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Skolem labelled trees and $P_s \square P_t$ Cartesian products

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Abstract

The concept of the Skolem labelled graph was introduced by Mendelsohn and Shalaby, (*Discrete Math.* 97 (1991), 301–317).

Translating into a graph theoretical context an idea introduced by Baker, Kergin and Bonato, (*Ars Combin.* 63 (2002), 97–107), we prove the necessary and sufficient conditions for maximum (hooked) Skolem labelling $P_s \square P_t$ Cartesian products. We also provide an algorithm used to generate Skolem labellings of trees as well as data generated using this algorithm.

1 Introduction

While studying Steiner triple systems in 1957, Thoralf Skolem had the idea to distribute the numbers $1, 2, \ldots, 2n$ into n distinct pairs (a_i, b_i) such that $b_i - a_i = i$ for each $i \in \{1, 2, \ldots, n\}$ [13]. Such a distribution is now known as a *Skolem sequence of order* n. Readers may wish to consult [4, 5, 8, 13] for the historical introduction and initial development of the subject.

The concept of applying a Skolem sequence to the labelling of a graph was introduced in 1990 by Mendelsohn and Shalaby in [6]. Since then, other papers have documented the Skolem labelling of windmills [7] and introduced the concept of the Skolem array [1], equivalent to the Skolem labelling of a ladder graph.

We expand on the results dealing with Skolem arrays by translating them into a graph theoretical context, and generalise the necessary and sufficient existence conditions to

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the case of generalised grid graphs. We also present an algorithm used for determining whether a particular tree can be Skolem labelled, as well as data pertaining to the number of Skolem labellable trees up to 20 vertices.

2 Preliminaries

A *Skolem sequence of order* n is a distribution of the numbers $1, 2, \ldots, 2n$ into n distinct pairs (a_i, b_i) such that $b_i - a_i = i$ for each $i \in \{1, 2, \ldots, n\}$ and a *hooked Skolem sequence of order* n is a distribution of the numbers $1, 2, \ldots, 2n-1, 2n+1$ into n distinct pairs (a_i, b_i) such that $b_i - a_i = i$ for each $i \in \{1, 2, \ldots, n\}$. These formulations are equivalent to sequences consisting of two copies of each of the numbers 1 through n inclusive arranged such that if s_i and s_j are respectively the first and second occurrences of the number k in the sequence, then $j - i = k$.

In the case of the hooked Skolem sequence under the second (and more common) formulation, the penultimate position in the sequence is left unfilled—this position is called the *hook* and is often denoted by an "$*$". The sequence $1, 1, 3, 4, 2, 3, 2, 4$ is a Skolem sequence of order 4 and the sequence $3, 1, 1, 3, 2, *, 2$ is a hooked Skolem sequence of order 3. There are now many variants on the Skolem sequence, denoted *Skolem-type sequences*. These are not relevant to the discussion at hand; some useful papers in this direction are [2, 3, 11, 12]. It is well-known that a Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$, and that a hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 \pmod 4$.

All graphs are assumed to be simple and undirected throughout. If u and v are vertices of a graph G, then the distance between them (the length of a minimal u, v-path) is denoted $d_G(u, v)$ or $d(u, v)$. The *eccentricity* of a vertex is the maximal distance between it and all other vertices in the graph, and the *diameter* of a graph is the maximal vertex eccentricity. The *Cartesian product* of two graphs G_1 and G_2, denoted $G_1 \Box G_2$, is the graph having vertex set $V(G_1) \times V(G_2)$ in which vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \Box G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1. The path on n vertices is referred to as P_n.

