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The genus distributions of 4-regular digraphs

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Abstract

An embedding of an Eulerian digraph in orientable surfaces was introduced by Bonnington et al. They gave some problems which need to be further studied. One of them is whether the embedding distribution of an embeddable digraph is always unimodal. In this paper, we first introduce the method of how to determine the faces and antifaces from a given rotation scheme of a digraph. The genus distributions of two new kinds of 4-regular digraphs in orientable surfaces are obtained. The genus distributions of one kind of digraph are strong unimodal, which gives a partial answer to the above problem.

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1 Introduction

Let D be a digraph. The symbol $\langle u, v\rangle$ denotes the arc of D whose direction is from a vertex u to a vertex v in D. The vertex u is called the *tail* of $\langle u, v\rangle$ and the vertex v is called the *head* of $\langle u, v\rangle$. The number of arcs whose tails are u, denoted by $\text{outdeg}(u)$, is the *outdegree* of u; The number of arcs whose heads are u, denoted by $\text{indeg}(u)$, is the *indegree* of u. A digraph D is called *Eulerian* if it is connected and the indegree equals the outdegree for each vertex. The *orientable surface* of genus h, denoted by S_h, is the sphere with h handles added. A graph is said to be *embedded* in a surface S if it is drawn in S so that edges intersect only at their common end vertices. A digraph D is said to be *embedded* in an orientable surface S if it is embedded as a graph and the directions of arcs are consistent in each region boundary. An embedding of a digraph D in a surface S is *cellular* if each component of $S\setminus D$ is homeomorphic to an open disk. In this paper, each embedding is a cellular embedding. A *rotation* at a vertex v is a cyclic permutation of arcs incident with v. in fact, a rotation at a vertex v corresponds to a cyclic permutation of semiarcs incident with v. A list of rotations, one for each vertex, is called a *rotation scheme* for the digraph. Rotation schemes are employed to represent embeddings. The consistency of the arc directions on each region boundary forces in-arcs and out-arcs alternating at each vertex rotation scheme.

Using the notation of [1], in the context of embeddable digraphs, each arc is on the boundary of exactly two regions: one is called a *face* which uses the arcs in the forward direction and the other is called an *antiface* where each arc is traversed against its given orientation.

Let D be a digraph and $g_i(D)$ be the number of different embeddings of D in the orientable surface with genus i. The sequence $g_0(D), g_1(D), g_2(D), \ldots$ is called the *genus distribution* of D. The *genus distribution polynomial* of D is as follows:

$$f_D(x) = \sum_{i=0}^{\infty} g_i(D)x^i.$$

A nonnegative sequence $\{a_m\}$ is said to be *unimodal* if there exists at least one integer N such that

$$a_{m-1} \leq a_m \text{ for all } m \leq N \text{ and } a_m \geq a_{m+1} \text{ for all } m \geq N.$$

A sequence $\{a_m\}$ is called *strong unimodal* [6], if its convolution with any unimodal sequence $\{b_m\}$ is unimodal. It has been proved that $\{a_m\}$ is strong unimodal if and only if $a_m^2 \geq a_{m+1}a_{m-1}$ for all m.

For genus distributions of graphs, there are some results known, such as for ladders, Mobius ladders, closed-end ladders, cobble-stone paths, necklaces and bouquets of circles et al. The reader may refer to the articles [4–9, 13, 14, 16] for details.

Although there are some results about genus distributions of graphs, little is known about those of digraphs up to now.

Bonnington et al. [1, 2] studied some basic results on digraph embedding and obtained the obstructions for directed embedding of digraphs in the plane. At the end of the article [1], they gave the following problem:

Is the embedding distribution of an embeddable digraph always unimodal, as is conjectured to be the case in the study of undirected graphs by Gross et al. [6]?

This problem motivates us to study the genus distributions of digraphs on orientable surfaces.

In this paper, we first give the method to determine the faces and antifaces from the given rotation scheme of a digraph. Then the genus distributions of two new kinds of 4-regular digraphs in orientable surfaces are obtained based on the technique of joint trees introduced by Liu. The genus distributions of one kind of digraph are strong unimodal which give the partial answer to problems in [1].

2 Determining faces and antifaces

The following proposition is justified by Euler's formula for graph embeddings.

