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Antimedian graphs*

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Abstract

Antimedian graphs are introduced as the graphs in which for every triple of vertices there exists a unique vertex x that maximizes the sum of the distances from x to the vertices of the triple. The Cartesian product of graphs is antimedian if and only if its factors are antimedian. It is proved that multiplying a non-antimedian vertex in an antimedian graph yields a larger antimedian graph. Thin even belts are introduced and proved to be antimedian. A characterization of antimedian trees is given that leads to a linear recognition algorithm.

1 Introduction

Location models are of immense use in our daily life. It is a very important branch of optimization theory and in particular of combinatorial optimization. Usually in location theory, we consider the problem of locating facilities in a network, where the distance from the customers to the facility is minimum or the sum of the distances from the customers to the facility is minimum; both problems are well-known in the literature as the center problem and the median problem, respectively [4].

But there is also the need to locate undesirable facilities such as nuclear reactors, hazardous waste disposal units, chemical plants, water purification plants, electric power supplier networks, etc. The main objective is to locate obnoxious or undesirable facilities as far away as possible from "users" of the corresponding network. These location problems are known as obnoxious facility location problems. They also form a wide area of research in location, network, and optimization theory; see for example the surveys by Cappanera [5] and Plastria [11], recent papers [1, 6, 13, 17] and references therein. In graphs, particularly for trees, such problems have been considered earlier by Zelinka [16] and Ting [14], and see also Tamir [12] for a survey on obnoxious facility locations on graphs.

Let $G = (V, E)$ be a connected graph and $X \subseteq V$. For a vertex $v \in V$ let $D(v, X) = \sum_{x \in X} d(v, x)$ be the summed distance between v and X. A vertex v is called an *antimedian* vertex of X provided that $D(v, X)$ is maximized [7]. Diané and Plesník used antimedians to design an antimedian heuristic for the Steiner problem in graphs. When $X = V$ the set of all antimedians of V is called the *antimedian of* G [15]. The concept of medians is defined analogously, except that maximization is replaced with minimization.

In graph theory there is another, closely related concept that is also called the median. Let u, v, w be vertices of a graph G. These vertices need not be different, hence we will use the notation (u, v, w) to denote such a triple. Then a vertex x is called a *median* of (u, v, w) if x lies simultaneously on a shortest u, v-path, a shortest u, w-path, and a shortest v, w-path. Note that such a vertex need not exist in general. However, if every triple of vertices of G has a unique median, the graph G is called a *median graph*. For instance, trees and n-cubes are median. Median graphs form a closely investigated and well understood class of graphs; see the survey [9] and recent papers [2, 3, 10].

Note that if a median x of a triple $\pi = (u, v, w)$ exists, then it minimizes $D(x, \pi)$. Hence it is natural to study graphs in which every triple π of vertices admits a unique vertex x that maximizes $D(x, \pi)$. More precisely, let G be a connected graph and $\pi = (u, v, w)$ a triple of vertices of G. Then $a \in V(G)$ is an *antimedian* of π if

$$d(a, u) + d(a, v) + d(a, w) \geq d(y, u) + d(y, v) + d(y, w)$$

for all $y \in V(G)$. We say that G is an *antimedian graph* if every triple of G has a unique antimedian and that a vertex x is an *antimedian vertex* in G if x is an antimedian of some triple of G.

In this paper we are interested in constructions, characterizations and recognition of antimedian graphs. In the next section we observe that even paths and Cartesian products of antimedian graphs are antimedian. In particular, hypercubes are such. In Section 3 we prove that given an antimedian graph and a vertex x that is not antimedian of any triple of vertices, the multiplication with respect to x gives a larger antimedian graph. In the subsequent section we introduce thin even belts as graphs obtained from an even path by attaching graphs with properly bounded depths to its vertices. We prove that any such graph is antimedian. In Section 5 we prove that among trees, thin even belts characterize antimedian graphs and conclude with a linear recognition algorithm for antimedian trees.