Introduced by Mendelsohn and Shalaby in [6], a *d-Skolem labelled graph*, by a slight abuse of notation, is a triple $(G, \mathcal{L}, d)$ that satisfies the following four conditions:

(i) $G = (V, E)$ is a graph (on $2n$ vertices),

(ii) $\mathcal{L} : V \to \{d, d+1, \ldots, d+n-1\}$ is a surjective vertex-labelling function,

(iii) for each $i \in \{d, d+1, \ldots, d+n-1\}$, there exist exactly two vertices $u, v \in V$ having $d_G(u, v) = d + i$ and labels $\mathcal{L}(u) = \mathcal{L}(v) = d + i$, and,

(iv) if $G' = (V, E')$ and $E' \subsetneq E$, then $(G', \mathcal{L}, d)$ violates condition (iii).

Item (iv) is simply the requirement that each edge of G must lie on a *labelling path*, that is, a path between a pair of similarly labelled vertices. The element d is called the *defect* of the labelling. We have need only for the case that $d = 1$, though for the sake of completeness, we include the general form in the following definitions. New definitions are displayed in block form.

A graph with an odd number of vertices may form the base for a *d-hooked Skolem labelled graph*; it is a triple $(G, \mathcal{L}, d)$ satisfying condition (iv), as well as conditions

(i′) $G = (V, E)$ is a graph (on $2n + 1$ vertices),

(ii′) $\mathcal{L} : V \to \{0\} \cup \{d, d+1, \ldots, d+n-1\}$ is a surjective vertex-labelling function,

(iii′) for $u \in V$, if $\mathcal{L}(u) \neq 0$ then condition (iii) applies,

(v) there exists exactly one $v \in V$ such that $\mathcal{L}(v) = 0$.

Note that, as $d = 1$, hooked Skolem labelled graphs have exactly one 0-labelled vertex. Furthermore, all Skolem labelled graphs (resp. hooked Skolem labelled graphs) have exactly $2n$ (resp. $2n + 1$) vertices by conditions (i) and (ii) (resp. (i′) and (ii′)) and include each edge on a path between two similarly labelled vertices by condition (iv). To describe the case where condition (iv) is not possible, we formulate

Definition 1. A *weak d-(hooked) Skolem labelled graph* is one which satisfies conditions (i), (ii), and (iii) (resp. (i′), (ii′), (iii′), and (v)) of the (hooked) Skolem labelled graph definition, as well as condition

(iv′) there exists an edge $e \in E(G)$ whose removal does not affect the labelling.

We will refer, from time to time, to (hooked) Skolem labelled graphs as *strong (hooked) Skolem labelled graphs* to differentiate them clearly from *weak* (hooked) Skolem labelled graphs.

We expand the definitions above to graphs that do not permit (weak) (hooked) Skolem labellings:

Definition 2. A *maximum d-hooked Skolem labelled graph* is a triple $(G, \mathcal{L}, d)$ satisfying conditions

(i″) $G = (V, E)$ is a graph on either $2n$ or $2n + 1$ vertices,

(ii″) $\mathcal{L} : V \to \{0\} \cup \{d, d+1, \ldots, d+m-1\}$ is a surjective function, where $m \in \mathbb{N}$ is strictly smaller than n and the largest value possible given the surjectivity of $\mathcal{L}$,

(iii″) for each $i \in \{d, d+1, \ldots, d+m-1\}$, there exist exactly two vertices $u, v \in V$ having $d_G(u, v) = d + i$ and labels $\mathcal{L}(u) = \mathcal{L}(v) = d + i$,

(iv″) if $G' = (V, E')$ and $E' \subsetneqq E$ then $(G', \mathcal{L}, d)$ violates (iii″).

There clearly exist at least two vertices $u, v \in V$ with $\mathcal{L}(u) = \mathcal{L}(v) = 0$ in a maximum hooked Skolem labelled graph by condition (ii$''$). An example of a maximum hooked Skolem labelled tree is pictured below in Figure 1.

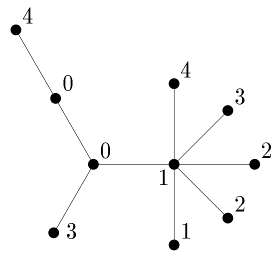


Figure 1: maximum hooked Skolem labelled 10-vertex tree

We also define, as above,

Definition 3. A *weak maximum d-hooked Skolem labelled graph* is one which satisfies condition (iv$'$) of the definition of weak d-(hooked) Skolem labelled graphs as well as conditions (i$''$), (ii$''$), and (iii$''$) of the definition of maximum d-hooked Skolem labelled graphs.