Proposition 2.1 *If $D = (V, A)$ is an embeddable digraph, then for any rotation scheme of D, $|V| - |A| + |R| = 2 - 2g$, where $|R|$ is the number of regions of the embedding and g is the genus of the embedding surface.*

Let $D = (V, A)$ be a digraph. For $x = \langle u, v\rangle \in A$, let x^+ be the semiarc of x with tail vertex u and x^- be the semiarc of x with head vertex v. $A^+ = \{x^+ : x \in A\}$; $A^- = \{x^- : x \in A\}$. It is easy to see that the cardinalities of A, A^+ and A^- are equal.

For example 1, if the rotation ρ at a vertex v is $(x_1^+ x_2^- x_3^+ \ldots)$, it means that $\rho_v(x_1^+) = \rho_v(x_1) = x_2^-$, and $\rho_v(x_2^-) = \rho_v(x_2) = x_3^+$. We say that x_2 is the next arc of x_1 under the rotation of the tail vertex of x_1 and x_3 is the next arc of x_2 under the rotation of the head vertex of x_2.

For brevity, for a given rotation scheme ρ of D on an orientable surface and for each $x \in A$, $\rho(x^+)$ means the next arc of x under the rotation of the tail vertex of x, while $\rho(x^-)$ means the next arc of x under the rotation of the head vertex of x. Let λ be the map such that $\lambda(x^+) = x^-$ and $\lambda(x^-) = x^+$.

As we can see in Figure 1, $\rho_u = (d^+, e^-, c^+, b^-), \rho_v = (f^+, d^-), \rho_w = (e^+, a^-), \rho_x = (b^+, c^-, a^+, f^-)$. Then $\rho(d^+) = \rho_u(d^+) = e^-$, and $\rho(d^-) = \rho_v(d^-) = f^+$.

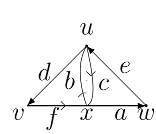


Figure 1

Using this notation, we have:

Proposition 2.2 *Let $D = (V, A)$ be a digraph and ρ be a given rotation scheme of D on orientable surfaces. A cycle means a connected Eulerian subgraph. Let τ be a function from $A^+ \cup A^-$ to A such that $\tau(x^+) = \tau(x^-) = x$ for each $x \in A$.*

(1) *If $(\rho\lambda)^{k+1}(x^+) = x^+$, and $(\rho\lambda)^i(x^+) \neq x^+$ for any $i < k+1$, then the cycle $(\tau(x^+), \tau(\rho\lambda(x^+)), \tau((\rho\lambda)^2(x^+)), (\tau(\rho\lambda)^3(x^+)), \ldots, \tau((\rho\lambda)^k(x^+)))$ is a face, where $(\rho\lambda)^i = \rho\lambda\rho\lambda\ldots\rho\lambda$.*

(2) *If $(\rho\lambda)^{l+1}(x^-) = x^-$, and $(\rho\lambda)^i(x^-) \neq x^-$ for any $i < l+1$, then cycle $(\tau(x^-), \tau(\rho\lambda(x^-)), \tau((\rho\lambda)^2(x^-)), \tau((\rho\lambda)^3(x^-)), \ldots, \tau((\rho\lambda)^l(x^-)))$ is an antiface.*

Proof We only prove (1) here. The proof of (2) is similar.

First the cycle is a directed cycle, and each arc is in the forward direction of the cycle's boundary, i.e., the direction of the cycle is consistent with that of each arc in this cycle boundary.

On the other hand, two consecutive occurrence of arcs in cycles is indeed consecutive in the embedding rotation scheme. Thus the cycle is a face. □

Since each arc is in only one face and one antiface in an embedding of a digraph on an orientable surface, we can find a face starting from any element $x^+ \in A^+$ by using Proposition 2.2(1). We repeat the procedure starting from any element which does not occur in faces found before. It terminates when all faces are found. Similarly we can find all antifaces.

In fact, if the digraph has a large number of arcs, we can use the ordered pair of vertices $\langle u, v\rangle$ to represent the arc $x = \langle u, v\rangle$. Let $\rho(x^+)$ denote $\rho_u(x^+)$ and $\rho\lambda(x^+) = \rho(x^-)$ denote $\rho_v(x^-)$.