2 Examples of antimedian graphs

In this section we give basic examples of antimedian graphs. The path P_n is antimedian if and only if n is even, while the cycle C_n, $n \geq 3$, is antimedian only for $n = 4$. Moreover, all hypercubes are antimedian, and, more generally, the Cartesian product is antimedian if and only if the factors are antimedian.

Clearly, an odd path P_{2n+1} is not an antimedian graph. Indeed, let x be the middle vertex of P_{2n+1}, then the triple $\pi = (x, x, x)$ has two antimedians: the endvertices of P_{2n+1}. On the other hand:

Proposition 1 *For any $n \geq 1$, P_{2n} is an antimedian graph.*

Proof. Let $\pi = (u, v, w)$ be a triple of vertices of P_{2n}. It is straightforward to observe that the only two candidates for an antimedian of π are the two endvertices v_1 and v_{2n} of P_{2n}. Now suppose that $D(v_1, \pi) = D(v_{2n}, \pi)$. Since for all $x \in P_{2n}$, $d(x, v_1) = 2n - 1 - d(x, v_{2n})$, we find that $D(v_1, \pi) = 6n - 3 - D(v_{2n}, \pi)$. Therefore $2D(v_1, \pi) = 6n - 3$, a contradiction. □

Proposition 2 *C_n is antimedian if and only if $n = 4$.*

Proof. Clearly, C_4 is an antimedian graph. If n is odd, then let $\pi = (u, u, u)$, where u is an arbitrary vertex of C_n. Let x and y be the antipodal vertices of u on C_n. Then $d(u, x) = d(u, y)$ and π has two antimedians. Let $n > 4$ be even and

$C_n = v_1 \dots v_n$. Then select $\pi = (v_1, v_3, v_{n/2+2})$. It is easy to see that $D(x, \pi)$ will be maximal for $x = v_{n/2+1}$ and $x = v_{n/2+3}$. □

We next show that hypercubes are antimedian. (Recall the vertex set of the n-cube Q_n consists of all binary n-tuples, two vertices being adjacent whenever they differ in precisely one position.) Although this is a consequence of Proposition 4, we include a direct proof that might be of independent interest.

Proposition 3 *For any $n \geq 1$, Q_n is an antimedian graph.*

Proof. Let $u = u_1 \dots u_n$, $v = v_1 \dots v_n$, $w = w_1 \dots w_n$ be an arbitrary triple of vertices of Q_n. Define $x = x_1 \dots x_n$ by the minority rule: set $x_i = 0$, if at least two of the u_i, v_i, w_i equal 1, otherwise set $x_i = 1$. Let y be an arbitrary vertex of Q_n. For $b, b' \in \{0, 1\}$ set $\delta(b, b') = 0$ if $b = b'$ and $\delta(b, b') = 1$, otherwise. Then,

$$\begin{aligned} d(y,u) + d(y,v) + d(y,w) &= \sum_{i=1}^{n} (\delta(y_i, u_i) + \delta(y_i, v_i) + \delta(y_i, w_i)) \\ &\leq \sum_{i=1}^{n} (\delta(x_i, u_i) + \delta(x_i, v_i) + \delta(x_i, w_i)) \\ &= d(x,u) + d(x,v) + d(x,w)\,. \end{aligned}$$

Moreover, equality holds if and only if $x = y$. It follows that x is the unique antimedian of (u, v, w). □

Recall that the *Cartesian product* $G \,\Box\, H$ of graphs G and H has vertex set $V(G \,\Box\, H) = V(G) \times V(H)$; and $E(G \,\Box\, H)$ consists of pairs $(g, h)(g', h')$ where either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. The most important metric property of the Cartesian product is that its metric is an additive function. More precisely,

$$d_{G \Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h')\,. \tag{1}$$

For more information on the Cartesian product of graphs see [8].