We will occasionally refer to maximum hooked Skolem labelled graphs as *strong maximum hooked Skolem labelled graphs* to differentiate them from *weak* maximum hooked Skolem labelled graphs. A weak maximum hooked Skolem labelled tree is pictured below in Figure 2.

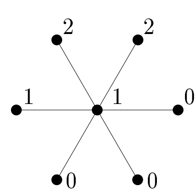


Figure 2: weak maximum hooked Skolem labelled 7-vertex tree

3 Skolem labelled $P_s \square P_t$ grids

3.1 Reformulation and generalisation of the Skolem array problem

The concept of the *Skolem array* was introduced by Baker, Bonato, and Kergin in [1].

Definition 4 [1]. A *Skolem array of order* n is a $2 \times n$ array A in which the *distance* between positions (a, b) and (c, d) is defined to be $|c - a| + |d - b|$, and in which each $i \in \{1, \ldots, n\}$ occurs in two positions which are distance i apart.

From this definition, Baker, Bonato, and Kergin proved, among other results, that a Skolem array of order n exists if and only if $n \equiv 0, 1 \pmod 4$ [1].

Due to the fact that distance in graphs corresponds to Baker, Bonato, and Kergin's concept of distance in Skolem arrays, it is clear that a $2 \times n$ Skolem array is equivalent to a Skolem labelled *ladder graph of length* n, which graph being the Cartesian product $P_2 \Box P_n$. And it is only natural, after having reformulated the issue in a graph theoretical context, to pose the following

Question. *Is it possible to Skolem label the graph* $P_s \Box P_t$ *for all* $s, t \in \mathbb{N}$*?*

We use the term *Skolem array* throughout when referring to the subject matter presented in [1], but will situate our results as dealing with the Skolem labelling of *grid graphs* $P_s \Box P_t$. In the following two subsections, we will first present the necessary and sufficient conditions to the solution for the above problem (ignoring whether the labelling is strong or weak), and second prove that all Skolem labelled grid graphs $P_s \Box P_t$ with $s, t \geq 2$ are weak.

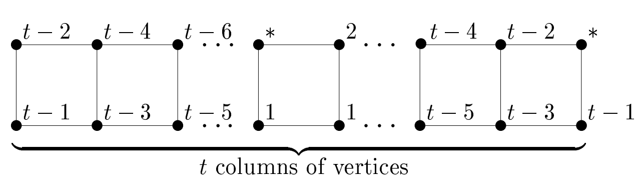
3.2 Necessity and sufficiency

We begin with a lemma that parallels the idea of Skolem and hooked Skolem sequences, and provides the optimal (maximum) Skolem labellings for ladder graphs. It should be noted that it is a triviality that the path $P_t \cong P_1 \Box P_t$ is Skolem labellable if and only if $t \equiv 0, 1 \pmod 4$, and hooked Skolem labellable if and only if $t \equiv 2, 3 \pmod 4$—this is simply the existence result for Skolem and hooked Skolem sequences [4, 13].

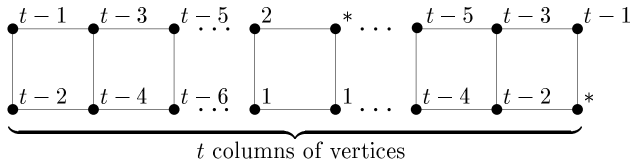
Lemma 1. *The ladder graph of length* t*,* $P_2 \Box P_t$*, is Skolem labellable if and only if* $t \equiv 0, 1 \pmod 4$*. It is maximum Skolem labellable with two hooks if and only if* $t \equiv 2, 3 \pmod 4$*.*

Proof. The case when $t \equiv 0, 1 \pmod 4$ was dealt with by Baker, Bonato, and Kergin in [1]. When $t \equiv 2, 3 \pmod 4$, the necessity is clear. For the sufficiency, there are two cases.