For example 2, let $D = K_5 = (V, A)$. Supposing $V = \{1, 2, 3, 4, 5\}$ and the rotation scheme of D is determined by ρ_i of rotation at vertex i as follows:

$\rho_1 : (\langle 1, 2\rangle^+\langle 4, 1\rangle^-\langle 1, 3\rangle^+\langle 5, 1\rangle^-)$

$\rho_2 : (\langle 1, 2\rangle^-\langle 2, 3\rangle^+\langle 5, 2\rangle^-\langle 2, 4\rangle^+)$

$\rho_3 : (\langle 3, 4\rangle^+\langle 1, 3\rangle^-\langle 3, 5\rangle^+\langle 2, 3\rangle^-)$

$\rho_4 : (\langle 3, 4\rangle^-\langle 4, 1\rangle^+\langle 2, 4\rangle^-\langle 4, 5\rangle^+)$

$\rho_5 : (\langle 3, 5\rangle^-\langle 5, 2\rangle^+\langle 4, 5\rangle^-\langle 5, 1\rangle^+)$

Then, from Proposition 2.2, we can find a face which contains the arc $\langle 1, 2\rangle$.

Let $x^+ = \langle 1, 2\rangle^+$; then $\rho\lambda(x^+) = \rho(\langle 1, 2\rangle^-) = \rho_2(\langle 1, 2\rangle^-) = \langle 2, 3\rangle^+$ and $(\rho\lambda)^2(x^+) = \rho\lambda(\langle 2, 3\rangle^+) = \rho(\langle 2, 3\rangle^-) = \rho_3(\langle 2, 3\rangle^-) = \langle 3, 4\rangle^+, \cdots$. Thus we have (from left to right, up to down):

$x^+ = \langle 1, 2\rangle^+$, $\rho\lambda(x^+) = \langle 2, 3\rangle^+$,

$(\rho\lambda)(\langle 2,3\rangle^+) = \langle 3,4\rangle^+$, $(\rho\lambda)(\langle 3,4\rangle^+) = \langle 4,1\rangle^+$,
$(\rho\lambda)(\langle 4,1\rangle^+) = \langle 1,3\rangle^+$, $(\rho\lambda)(\langle 1,3\rangle^+) = \langle 3,5\rangle^+$,
$(\rho\lambda)(\langle 3,5\rangle^+) = \langle 5,2\rangle^+$, $(\rho\lambda)(\langle 5,2\rangle^+) = \langle 2,4\rangle^+$,
$(\rho\lambda)(\langle 2,4\rangle^+) = \langle 4,5\rangle^+$, $(\rho\lambda)(\langle 4,5\rangle^+) = \langle 5,1\rangle^+$,
$(\rho\lambda)(\langle 5,1\rangle^+) = \langle 1,2\rangle^+ = x^+$.

From Theorem 2.2(1), arcs $\langle 1,2\rangle$, $\langle 2,3\rangle$, $\langle 3,4\rangle$, $\langle 4,1\rangle$, $\langle 1,3\rangle$, $\langle 3,5\rangle$, $\langle 5,2\rangle$, $\langle 2,4\rangle$, $\langle 4,5\rangle$, $\langle 5,1\rangle$ compose a face (1234135245) which contains ten arcs. Also since $|A(D)| = 10$, this rotation scheme only has one face.

The antiface containing the arc $\langle 1,2\rangle$ is found as follows (from left to right, up to down):

$x^- = \langle 1,2\rangle^-$, $\rho\lambda(\langle 1,2\rangle^-) = \rho(\langle 1,2\rangle^+) = \rho_1(\langle 1,2\rangle^+) = \langle 4,1\rangle^-$, $\rho\lambda(\langle 4,1\rangle^-) = \langle 2,4\rangle^-$, $\rho\lambda(\langle 2,4\rangle^-) = \langle 1,2\rangle^- = x^-$.

From Theorem 2.2(2), arcs $\langle 1,2\rangle, \langle 4,1\rangle, \langle 2,4\rangle$ compose an antiface (214) which contains three arcs. Similarly we can find the other antiface (3153254) which contains seven arcs. Thus this rotation scheme has two antifaces.