Proposition 4 *The Cartesian product of graphs is antimedian if and only if both factors are antimedian.*

Proof. Suppose G and H are antimedian graphs. Let $\pi = ((g, h), (g', h'), (g'', h''))$ be a triple from $V(G \,\Box\, H)$. Let a be the unique antimedian of (g, g', g'') in G and let b be the unique antimedian of (h, h', h'') in H. Then (1) implies that (a, b) is the unique antimedian of π.

Conversely, if one of the factors is not antimedian, say G, then there is a triple (g, g', g'') of G with at least two antimedians a_1 and a_2. Let h be an arbitrary vertex of H and consider the triple $((g, h), (g', h), (g'', h))$. If b is an antimedian of (h, h, h) in H, then (a_1, b) and (a_2, b) are antimedians of $((g, h), (g', h), (g'', h))$, hence the product is not antimedian. □

Since Q_n can be represented as the Cartesian product of n copies of K_2, Proposition 3 immediately follows from Proposition 4.

3 Multiplying non-antimedian vertices

Let G be a graph and $u \in V(G)$. Then the *multiplication* of G with respect to u is the graph obtained from G by replacing u by two adjacent vertices u' and u'' and joining them by an edge with all the neighbors of u.

Theorem 5 *Let G be an antimedian graph and u a non-antimedian vertex of G. Then the multiplication of G with respect to u is antimedian.*

Proof. Let G_u be the multiplication of G with respect to u and suppose to the contrary that G_u is not antimedian. Then there are three vertices $x, y, z \in V(G_u)$ which do not have a unique antimedian. Clearly, $d_{G_u}(u', u'') = 1$, $d_{G_u}(a, b) = d_G(a, b')$ and $d_{G_u}(u', a) = d_G(u, a)$ if $a, b \neq u', u''$.

If $x, y, z \notin \{u', u''\}$ and they have two antimedians in G_u then (x, y, z) have two antimedians in G, a contradiction. Therefore assume that $x = u'$ and $y, z \notin \{u', u''\}$. Let a be the unique antimedian of (u, y, z) in G. Since (u', y, z) has more than one antimedian in G_u we infer that these must be u'' and a. Therefore

$$d_{G_u}(u', a) + d_{G_u}(y, a) + d_{G_u}(z, a) = d_{G_u}(u', u'') + d_{G_u}(y, u'') + d_{G_u}(z, u''),$$

and hence

$$d_G(u, a) + d_G(y, a) + d_G(z, a) = 1 + d_G(y, u) + d_G(z, u).$$

We claim that then u is an antimedian of (y, y, z) in G. If not, then there is a vertex $w \in V(G)$, such that

$$2d_G(w, y) + d_G(w, z) > 2d_G(y, u) + d_G(z, u).$$

Then we have the following

$$\begin{aligned} 1 + d_G(y, u) + d_G(z, u) &= d_G(u, a) + d_G(y, a) + d_G(z, a) \\ &\geq d_G(u, w) + d_G(y, w) + d_G(z, w) + 1 \\ &= 2d_G(w, y) + d_G(w, z) - d_G(w, y) + d_G(w, u) + 1 \\ &> 2d_G(y, u) + d_G(z, u) - d_G(y, w) + d_G(u, w) + 1. \end{aligned}$$

Comparing the beginning and the end we get $d_G(w, y) > d_G(y, u) + d_G(u, w)$, a contradiction.

Suppose that $x, y \in \{u', u''\}$ and $z \notin \{u', u''\}$. We claim that the unique antimedian of (x, y, z) in G_u is the unique antimedian a of (u, u, z) in G. If there is an antimedian of (x, y, z) in G_u different from a, then this must be u' or u''. Without loss of generality assume this is u'. Then we have

$$d_{G_u}(x, u') + d_{G_u}(y, u') + d_{G_u}(z, u') \geq d_{G_u}(x, a) + d_{G_u}(y, a) + d_{G_u}(z, a)$$

and therefore (since $x, y \in \{u', u''\}$)

$$2 + d_G(z, u) \geq 2d_G(u, a) + d_G(z, a) \geq d_G(u, a) + d_G(z, u). \tag{2}$$

If there is a vertex $t \neq a$ such that $d(t,u) \geq 2$, then

$$2d_G(u,t) + d_G(z,t) \geq 2 + d_G(z,u) \geq 2d_G(u,a) + d_G(z,a)\,,$$

which is a contradiction, since this implies that a is not the unique antimedian of (u,u,z) in G.