(i) $t \equiv 2 \pmod 4$. The construction is:



(ii) $t \equiv 3 \pmod 4$. The construction is:



□

Theorem 2. *The grid graph $P_3 \square P_3$ has a hooked Skolem labelling and any other grid graph $P_s \square P_t$ with $s, t \geq 3$ has a maximum Skolem labelling with $st - 2s - 2t + 4$ hooks.*

Proof. First of all, a hooked Skolem labelling for $P_3 \square P_3$ is displayed. Though this is a graph theoretical result, for easy legibility, we borrow the table and cell method of construction used by Baker, Bonato, and Kergin in [1]. Here is the labelling in table format:

4	1	1
2	*	3
3	2	4

It should be kept in mind that in the table above, and in each table below, each cell represents a vertex in the corresponding grid graph and that, furthermore, horizontally and vertically adjoining cells represent adjacent vertices.

Now for the general case. Both s and t are assumed to be greater than 2 and at least one is assumed to be greater than 3.

There are $s \cdot t$ vertices in a $P_s \square P_t$ grid. The maximum label value is $s + t - 2$ since the maximum length path has $(s-1)+(t-1) = s+t-2$ edges. Hence, the minimum number of unlabelled vertices, or hooks, is

$$st - 2(s + t - 2) = st - 2s - 2t + 4 = (s - 2)(t - 2)$$

since each label is represented on two separate vertices. This conclusion is consistent, as it is clear that for all values of s and t, the number of hooks will always be a positive number strictly less than the total number of vertices.

We consider two cases for the construction: first, when exactly one of s and t equals 3 and the other is greater than 3; second, when s and t are both greater than 3.

1) Exactly one of s and t equals 3 and the other is greater than 3.

   Without loss of generality, we need only provide a construction for the case when $s = 3$ and $t > 3$. We subdivide the $P_3 \square P_t$ grid case into two subcases; $t \equiv 0 \pmod 2$ and $t \equiv 1 \pmod 2$.

(i) $t \equiv 0 \pmod 2$. The construction is:

$t+1$	$*$	$t-4$	$\cdots$	2	$*$	2	$\cdots$	$t-6$	$t-4$	t
$t-1$	$t-3$	$t-5$	$\cdots$	1	1	3	$\cdots$	$t-5$	$t-3$	$t-1$
$t-2$	t	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$t-2$	$t+1$

(ii) $t \equiv 1 \pmod 2$. The construction is:

$t+1$	$*$	$t-4$	$\cdots$	3	1	1	$\cdots$	$t-6$	$t-4$	t
$t-1$	$t-3$	$t-5$	$\cdots$	2	$*$	2	$\cdots$	$t-5$	$t-3$	$t-1$
$t-2$	t	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$t-2$	$t+1$

2) Both s and t are greater than 3.

We consider all combinations of even and odd values for s and t. The diagrams representing the cases s and t both even, s and t both odd, and, without loss of generality, s odd and t even follow in the same format as those for $s = 3$ previously.

$s+t-2$	$s+t-4$	$\cdots$	t	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$t+1$	$t+3$	$\cdots$	$s+t-5$	$s+t-3$
$t-1$	$t-3$	$\cdots$	$t-s+1$	$t-s-1$	$\cdots$	3	1	1	$\cdots$	$t-s+3$	$t-s+5$	$\cdots$	$t-3$	$t-1$
$t-2$	$t-4$	$\cdots$	$t-s$	$t-s-2$	$\cdots$	2	$*$	2	$\cdots$	$t-s+4$	$t-s+6$	$\cdots$	$t-2$	$*$
$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	t
$*$	$t+1$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$t+2$
$*$	$t+3$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$t+4$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$*$	$s+t-7$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-6$
$*$	$s+t-5$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-4$
$*$	$s+t-3$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-2$