3 Genus distributions of some 4-regular digraphs

In this section, we use the method of a joint tree, introduced by Liu [10], to compute the genus distributions of digraphs. The genus distributions of two new kinds of 4-regular digraphs are obtained.

For a surface S, let $o(S)$ denote the genus of S. An orientable surface can be written as a cyclic sequence (i.e., a string of letters) that each letter appears exactly twice and the two occurrences of each letter with distinct powers "+" (which is always omitted) and "−". Two orientable surfaces are treated as the same if one can be obtained from the other by starting from any letter in cyclic order, inversing the cyclic order, permuting some letters or replacing a letter by its inverse. For example, if $S = a^-bc^-b^-ac$, then $S = bc^-b^-aca^- = c^-b^-aca^-b = aca^-bc^-b^-$, $S = cab^-c^-ba^-$, $S = d^-bc^-b^-dc$ and $S = abc^-b^-a^-c$.

The genus under the following equivalence relation $\sim$, determined by the following three operations, are equal; for details see [11, 12].

OP1 $AB \sim (Ae)(e^-B)$, where $e \notin AB$;

OP2 $Ae_1e_2Be_2^-e_1^- \sim AeBe^- = Ae^-Be$;

OP3 $Aee^-B \sim AB$, where $AB \neq \emptyset$;

Here A, B are linear sequences, and $e, e^-, e_1, e_1^-, e_2, e_2^- \notin A \cup B$.

It can be seen that each embedding surface is equivalent to only one of the following canonical forms of surfaces:

$$O_i = \begin{cases} a_0 a_0^-, & \text{if } i = 0 \\ \prod_{k=1}^{i} a_k b_k a_k^- b_k^-, & \text{if } i \geq 1 \end{cases}$$

O_i means the orientable surface of genus i.

This section now contains some review.

Let D be a digraph and T be a spanning tree of D. For each non-tree edge e, e is split into two semi-edges e^+ and e^-. (This notation is different from x^+ and x^- in Section 2.) It is obvious that the graph obtained by splitting all co-tree edges is a tree, which is called a *joint tree* $\widetilde{T}$. (We shall use e for e^+ for the sake of convenience without confusion in this paper.) It is known that the distribution of genus is independent of the choice of a tree, that is:

Proposition 3.1 [10] *Let T and T' be the two distinct spanning trees of a graph G, Σ be the rotation set of G. For $\sigma \in \Sigma$, let G_σ be an embedding determined by σ, and let $\widetilde{T}_\sigma$ be the joint tree corresponding to G_σ. Then there exists a bijection between $(\Sigma, \widetilde{T})$ and $(\Sigma, \widetilde{T'})$, where $(\Sigma, \widetilde{T}) = \{\widetilde{T}_\sigma \mid \sigma \in G\}$, $(\Sigma, \widetilde{T'}) = \{\widetilde{T'}_\sigma \mid \sigma \in G\}$.*

For a joint tree $\widetilde{T}_\sigma$ of D, all semi-edges clockwise (or anticlockwise) are regarded as the *embedding surfaces* of $\widetilde{T}_\sigma$.

Proposition 3.2 [12, 15] *Let A, B, C, D and E be linear sequences, where x, y, z are mutually distinct and $x, y, z, x^-, y^-, z^- \notin ABCDE$. Then*

(1) $AxByCx^-Dy^-E \sim ADCBExyx^-y^-$;
(2) $xABx^-CD \sim xBAx^-CD \sim xABx^-DC$;
(3) $AxBx^-yCy^-zDz^- \sim xBx^-AyCy^-zDz^- \sim xBx^-yCy^-AzDz^-$
$\sim BxAx^-yCy^-zDz^- \sim CxAx^-yBy^-zDz^- \sim DxAx^-yBy^-zCz^-$.

Proposition 3.3 [10, 12] *Let S and S' be surfaces. If $S \sim S'xyx^-y^-$, where $x \neq y$ and $x, y, x^-, y^- \notin S'$, then $o(S) = o(S') + 1$.*

Let C be a di-circuit with $2n$ vertices $\{u_i, v_i \mid i = 1, 2, \ldots, n\}$. The digraph G is obtained from C by adding two arcs (called *digons*) between u_i and v_i. If the circuit C is $u_1u_2 \ldots u_n v_n v_{n-1} \ldots v_1$, then the graph G obtained in this way is denoted by $D1_n$. If the circuit C is $u_1u_2 \ldots u_n v_1 v_2 \ldots v_n$, then the graph G is denoted by $D2_n$. The graph in Figure 2 is a $D1_4$ and the graph in Figure 3 is a $D2_4$, where the direction of C is clockwise.