By equation (2) we have that either $d(u,a) = 1$ or $d(u,a) = 2$. If $d(u,a) = 1$, we find that there is no vertex $t \in G$, such that $d(t,u) \geq 2$, hence $K_{1,n}$ is a spanning subgraph of G, which is a contradiction to the hypothesis that G is an antimedian graph (namely, (u,u,u) does not have a unique antimedian). If $d(u,a) = 2$, then a lies on a shortest path from u to z (this follows from (2)) and since a is the antimedian of (u,u,z) in G, we find that $a = z$. Observe that since u is not an antimedian of (a,a,a), there is exactly one neighbor t of u, such that $d_G(t,a) = 3$. Clearly, then u' cannot be an antimedian of (x,y,z) in G_u (since the sum of distances to t is greater).

If $x,y,z \in \{u',u''\}$ and u' is an antimedian of (x,y,z), then every vertex from G is adjacent to u, hence $G = K_{1,n}$, a contradiction. □

Note that the proof of Theorem 5 implies that the set of all antimedian vertices of G is preserved after the described expansion is performed. It follows that we can apply the operation several times and in this way many interesting antimedian graphs can be constructed. For instance, by Propositions 1 and 4 we know that $P_4 \,\square\, P_4$ is antimedian. Moreover, the four vertices of degree 2 are its antimedian vertices. So we may apply the construction on any of the other vertices. The graph that is obtained in this way by expanding the four vertices of degree 4 is shown in Fig. 1.

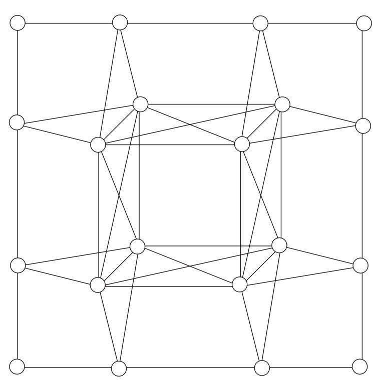

Figure 1: An antimedian graph.

4 Thin even belts are antimedian

Let $v_1, \ldots, v_n$ be the vertex set of the path on n vertices $P = P_n$ and let G_i, $2 \leq i \leq n-1$, be rooted graphs with roots y_i, respectively. Let G be the graph obtained from the disjoint union of P and the graphs G_i such that for $i = 2, \ldots, n-1$, y_i is identified with v_i. Let us call G a *belt*, P the *support* of the belt, and the graphs G_i the *ears* of the belt. A belt is *even*, if the support is an even path. If, in addition, the depth of G_i is at most $\lfloor (i-2)/3 \rfloor$ for $i \leq n/2$ and at most $\lfloor (n-i-1)/3 \rfloor$ for $i > n/2$, we speak of a *thin belt*. Note that in a thin belt, the graphs G_i and G_{n-i+1}, $2 \leq i \leq 4$, are all isomorphic to K_1. For a set $X \subseteq V(G)$ and a vertex $u \in V(G)$ we use $d_G(u, X)$ to denote the minimum distance between u and a vertex $x \in X$.

Theorem 6 *Let G be a thin even belt. Then G is antimedian.*

Proof. Let $P = v_1, v_2, \ldots, v_{2k}$ be the support of G and let G_i, $2 \leq i \leq 2k-1$, be the ears of G. Let in addition G' be the subgraph of G induced by the graphs $G_2, \ldots, G_k$, and let $G'' = G - G'$. For a vertex x of G set $d_x = d_G(x, P)$.