Maximum Skolem labelled $P_s \square P_t$ grid, $s, t \equiv 0 \pmod 2$

$s+t-2$	$s+t-4$	$\cdots$	$t+1$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	t	$t+2$	$\cdots$	$s+t-5$	$s+t-3$
$t-1$	$t-3$	$\cdots$	$t-s+2$	$t-s$	$\cdots$	2	$*$	2	$\cdots$	$t-s+2$	$t-s+4$	$\cdots$	$t-3$	$t-1$
$t-2$	$t-4$	$\cdots$	$t-s+1$	$t-s-1$	$\cdots$	1	1	3	$\cdots$	$t-s+3$	$t-s+5$	$\cdots$	$t-2$	$*$
$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$*$	t	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$t+1$
$*$	$t+2$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$t+3$
$*$	$t+4$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$t+5$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$*$	$s+t-7$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-6$
$*$	$s+t-5$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-4$
$*$	$s+t-3$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-2$

Maximum Skolem labelled $P_s \Box P_t$ grid, $s, t \equiv 1 \pmod 2$

$s+t-2$	$s+t-4$	$\cdots$	$t+1$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	t	$t+2$	$\cdots$	$s+t-5$	$s+t-3$
$t-1$	$t-3$	$\cdots$	$t-s+2$	$t-s$	$\cdots$	3	1	1	$\cdots$	$t-s+2$	$t-s+4$	$\cdots$	$t-3$	$t-1$
$t-2$	$t-4$	$\cdots$	$t-s+1$	$t-s-1$	$\cdots$	2	$*$	2	$\cdots$	$t-s+3$	$t-s+5$	$\cdots$	$t-2$	$*$
$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$*$	t	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$t+1$
$*$	$t+2$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$t+3$
$*$	$t+4$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$t+5$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$*$	$s+t-7$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-6$
$*$	$s+t-5$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-4$
$*$	$s+t-3$	$\cdots$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	$s+t-2$

Maximum Skolem labelled $P_s \Box P_t$ grid, $s \equiv 1 \pmod 2$, $t \equiv 0 \pmod 2$

□

We group the initial existence result for Skolem and hooked Skolem sequences, Lemma 1, and Theorem 2 together to give the full set of necessary and sufficient conditions as

Theorem 3. *The generalised grid graph is (maximum) (hooked) Skolem labellable under the following conditions:*

(i) *the path* $P_t \cong P_1 \Box P_t$ *is Skolem labellable if and only if* $t \equiv 0, 1 \pmod 4$ *and hooked Skolem labellable if and only if* $t \equiv 2, 3 \pmod 4$,

(ii) *the ladder graph* $P_2 \Box P_t$ *is Skolem labellable if and only if* $t \equiv 0, 1 \pmod 4$ *and maximum Skolem labellable with two hooks if and only if* $t \equiv 2, 3 \pmod 4$,

(iii) *the grid graph* $P_3 \Box P_3$ *has a hooked Skolem labelling and any other grid graph* $P_s \Box P_t$ *with* $s, t \geq 3$ *has a maximum Skolem labelling with* $st - 2s - 2t + 4$ *hooks.*

3.3 Weak grid graph Skolem labellings

We now prove two lemmata necessary to showing that when $s, t \geq 2$, any grid graph $P_s \square P_t$ has only weak Skolem labellings.[1] We first prove that any $P_s \square P_t$ grid with a hook on a corner or side vertex must have a weak labelling.

Lemma 4. *For $s, t \geq 2$, any maximum Skolem labelling of $P_s \square P_t$ with a hook on a corner or side vertex is weak.*

Proof. Suppose, to the contrary, that there exists a strongly labelled grid $P_s \square P_t$ having a hook, without loss of generality, on its right side. We will show that one of the horizontal edges connecting the side vertices to the rest of the grid does not lie on a labelling path. This edge may therefore be removed without affecting the labelling.