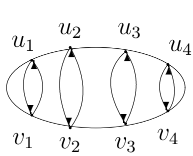


Figure 2

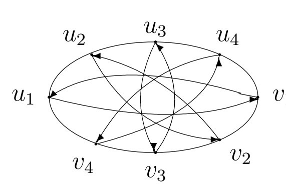


Figure 3

We first compute the genus distribution of $D1_n$.

Proposition 3.4 *The genus distribution of a digraph* $D1_n$ *is* $f_{D1_n}(x) = 2^n(1+x)^n$. *Furthermore, this genus distribution is strong unimodal.*

Proof For convenience, the arc from u_i to v_i is denoted by a_i while the arc from v_i to u_i is denoted by b_i for each $i \in \{1, 2, \ldots, n\}$. The path obtained from u_1 to v_1 which goes along C clockwise is chosen as a spanning tree.

Since the degree of each vertex is 4, the two semi-edges that are adjacent to the same vertex have only two possibilities in order to preserve exchange of the in-arc and the out-arc: they must lie on one side of the tree. Thus there are only four cases from each embedding AB of $D1_{n-1}$ to $D1_n$. They are

$$ABb_na_na_n^-b_n^-, \quad Ab_n^-a_n^-a_nb_nB, \quad Ab_n^-a_n^-Bb_na_n, \quad Aa_nb_nBa_n^-b_n^-.$$

From OP3, $ABb_na_na_n^-b_n^- \sim AB, Ab_n^-a_n^-a_nb_nB \sim AB$.

From Proposition 3.2,

$$Ab_n^-a_n^-Bb_na_n \sim ABb_n^-a_n^-b_na_n \sim ABb_na_nb_n^-a_n^- \sim ABa_nb_na_n^-b_n^-.$$

Similarly, $Aa_nb_nBa_n^-b_n^- \sim ABa_nb_na_n^-b_n^-$. We can see that the genera remain the same as $o(AB)$ for two cases and increase by 1 for the other two cases. Thus

$f_{D1_n}(x) = 2f_{D1_{n-1}}(x) + 2f_{D1_{n-1}}(x)x = 2(1+x)f_{D1_{n-1}}(x)$.

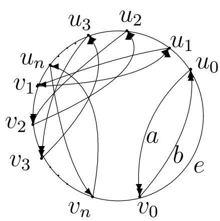
By using this recursion repeatedly,

$f_{D1_n}(x) = 2(1+x)f_{D1_{n-1}}(x) = \cdots = 2^{n-1}(1+x)^{n-1}f_{D1_1}(x)$.

It is easy to check $f_{D1_1}(x) = 2+2x$; thus $f_{D1_n}(x) = 2^n(1+x)^n$.

By the definition of the genus distribution, $g_i(D1_n) = 2^n\binom{n}{i}$. Since $\binom{n}{i}^2 \geq \binom{n}{i-1}\binom{n}{i+1}$, we have $g_i^2(D1_n) \geq g_{i-1}(D1_n)g_{i+1}(D1_n)$ for $2 \leq i \leq n-1$. The genus distribution of $D1_n$ is strong unimodal. The minimal genus is 0 and the maximal genus is n. □

Let D_n be the graph obtained from $D2_n$ by adding two vertices u_0 and v_0 along the edge u_1v_n and two parallel arcs $a = \langle u_0, v_0\rangle$ and $\langle v_0, u_0\rangle$. (See Figure 4, where the outer cycle direction is clockwise.) Now we calculate the genus distribution of D_n.



D_n

Figure 4

For a finite set M of surfaces, let $g_i(M)$ be the number of surfaces with genus i

in M. The genus distribution polynomial of M is as follows:

$$f_M(x) = \sum_{i=0}^{\infty} g_i(M)x^i.$$

Let e, a, b and a_l, b_l $(1 \le l \le n)$ be non-tree edges for the digraph D_n.