We are going to show that for an arbitrary triple of vertices of G, either v_1 or v_{2k} is its unique antimedian. Hence in the rest of the proof let $\pi = (u, v, w)$ be an arbitrary fixed triple, where $u \in G_i$, $v \in G_j$, and $w \in G_r$.

Note first that

$$D(v_1, \pi) = ((i-1) + d_u) + ((j-1) + d_v) + ((r-1) + d_w) \tag{3}$$

and

$$D(v_{2k}, \pi) = ((2k-i) + d_u) + ((2k-j) + d_v) + ((2k-r) + d_w)\,. \tag{4}$$

Then $D(v_1, \pi) = D(v_{2k}, \pi)$ implies $6k + 3 = 2(i + j + r)$ which is not possible. Therefore, $D(v_1, \pi) \neq D(v_{2k}, \pi)$. Hence it suffices to prove that for an arbitrary vertex $x \notin \{v_1, v_{2k}\}$ of G, one of $D(v_1, \pi) > D(x, \pi)$ or $D(v_{2k}, \pi) > D(x, \pi)$ holds.

Let $x \in G_s$. Then we distinguish the following cases.

Case 1: $u, v, w \in G'$ or $u, v, w \in G''$.
We may without loss of generality assume that $u, v, w \in G'$. Clearly, if $x \in G''$ then $D(v_{2k}, \pi) > D(x, \pi)$. Let $x \in G'$. Then $d(x, u) \leq k + d_u$. On the other hand, $d(v_{2k}, u) \geq (k+1) + d_u$. Analogous conclusions hold for v and w as well, hence also in this case $D(v_{2k}, \pi) > D(x, \pi)$.

Case 2: Two of u, v, w belong to G' and one to G'' or vice versa.
By symmetry we may without loss of generality assume that $u, v \in G', w \in G''$. Furthermore, we may also assume that $i \leq j$. Hence $i \leq j \leq k < r$.

Case 2.1: $s \leq i$.
In this case we clearly have $D(v_1, \pi) > D(x, \pi)$. Hence if $D(v_1, \pi) > D(v_{2k}, \pi)$ we infer that v_1 is the antimedian of π and if $D(v_1, \pi) < D(v_{2k}, \pi)$ then $D(x, \pi) < D(v_1, \pi) < D(v_{2k}, \pi)$ and v_{2k} is the antimedian of π.

Case 2.2: $i < s \leq j$.
We claim that in this subcase $D(v_1, \pi) > D(x, \pi)$. Suppose first that $s < j$. Then

$$D(x, \pi) = 3d_x + (s-i) + d_u + (j-s) + d_v + (r-s) + d_w\,, \tag{5}$$

and the claimed inequality will hold, comparing (3) and (5), provided that $2i + s > 3d_x + 3$. This is indeed the case since $d_x \leq \lfloor (s-2)/3 \rfloor$ and therefore $3d_x \leq s - 2$.

Suppose $s = j$. Then the computation is just as above, except that we use the fact that $d(x, v) \le d_x + d_v$ holds in this case.

Case 2.3: $j < s < r$.
In this subcase we claim that $D(v_{2k}, \pi) > D(x, \pi)$. Now,

$$D(x, \pi) = 3d_x + (s - i) + d_u + (s - j) + d_v + (r - s) + d_w\,, \tag{6}$$

and the claimed inequality will hold provided that $3d_x + s + 2r < 6k$. If $s \le k$ then, using the fact that $3d_x \le s - 2$, we infer $3d_x + s + 2r \le 2s + 2r - 2 \le 6k - 2$. And if $s > k$, then $3d_x \le 2k - s - 1$ and hence $3d_x + s + 2r \le 2k - s + s + 2r - 1 \le 6k - 1 < 6k$.