As labelling paths must be of minimal length and there are s vertices on the right side of the grid, there can be at most s labelling paths which enter the right side of the grid on one of the s horizontal edges joining the side to the rest of the grid. Each of these labelling paths must have as one endpoint one of the s vertices on the right side. But there are only $s - 1$ labelled vertices on the side, and hence only $s - 1$ labelling paths for these vertices. Hence, one of the s horizontal edges connecting the side to the rest of the grid does not lie on a labelling path and may be removed without affecting the labelling, thus yielding a contradiction. □

With this lemma, we can now prove that all Skolem labelled ladder graphs are weak.

Theorem 5. *All (hooked) Skolem labelled ladder graphs $P_2 \square P_t$ are weak (hooked) Skolem labelled.*

Proof. We consider cases.

(i) $t \equiv 2, 3 \pmod 4$.
Any labelling requires at least two hooks by Lemma 1. Since each vertex in the graph is a side or corner vertex, the labelling is weak by Lemma 4.

(ii) $t \equiv 0, 1 \pmod 4$.
The maximum label in a $P_2 \square P_t$ ladder graph when $t \equiv 0, 1 \pmod 4$ is t since there is 1 vertical edge in each column of the graph, and $t - 1$ horizontal edges in each row of the graph.

There are only four possible ways to place the t and $t - 1$ labels; notably, the position shown below in Figure 3 and its three images following vertical or horizontal reflection, or 180° rotation. We need only discuss the case depicted in Figure 3. Note that the top-right vertex has no label in the diagram but that any label other than 1, $t - 1$, or t may be placed there.

[1] It was proved in Theorem 12 in [6] that all (hooked) Skolem labelled paths have strong labellings.

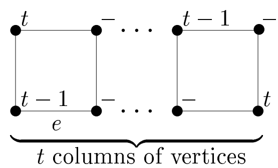


Figure 3: Possible position of t and $t-1$ labels

If the labelling path from the right-hand t label begins horizontally, moving left, then the label in the top-right position has two possibilities for its labelling path. One of these will form part of the labelling path for the top-right vertex, but there is no way for another labelling path to use the other edge. This would create a weak labelling.

Hence, the labelling path emanating from the right-hand t label must use the right-most vertical edge of the graph and then travel left along the upper horizontal edges of the graph. Because of this, the path reaches the left-hand t label on a horizontal edge; there is no way that the path can use the left-most vertical edge of the graph, i.e., the edge joining the left-hand t and $t-1$ labels.

The only other labelling path that may use this left-most vertical edge is the path between the two $t-1$ labelled vertices. To create a strong labelling, this is necessary. However, if the labelling path between the two $t-1$ labelled vertices uses that edge, it begins from the left-hand $t-1$ labelled vertex and proceeds upwards to the top-left vertex of the graph and then right along the upper horizontal edges of the graph. However, this leaves the left-most bottom horizontal edge e unlabelled, as the only two labelling paths which may use the edge are the paths joining the $t-1$ and t labelled vertices, and from above, these paths do not use it.

Hence, no $P_2 \square P_t$ ladder graph may be strong Skolem labelled. □

We can now prove that there can be no strong Skolem labelling of a $P_s \square P_t$ grid graph with $s, t \geq 2$.

Theorem 6. *There exists no strong Skolem or strong maximum Skolem labelled $P_s \square P_t$ grid graph for $s, t \geq 2$.*

Proof. We assume $s, t > 2$ by Theorem 5.

Note that the maximum path length in a $P_s \square P_t$ grid is $s+t-2$ and that there are $st-2s-2t+4$ hooks in the grid. To have a strong labelling, none of these hooks may sit on a side or corner vertex of the graph by Lemma 4. Note that there are $(s-2)\cdot(t-2) = st-2s-2t+4$ central (that is, non-side or corner) vertices in a $P_s \square P_t$ grid; exactly the number of hooks. Hence, every central vertex of the grid must be used by a hook for the grid to have a strong labelling.

This forces all labels to sit on the outside vertices of the graph and those vertices only. Specifically, the two 1s must sit on outside vertices of the grid.