Let $Y_1 = y_{k_1}y_{k_2}y_{k_3}\dots y_{k_r}$, $Y_2 = y_{k_{r+1}}y_{k_{r+2}}y_{k_{r+3}}\dots y_{k_n}$, $Y_3 = y^-_{m_1}y^-_{m_2}y^-_{m_3}\dots y^-_{m_s}$, and $Y_4 = y^-_{m_{s+1}}y^-_{m_{s+2}}y^-_{m_{s+3}}\dots y^-_{m_n}$, where $n \ge k_1 > k_2 > k_3 > \dots > k_r \ge 1$, $1 \le k_{r+1} < k_{r+2} < k_{r+3} < \dots < k_n \le n$, $1 \le m_1 < m_2 < m_3 < \dots < m_s \le n$, $n \ge m_{s+1} > m_{s+2} > m_{s+3} > \dots > m_n \ge 1$, $0 \le r, s \le n$, $k_p \ne k_q, m_p \ne m_q$ for $p \ne q$ and y_{ij} represents two semi-edges as follows:

$$y_{i_j} = \begin{cases} a_{i_j}b_{i_j}, & i_j \in \{k_1, \dots, k_r\} \\ b_{i_j}a_{i_j}, & i_j \in \{k_{r+1}, \dots, k_n\}; \end{cases}$$

$$y^-_{i_j} = \begin{cases} b^-_{i_j}a^-_{i_j}, & i_j \in \{m_{s+1}, \dots, m_n\} \\ a^-_{i_j}b^-_{i_j}, & i_j \in \{m_1, \dots, m_s\}. \end{cases}$$

Let

$$M_1^n = Y_4pY_1p^-qY_2q^-cY_3c^-, M_2^n = Y_1Y_4pY_2p^-qY_3q^-, M_3^n = Y_1pY_2p^-qY_3q^-cY_4c^-,$$

$$M_4^n = Y_3Y_2pY_1p^-qY_4q^-,\ M_5^n = Y_4Y_1pY_2p^-qY_3q^- \text{ and } M_6^n = pY_1Y_4p^-Y_2Y_3,$$

where p, q, c are three letters and M_i^n $(1 \le i \le 6)$ means the surfaces consist of $4n+2$ semi-edges. From Propositions 3.2–3.3, it can be proved that $M_1^n \sim M_3^n$ and $M_2^n \sim M_5^n$; thus we just need to calculate the genus for $i = 2, 3, 4, 6$.

Proposition 3.5 *Let $g_{i_j}(n)$ be the number of surfaces with genus i in M_j^n for $n \ge 0$, $i \ge 0$ and $j = 2, 3, 4, 6$. Then, for $n \ge 1$, $g_{i_j}(n) =$*

$$\begin{array}{ll} 2g_{(i-1)_2}(n-1) + g_{(i-1)_6}(n-1) + g_{i_3}(n-1), & \text{if } j = 2 \text{ and } 1 \le i \le n+1; \\ g_{(i-1)_2}(n-1) + 2g_{(i-1)_3}(n-1) + g_{(i-1)_4}(n-1), & \text{if } j = 3 \text{ and } 1 \le i \le n+1; \\ g_{(i-1)_6}(n-1) + g_{i_3}(n-1) + 2g_{(i-1)_4}(n-1), & \text{if } j = 4 \text{ and } 1 \le i \le n+1; \\ 2g_{(i-1)_6}(n-1) + 2g_{i_2}(n-1), & \text{if } j = 6 \text{ and } 1 \le i \le n; \end{array}$$

Proof Since the proofs are similar for each j, here we just check the result for $j = 2$. The others can be checked from OPs 1–3 and Propositions 3.2–3.3.