Case 2.4: $r \le s$.
In this subcase it is clear that $D(v_{2k}, \pi) > D(x, \pi)$.

We have thus proved, that for a fixed triple π and an arbitrary vertex $x \notin \{v_1, v_{2k}\}$ of G, at least one of $D(v_1, \pi) > D(x, \pi)$ or $D(v_{2k}, \pi) > D(x, \pi)$ holds. Now, since $D(v_1, \pi) \ne D(v_{2k}, \pi)$, v_1 is the unique antimedian of π if $D(v_1, \pi) > D(v_{2k}, \pi)$, otherwise v_{2k} is the unique antimedian of π. □

There exist other "thin" graphs that are also antimedian. For instance, given a thin even belt, one may add several edges that connect vertices in neighboring ears that are at the same depth. An example of such an antimedian graph is the upper graph from Fig. 2.

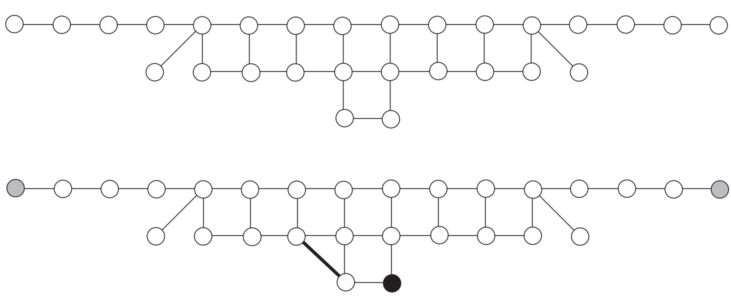

Figure 2: Antimedian graph and not antimedian graph.

On the other hand, adding the bold edge to this graph as shown on the below graph from the same figure, the obtained graph is no longer antimedian: the triple (x, x, x), where x is the black vertex from the figure has two antimedians—the grey vertices.

An ear of a thin belt can be an arbitrary graph, as long as its depth is small with respect to the length of the support. Hence Theorem 6 implies:

Corollary 7 *There is no forbidden subgraphs characterization of antimedian graphs.*

5 Characterization of antimedian trees

In this section we prove that among trees, thin even belts are precisely the antimedian graphs. For this sake we first show:

Lemma 8 *Let T be an antimedian tree. Then T contains exactly two diametrical vertices a and b. Moreover, $d(a,b)$ is odd, and for any triple of vertices either a or b is its antimedian.*

Proof. Let a and b be arbitrary diametrical vertices in T and let P be the a,b-path in T. Suppose that $d(a,b)$ is even. Let y be the middle vertex of P, then the triple (y,y,y) has at least two antimedians, a and b. Indeed, since a and b are diametrical vertices, no vertex is further away from y than a and b. Hence $d(a,b)$ must be odd.

Let x be an arbitrary vertex of T different from a,b. Let $d_x = d(x,P)$ and let y be the vertex of P with $d(x,y) = d_x$. Assuming without loss of generality that $d(y,a) < d(y,b)$ we infer that $d(x,y) < d(y,a)$, for otherwise either (b,b,b) would have at least two antimedians (in the case $d(x,y) = d(y,a)$) or a would not be the diametrical vertex (in the case $d(x,y) > d(y,a)$). It follows that x is not a diametrical vertex of T and hence a and b are the only diametrical vertices.

Let $\pi = (u,v,w)$ be a triple of T and suppose that $z \notin \{a,b\}$ is the antimedian of π. Suppose z is not a leaf. Clearly, if z is of degree 2, it cannot be the antimedian. Suppose the degree of z is at least 3. But then z has a neighbor x such that $D(x,\pi) > D(z,\pi)$, which is not possible. So z is a leaf of T. Let z' be the first vertex of the z,a-path that is on P.