Without loss of generality, suppose the two 1-labelled vertices are on the right side of the grid, so that there are s horizontal edges joining the vertices of the right side to the rest of the grid. For the labelling to be strong, each of these horizontal edges must lie on a labelling path having one endpoint among the s vertices of the right side.

However, the 1-labelled vertices have one edge joining them and so their labelling path uses none of these horizontal edges joining the right side to the rest of the graph. Hence, at most $s-2$ of the horizontal edges connecting the right side to the rest of the graph are used by labelling paths. Since two edges have not been used by labelling paths, the labelling must be weak. □

4 New data

In [6] it was shown that any tree can be embedded in a Skolem labelled tree with $O(v)$ vertices and any graph can be embedded in a Skolem labelled graph with $O(v^3)$ vertices. We note that this graph is also a tree-like graph (with long branches to accommodate the Skolem labels). These embedding results demonstrate the importance of Skolem labellings of trees for generating Skolem labellings of other graphs. For this reason, an exhaustive study of the Skolem labellings for trees of small order was undertaken.

There were two major steps in the gathering of the data to generate Skolem, hooked Skolem, or maximum Skolem labellings for trees of small orders. From a given file of all trees of order n, an initial `C` program generated all possible trees of order $n+1$ using a leaf-adding function. To remove excess isomorphic copies arising from this process, these trees were then parsed after having been assigned canonical labellings using an implementation of Brendan McKay's NAUTY program.

A second `C` program, using Algorithm 1 shown below, then attempted to either Skolem or hooked Skolem, and if necessary, maximum Skolem label each tree using an exhaustive backtrack approach. Each tree's candidacy was considered first for (hooked) Skolem labelling with largest label size equal to $\lfloor \frac{n}{2} \rfloor$; if this failed, maximum Skolem labellings were attempted using decreasing maximum labels $\lfloor \frac{n}{2} \rfloor - i$ so as to generate the maximum Skolem labelling with largest possible labels.

At each labelling level, the diameter of the tree was first checked; if it was strictly smaller than the largest label, then the tree could not be labelled at that level. If the diameter was at least equal to the largest label, the parity[2] of the tree would be checked if the tree was of even order.

[2]The *Skolem parity* of a vertex u in a tree $T=(V,E)$ is the modulus two sum $\sum_{v\in V} d(u,v) \pmod 2$ of the distances between u and all other vertices $v \in V$. If $|V|$ is even, the parity is independent of u and is denoted the Skolem parity of the tree [7].

Algorithm 1.

```
Given: Tree T, V(T) = {1,2,...,n}, LABEL = ⌊n/2⌋, LABELLED = false.
All vertices are given an initial label value of 0 since hooks
will be labelled with a 0.

  for ((u = 1,...,n) and (LABELLED == false))
      if ((eccentricity of u ≥ LABEL) and (u is unlabelled))
         for ((v = 1,...,n) and (LABELLED == false))
             if ((d(u,v) == LABEL) and (v is unlabelled))
               {
               assign LABEL to u and v
               if (LABEL == 1)
                 {
                 LABELLED = true
                 output the labelling
                 }
               else
                 recurse with LABEL = LABEL - 1
               unlabel u and v
               }
  return(LABELLED)
```

Algorithm 1: Tree Skolem labelling algorithm

Any tree of order $n = 2k$ with even parity and $k \equiv 0, 3 \pmod 4$ was still potentially Skolem labellable, as was any tree of order $n = 2k$ with odd parity and $k \equiv 1, 2 \pmod 4$ [7]. If this was the case, then a standard backtrack labelling attempt would start. If the labelling attempt failed, the labelling level would decrease by one, and the process would repeat.

We present a table containing labelling information for trees up to 20 vertices in the Appendix. This table lists alongside the number of trees of each order the number of (hooked) Skolem labellable trees, the number of strong and weak Skolem labellings, and the number of strong and weak maximum Skolem labellings.