D_n is obtained from D_{n-1} by adding two vertices u_n, v_n and two arcs $d = \langle v_n, u_n\rangle$ and $t = \langle u_n, v_n\rangle$. Let $M_2^{n-1} = X_1X_4pX_2p^-qX_3q^-$, where X_i are letters such that M_2^{n-1} has the same form as M_2^n, but the number of semiedges in M_2^{n-1} is four less than that in M_2^n. From the relation between the joint tree of D_n and that of D_{n-1}, there are four possibilities to get M_2^n from M_2^{n-1}:

$$(I) = X_1X_4pX_2dtp^-qX_3t^-d^-q^-;\quad (II) = X_1d^-t^-X_4pX_2dtp^-qX_3q^-;$$

$$(III) = tdX_1X_4pX_2p^-qX_3t^-d^-q^-;\ (IV) = tdX_1d^-t^-X_4pX_2p^-qX_3q^-.$$

From OP2 and Propositions 3.2–3.3,
$(I) \sim X_1X_4pX_2cp^-qX_3c^-q^- \sim X_1X_4qX_3X_2q^-pcp^-c^-$, so $o(I) = o(M_6^{n-1}) + 1$. Similarly, $o(II) = o(III) = o(M_2^{n-1}) + 1$ and $o(IV) = o(M_3^{n-1})$. As a consequence, $g_{i_2}(n) = 2g_{(i-1)_2}(n-1) + g_{(i-1)_6}(n-1) + g_{i_3}(n-1)$. □

Proposition 3.6 *We have $g_i(D_n) = 2g_{i_6}(n) + 2g_{(i-1)_6(n)}$, where $g_i(D_n)$ is the number of embedding of the digraph D_n in the surface with genus i.*

Proof Since a_l and b_l $(1 \le l \le n)$ in D_n are either both inside or both outside the circuit C_{2n}, i.e., either clockwise or anticlockwise in each vertex, a_{i_j} and b_{i_j} (or $a^-_{i_j}$ and $b^-_{i_j}$) are either both on the left or both on the right of the tree T. Thus embedding surfaces determined by semi-edges a, b, e, a^-, b^-, e^- for fixed Y_1, Y_2, Y_3, Y_4 have four types as follows: (See Figure 5, one of four types, where the rotation at each vertex of the joint tree is clockwise.)

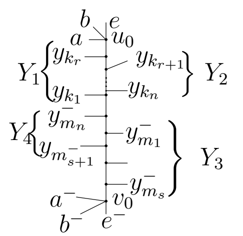


Figure 5

$Y_4Y_1abeY_2Y_3e^-b^-a^-, Y_4Y_1abeY_2Y_3a^-b^-e^-, Y_4Y_1ebaY_2Y_3a^-b^-e^-, Y_4Y_1ebaY_2Y_3e^-b^-a^-$.

By OPs 2–3 and Propositions 3.2–3.3, we have

$Y_4Y_1abeY_2Y_3e^-b^-a^- \sim Y_4Y_1aY_2Y_3a^- \sim aY_4Y_1a^-Y_2Y_3$, thus, $o(Y_4Y_1abeY_2Y_3e^-b^-a^-) = o(aY_4Y_1a^-Y_2Y_3)$;

$Y_4Y_1abeY_2Y_3a^-b^-e^- \sim Y_4Y_1eY_2Y_3e^-aba^-b^- \sim eY_2Y_3e^-aba^-b^-Y_4Y_1 \sim eY_2Y_3e^-Y_4Y_1aba^-b^-$, thus

$$o(Y_4Y_1abeY_2Y_3a^-b^-e^-) = o(aY_2Y_3a^-Y_4Y_1) + 1 = o(aY_4Y_1a^-Y_2Y_3) + 1;$$

$Y_4Y_1ebaY_2Y_3a^-b^-e^- \sim Y_4Y_1aY_2Y_3a^- \sim aY_4Y_1a^-Y_2Y_3$, thus $o(Y_4Y_1ebaY_2Y_3a^-b^-e^-) = o(aY_4Y_1a^-Y_2Y_3)$;

$Y_4Y_1ebaY_2Y_3e^-b^-a^- \sim Y_4Y_1aY_2Y_3a^-ebe^-b^-$, thus, $o(Y_4Y_1ebaY_2Y_3e^-b^-a^-) = o(Y_4Y_1aY_2Y_3a^-) + 1 = o(aY_4Y_1a^-Y_2Y_3) + 1$.

