $\mathcal{L}'$. Note that we have chosen D' to measure the number of common edges in ordered one-factorizations of K_{2n}, and this departs from the definitions of D and D_i given earlier.

A similar argument as in Lemma 2.2 establishes the following upper bound on D'.

Lemma 5.1. *For $0 < k < n$, $D'(2n-1, 2(n+k)-1) \le \binom{2n}{2} - 2nk + 2k^2$.*

We now give a construction of a quasi-embedding of idempotent symmetric latin squares of side $2n-1$ into ones of side $3n-1$. We will use the language of one-factorizations of K_{2n} and K_{3n}. Note that $3n$ is exactly in the middle of the relevant range: one-factorizations of K_{2n} always embed [3] in one-factorizations of K_m for $m \ge 4n$. It should be mentioned that there is presently no known construction of such 'mid-range' quasi-embeddings of Steiner triple systems. See [6].

Theorem 5.2. *If n is even, then $D'(2n-1, 3n-1) = \frac{3}{2}n^2 - n$.*

Proof: First, observe that by Lemma 5.1, the right side is an upper bound on $D'(2n-1, 3n-1)$. Suppose the vertex set of K_{2n} is $A \cup B$, where $A = A_0 \cup A_1$, $B = B_0 \cup B_1$, and $|A_i| = |B_i| = n/2$ for $i = 0, 1$. Let $F_1^A, \ldots, F_{n-1}^A$ and $F_1^B, \ldots, F_{n-1}^B$ be one-factorizations of the complete graphs on vertices A and B, respectively. Put $F_i = F_i^A \cup F_i^B$ for $i = 1, \ldots, n-1$. For $0 \le k, l \le 1$, let $\{F_1^{kl}, \ldots, F_{n/2}^{kl}\}$ be one-factorizations of the complete bipartite graph with vertex bipartition $A_k \cup B_l$. Put $F_{n-1+j} = F_j^{00} \cup F_j^{11}$ and $F_{n-1+n/2+j} = F_j^{01} \cup F_j^{10}$ for $j = 1, \ldots, n/2$. It is easy to see that $\{F_1, \ldots, F_{2n-1}\}$ is a one-factorization of K_{2n}. This is the standard 'doubling construction' for one-factorizations. We illustrate this with the table below, where each cell represents a sub-one-factorization of the indicated complete graph or complete bipartite graph. The edges within a one-factor are to be aligned in columns. For instance, when the entries labeled 'A' and 'B' are replaced by one-factorizations on points A, B, respectively, the first $n-1$ columns of the table exhaust the one-factors with no edge crossing the partition $A \cup B$.

	$n-1$	$n/2$	$n/2$
$n/2$	A	A_0B_0	A_0B_1
$n/2$	B	A_1B_1	A_1B_0

Now consider K_{3n} with vertices $A \cup B \cup C$, where $C = C_0 \cup C_1$ and $|C_i| = n/2$. Construct a one-factorization of K_{3n} as above according to the following table, where the same sub-one-factorizations as above are repeated.

	$n-1$	$n/2$	$n/2$	$n/2$	$n/2$
$n/2$	A	A_0B_0	A_0B_1	A_0C_0	A_0C_1
$n/2$	B	A_1C_0	A_1C_1	A_1B_1	A_1B_0
$n/2$	C	B_1C_1	B_0C_0	B_0C_1	B_1C_0

These two (ordered) one-factorizations intersect in precisely $n(n-1)+n(n/2) = \frac{3}{2}n^2 - n$ edges. $\square$

Quasi-embeddings of one-factorizations of K_{2n} which add two points are not hard to obtain.

Theorem 5.3. $D'(2n-1, 2n+1) = 2n^2 - 3n + 2.$

Proof: Consider the standard one-factorization $\mathcal{F} = \{F_i : i = 0, 1, \ldots, 2n-2\}$ of K_{2n} obtained from the so-called 'patterned starter' on $\mathbb{Z}_{2n-1} \cup \{\infty\}$. Then $\mathcal{F}$ enjoys a one-rotational automorphism and the cycle $0, 1, 2, \ldots, 2n-2, 0$ has each edge $\{i, i+1\}$ (mod $2n-1$) in a distinct one-factor, say F_i. See [2] for further details on the patterned starter. We now construct a one-factorization of K_{2n+2}. Add two new points X, Y and replace edge $\{i, i+1\}$ in F_i by $\{i, X\}$, $\{i+1, Y\}$ for all $i \neq 0$. To F_0, add $\{X, Y\}$. Finally, include two new one-factors, where $\{\infty, X\}$ goes with $\{1, Y\}, \{2, 3\}$, $\{4, 5\}, \ldots$ and similarly $\{\infty, Y\}$ goes with $\{0, X\}, \{1, 2\}$, $\{3, 4\}, \ldots$. This destroys exactly $2n-2$ edges, and it is easy to see that this is the required number for the bound in Lemma 5.1. $\square$

It should be remarked that repeated application of the construction in Theorem 5.3 fails to result in quasi-embeddings. However, quasi-embeddings which add 4 or more points can in many cases be taken from another context. Recall that Steiner triple systems of order u are equivalent to idempotent latin squares of order u which are *totally symmetric*; that is, they have self-conjugate quasigroups with respect to rows, columns and entries. See Colbourn and Rosa's book [2] for more details. In particular, we mention the well-known fact that Steiner triple systems of order u exist if and only if $u \equiv 1, 3 \pmod 6$, $u \geq 1$. Various constructions for Steiner triple systems of order $2n-1$ in [6] yield quasi-embeddings of one-factorizations of K_{2n}. For instance, the cases $2 \leq k \leq 5$ are settled for many values of n.

Theorem 5.4. ([6]) *Suppose either*

- $k = 2$ *and* $n \equiv 2 \pmod 3$,
- $k = 3$ *and* $n \equiv 1, 2 \pmod 3$, $n \geq 50$,
- $k = 4$ *and* $n \equiv 1 \pmod 3$, $n \geq 52$, *or*
- $k = 5$ *and* $n \equiv 2 \pmod 3$, $n \geq 41$.

Then $D'(2n-1, 2(n+k)-1) = \binom{2n}{2} - 2nk + 2k^2$.

6 Conclusion

We have introduced the notion of quasi-embeddings and an intersection problem for latin squares of different orders. Together with [4] and [6], which consider Steiner triple systems, this gives a spectrum of problems with very different properties. The upper bounds for quasi-embeddings are summarized in the table below.

object	orders	upper bound
latin squares	$u \le v$	$u^2 - (v-u)(2u-v)$
idempotent latin squares	$u \le v$	$u^2 - (v-u)(2u+1-v)$
one-factorizations	$u \le v$	$\binom{u}{2} - \frac{1}{2}(v-u)(2u-v)$
Steiner triple systems	$u \le v$	$\frac{1}{3}\binom{u}{2} - \frac{1}{6}(v-u)(2u+1-v)$

As we have seen in section 2, quasi-embeddings for latin squares are relatively easy to characterize. On the other hand, the same problem for idempotent latin squares remains open for roughly two-thirds of all parameters, while quasi-embeddings of one-factorizations of K_{2n} and Steiner triple systems have not been settled for any positive density of parameters. These remain interesting open problems.

Another question we hope receives some attention is the complete determination of $I(n, n+k)$.

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