Note that there are many more weak hooked Skolem labellings for odd graphs than for their even counterparts. This has to do with the fact that there is one more edge on an odd tree than an even tree, when compared with the number of labels used, hence more labelling possibilities.

Note also that there are more hooked Skolem labelled trees for odd orders than there are Skolem labelled trees for the corresponding even orders. This also has to do with the fact that it is easier to label a tree that has an extra vertex and edge. In fact, the odd orders have more labelled trees than the even orders one greater than them, which suggests that it is much easier to find hooked Skolem labellings on odd trees than it is to find Skolem labellings on even trees.

5 Conclusion

We group Theorems 3 and 6 together to produce the final

Theorem 7. *The generalised grid graph is (weak) (maximum) (hooked) Skolem labellable under the following conditions:*

(i) *the path* $P_t \cong P_1 \Box P_t$ *is strong Skolem labellable if and only if* $t \equiv 0, 1 \pmod 4$ *and strong hooked Skolem labellable if and only if* $t \equiv 2, 3 \pmod 4$,

(ii) *the ladder graph* $P_2 \Box P_t$ *is weak Skolem labellable if and only if* $t \equiv 0, 1 \pmod 4$ *and weak maximum Skolem labellable with two hooks if and only if* $t \equiv 2, 3 \pmod 4$,

(iii) *the grid graph* $P_3 \Box P_3$ *is weak hooked Skolem labellable and any other grid graph* $P_s \Box P_t$ *with* $s, t \geq 3$ *is weak maximum Skolem labellable with* $st - 2s - 2t + 4$ *hooks.*

Two-dimensional grid graphs are the second class of graphs whose Skolem labelling has been fully classified, after the windmills, which Mendelsohn and Shalaby completed in [7]. A simple generalisation of Theorem 7 would be to k-dimensional grids $P_{s_1} \Box P_{s_2} \Box \cdots \Box P_{s_k}$. As such graphs would contain $s_1 s_2 \cdots s_k$ vertices, the minimum number of unlabelled vertices, clearly a positive number, would be

$$s_1 s_2 \cdots s_k + 2k - 2(s_1 + s_2 + \cdots + s_k).$$

It may prove useful to incorporate the new ideas of *weak* and *maximum* Skolem labellings into the existing literature. In this vein, another obvious problem to tackle would be expanding the windmill results. It would be then hoped that all new Skolem labelling results could include, if possible, these concepts.

A major unresolved question is the missing necessary condition for Skolem labelling trees. It is hoped that the data presented herein should help in the search for finding such a necessary condition. At the moment, this is one of the main unsolved problems in the area of Skolem labelled graphs.

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Appendix

n	Number of trees	Number of (hooked) Skolem labellable trees	Skolem labellings[a]		Maximum Skolem labellings[b]	
			Total strong	Total weak	Total strong	Total weak
2	1	1	1	1	– –	– –
3	1	1	0	2	– –	– –
4	2	1	3	0	0	3
5	3	3	0	18	– –	– –
6	6	2	12	0	0	53
7	11	10	28	138	0	60
8	23	10	153	10	0	912
9	47	42	968	1 876	0	1 524
10	106	41	2 060	108	1 190	23 050
11	235	193	22 795	29 893	276	70 742
12	551	174	34 062	1 828	81 897	793 292
13	1 301	978	618 798	639 180	82 129	3 043 643
14	3 159	848	730 187	43 744	3 401 517	27 663 559
15	7 741	5 102	17 779445	16 018 981	7 438 391	151 125 635
16	19 320	4 408	18 767 052	1 153 094	158 281 229	1 325 275 250
17	48 629	27 380	566 227 749	463 907 727	461 589 964	8 469 010 380
18	123 867	23 140	?	?	?	?
19	317 955	148 748	?	?	?	?
20	823 065	123 369	?	?	?	?

[a]Total number of distinct (hooked) Skolem labellings of all (hooked) Skolem labellable trees for each order.

[b]Total number of distinct maximum labellings for all trees which cannot be Skolem labelled.

Table 1: Skolem label data

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