It can be seen that the genera of four cases are all determined by embedding surfaces M_6^n. The genera are the same as $o(M_6^n)$ for two cases and the genera are bigger 1 than $o(M_6^n)$ for other two cases. Thus, $g_i(D_n) = 2g_{i_6}(n) + 2g_{(i-1)_6(n)}$. □

Corollary 3.7 *The coefficients of genus distribution of D_n satisfy the following equation:*
$g_{n+1}(D_n)-g_n(D_n)+g_{n-1}(D_n)-g_{n-2}(D_n)+\cdots(-1)^{n-i+1}g_i(D_n)+\cdots+(-1)^{n+1}g_0(D_n) = 0.$

Proof From Proposition 3.6, we have

$$\begin{aligned}
&g_0(D_n) = 2g_{0_6}(n)\\
&g_1(D_n) = 2g_{1_6}(n) + 2g_{0_6}(n)\\
&g_2(D_n) = 2g_{2_6}(n) + 2g_{1_6}(n)\\
&g_3(D_n) = 2g_{3_6}(n) + 2g_{2_6}(n)\\
&\cdots \qquad\quad \cdots\\
&g_{n-2}(D_n) = 2g_{(n-2)_6}(n) + 2g_{(n-3)_6}(n)\\
&g_{n-1}(D_n) = 2g_{(n-1)_6}(n) + 2g_{(n-2)_6}(n)\\
&g_n(D_n) = 2g_{n_6}(n) + 2g_{(n-1)_6}(n)\\
&g_{n+1}(D_n) = 2g_{(n+1)_6}(n) + 2g_{n_6}(n)\\
&g_{n+2}(D_n) = 2g_{(n+2)_6}(n) + 2g_{(n+1)_6}(n)
\end{aligned}$$

Repeating the minus and plus alternatively starting from the last equation, the following equation holds.

$$g_{n+2}(D_n) - (g_{n+1}(D_n) - g_n(D_n) + g_{n-1}(D_n) - g_{n-2}(D_n) + \cdots + (-1)^{n-i+1}g_i(D_n) + \cdots + (-1)^{n+1}g_0(D_n)) = 2g_{(n+2)_6}(n)$$

From Proposition 3.5, $g_{i_6}(n) = 0$ for all $i > n$, and thus $g_{(n+2)_6}(n) = g_{(n+1)_6}(n) = 0$. From Proposition 3.6, $g_{n+2}(D_n) = 2g_{(n+2)_6}(n)+2g_{(n+1)_6(n)}$, so we have $g_{n+2}(D_n) = 0$, and the equation holds. □

Let $f_{M_j^0}(x) = 1$, that is, $g_{0_j}(0) = 1$ and $g_{i_j}(0) = 0$ for all $i \neq 0$. If we let $g_{i_j}(n) = 0$ for all $i < 0$, then by applying Propositions 3.5–3.6, the genus distribution polynomial of D_n for a given n can be easily computed. For example, distribution polynomials for $n = 0, 1, 2, \ldots, 11$ are as follows:

$f_{D_0}(x) = 2 + 2x;$

$f_{D_1}(x) = 4 + 8x + 4x^2;$

$f_{D_2}(x) = 4 + 24x + 28x^2 + 8x^3;$

$f_{D_3}(x) = 40x + 112x^2 + 88x^3 + 16x^4;$

$f_{D_4}(x) = 16x + 256x^2 + 464x^3 + 256x^4 + 32x^5;$

$f_{D_5}(x) = 224x^2 + 1344x^3 + 1760x^4 + 704x^5 + 64x^6;$

$f_{D_6}(x) = 64x^2 + 1856x^3 + 6272x^4 + 6208x^5 + 1856x^6 + 128x^7;$

$f_{D_7}(x) = 1152x^3 + 11904x^4 + 26880x^5 + 20608x^6 + 4736x^7 + 256x^8;$

$f_{D_8}(x) = 256x^3 + 11776x^4 + 65280x^5 + 107520x^6 + 65024x^7 + 11776x^8 + 512x^9;$

$f_{D_9}(x) = 5632x^4+90112x^5+321024x^6+405504x^7+196608x^8+28672x^9+1024x^{10};$

$f_{D_{10}}(x) = 1024x^4 + 68608x^5 + 574464x^6 + 1453056x^7 + 1453056x^8 + 573440x^9 + 68608x^{10} + 2048x^{11}$.

$f_{D_{11}}(x) = 26624x^5 + 612352x^6 + 3221504x^7 + 6150144x^8 + 4978688x^9 + 1622016x^{10} + 161792x^{11} + 4096x^{12}$.

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