Suppose $u \notin P$, and let u' be the first vertex on the u,a-path that is on P. Let $\pi' = (u',v,w)$. Since $D(a,\pi) < D(z,\pi)$ and since $D(a,\pi)$ is reduced at least as much as $D(z,\pi)$ when we change u to u' we find that $D(a,\pi') < D(z,\pi')$. So a is also not the antimedian of π'. Analogously, b is not the antimedian of π'. Therefore we can assume without loss of generality that $u,v,w \in P$.

If u,v,w would all lie on the z',a-path (resp. z',b-path) then $D(b,\pi) > D(z,\pi)$ (resp. $D(a,\pi) > D(z,\pi)$), which is not possible. So without loss of generality assume that u lies in the z',a-path and w lies in the z',b-path. Also, we can assume that $d(z,a)$ is odd and $d(z,b)$ is even.

Let u' be the neighbor of u on P with $d(u',a) = d(u,a) - 1$ and set $\pi' = (u',v,w)$. Then $D(z,\pi') = D(z,\pi) + 1$, hence z is also the antimedian of π'. It follows by induction that z is also the antimedian of (a,v,w). Analogously we can move w to b and conclude that z is the antimedian of (a,v,b). So without loss of generality set $\pi = (a,v,b)$. Now $D(z,\pi) - D(b,\pi)$ is an even number, because

$$\begin{aligned} D(z,\pi) - D(b,\pi) &= (d(z,a) + d(z,v) + d(z,b)) - (d(b,a) + d(b,v) + d(b,b)) \\ &= (d(z,a) - d(b,a)) + (d(z,v) - d(b,v)) + d(z,b), \end{aligned}$$

which is the sum of three even numbers. Of course, this difference is positive as z is the antimedian of π.

Case 1: v is on the a,z-path.
In this case, we can move w towards v in such a manner that the above difference

decreases. Note that in this movement no other leaf y can become the antimedian of π because the distance to y will start to increase only when all u, v, w are on the a, y-path. In that case $D(b, \pi) > D(y, \pi)$ since b is an antipodal vertex. To be more precise, let w' be the neighbor of b on P. Let $\pi' = (a, v, w')$. Then

$$D(z, \pi') - D(b, \pi') = (D(z, \pi) - D(b, \pi)) - 2\,.$$

Repeating this procedure we must arrive at the situation when $D(z, \pi') = D(b, \pi')$. But at this stage π' has two antimedians z and b, a contradiction.

Case 2: v is on the b, z-path.
If $v = w$, then v can be moved towards a at least once without making any other vertex as the antimedian of π. Whenever v is on the a, w-path, w can be moved towards v similarly. Each of this move reduces the difference $D(z, \pi) - D(b, \pi)$ by 2 and repeating this procedure, we arrive at the situation when $D(z, \pi) = D(b, \pi)$. But at this stage π has two antimedians z and b, the final contradiction. □

Note that the above lemma holds also for thin even belts.

Theorem 9 *Let T be a tree. Then T is an antimedian graph if and only if it is a thin even belt.*

Proof. By Theorem 6 we know that thin even belts are antimedian. It remains to prove that among trees no other graph is antimedian.

Let T be an arbitrary antimedian tree. By Lemma 8, T has exactly two diametrical vertices u and v and let $P : u = v_1, v_2, \ldots, v_r = v$ be the u, v-path in T. Let T_i, $1 \leq i \leq r$, be the maximal subtree of T that contains v_i and no other vertex of P. We can consider T_i as a rooted tree with the root v_i. Moreover, we can consider T as a belt, where P is its support and T_i are its ears. From Lemma 8 we also know that T is an even belt.

Let d_i be the depth of T_i, $1 \leq i \leq r$. Suppose that for some $i \leq n/2$ the condition $d_i \leq \lfloor (i-2)/3 \rfloor$ is not fulfiled. Hence $3d_i > i - 2$ and let w be a vertex from T_i with $3d(w, v_i) > i - 2$ or, more convenient for us, $3d(w, v_i) \geq i - 1$. Consider the triple $\pi = (u, v, v)$. Clearly $D(v, \pi) < D(u, \pi) = 2(r-1)$. However $D(w, \pi) = 3d_i + i - 1 + 2(r - i) \geq 2r - 2$. We have a contradiction with Lemma 8 since w is also an antimedian vertex. By symmetry we have an analogue proof for $i > r/2$.

Suppose T is not a thin belt. Then for at least some i, $d_i > \lfloor (i-2)/3 \rfloor$ if $i \leq r/2$, or $d_i > \lfloor (r-i-1)/3 \rfloor$ when $i > r/2$. Assume without loss of generality that x is a vertex of T_i with $d(x, P) > \lfloor (i-2)/3 \rfloor$, where $i \leq r/2$.

Consider the triple $\pi = (v_1, v_r, v_r)$. Then $D(v_1, \pi) = 2r - 2$ and $D(v_r, \pi) = r - 1$. On the other hand,

$$D(x, \pi) \geq 3(\lfloor (i-2)/3 \rfloor + 1) + (i-1) + 2(r-i) = \begin{cases} 2r-1; & i \equiv 0 \bmod 3, \\ 2r-2; & i \equiv 1 \bmod 3, \\ 2r; & i \equiv 2 \bmod 3. \end{cases}$$

Since by Lemma 8 exactly one of v_1 and v_r is the antimedian of (v_1, v_r, v_r), this contradiction shows that T is not antimedian as soon as it is not a thin belt. □

6 Recognizing antimedian graphs

Let G be a graph on n vertices. Then a direct computation (first compute the distance matrix and then check every triple of vertices whether it has a unique antimedian) yields an $O(n^4)$ recognition algorithm for antimedian graphs. On the other hand, Theorem 9 enables one to design a linear algorithm for recognizing antimedian trees as follows.

Algorithm AnitimedianTrees

Input: tree T.
Output: yes, if T is antimedian; no, otherwise.

Step 1. Find the center $C(T)$ of T. If $|C(T)| = 1$ then return "no" and finish.
Step 2. Let $C(T) = \{u, v\}$. Construct the bidirectional BFS from the edge uv. If there are more than two leaves at the last BFS-level, return "no" and finish.
Step 3. Let T_u and T_v be the BFS trees from u and v, respectively, and let x and y be the corresponding leaves. Check that the x, y-path in T is the support of an even belt. If so, return "yes", otherwise return "no".

Proposition 10 *Algorithm AntimedianTrees correctly determines whether a given tree T is antimedian and can be implemented in $O(|T|)$ time.*

Proof. By Theorem 9 a tree T is antimedian if and only if it is a thin even belt. Then the center of such a tree is an edge, otherwise the support of T would have odd number of vertices. Moreover, the endvertices of the support are the unique diametrical vertices of T. Hence the x, y-path, denote it with P, found by the algorithm is the support of T and Step 3 checks whether T is thin. It follows that the algorithm correctly recognizes antimedian trees.

It is well-known that the center of T can be found in linear time: repeatedly remove the sets of pendant vertices until a vertex or two vertices are obtained. The bidirectional BFS can be determined within the same time complexity.

The last step can be implemented as follows. Start from x (resp. y) and examine the x, u-subpath of P (resp. the y, v-subpath of P) in T_u (resp. T_v). Check whether any subtree at vertex w along this path has length greater than $\lfloor (d(x, w) - 2)/3 \rfloor$ (resp. $\lfloor (d(y, w) - 2)/3 \rfloor$) and if so, reject the graph. The distance of a vertex from T_u (resp. T_v) from u (resp. v) can be obtained along the way when T_u and T_v are constructed. All this can be done in linear time. □

7 Concluding remarks

As a variant of the problem considered in this paper one might also consider the graphs in which any triple of *different* vertices has a unique antimedian. Clearly, a larger class of graphs than the one treated here is obtained in this way. In particular, it would be interesting to relate both classes.

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