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The d-relaxed game chromatic index of k-degenerated graphs

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Abstract

The d-relaxed game chromatic number of a graph G is the minimum number k such that Alice has a winning strategy in the following game: Alice and Bob alternately colour the vertices of G with colours from a set X with $|X| = k$. We say a colour i is legal for an uncoloured vertex x if after colouring x with colour i, no vertex of colour i is adjacent to more than d vertices of colour i. Both players can only colour vertices with legal colours. Bob wins the game if at a certain step, there is an uncoloured vertex which has no legal colour. Alice wins the game if all the vertices are coloured with legal colours. The d-relaxed game chromatic index of a graph G denoted by d-$\chi_g'(G)$ is the d-relaxed game chromatic number of the line graph of G. This paper studies the d-relaxed game chromatic index of k-degenerate graphs. Suppose G is a k-degenerate graph with maximum degree Δ. This paper proves that if $d \geq 2k^2 + 5k - 1$, then d-$\chi_g'(G) \leq 2k + \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$.

1 Introduction

Let G be a finite graph, X a set of colours and d a non-negative integer. A d-relaxed colouring game on G with colour set X is played by two players Alice and Bob. The players take turns colouring the vertices of G with legal colours from X, with Alice moves first. Here a colour i is a legal colour for an uncoloured vertex v, if after colouring v with colour i, each vertex of colour i has at most d neighbours of colour i. In other words, colour i is legal for an uncoloured vertex v if (1) v has at most d

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neigbours coloured by colour i, and (2), if v' is a neighbour of v coloured by colour i, then v' has at most $d-1$ neighbours (other than v) coloured by colour i. If all the vertices are coloured after $|V(G)|$ turns, then Alice wins the game. Otherwise, at a certain step, some uncoloured vertices have no legal colour. In this case, Bob wins the game. So Alice's goal is to produce a legal colouring of all vertices of G and Bob tries to prevent this from happening. The *d-relaxed game chromatic number* d-$\chi_g(G)$ of a graph $G=(V,E)$ is the least cardinality of a colour set X for which Alice has a winning strategy in the d-relaxed colouring game on G. This parameter is well-defined, since Alice always wins if $|X| \geq |\Delta(G)|+1$. In case $d=0$, then the 0-relaxed colouring game is called a colouring game. The 0-relaxed game chromatic number is called the *game chromatic number* and is denoted by $\chi_g(G)$.

The game chromatic number and the relaxed game chromatic number of various classes of graphs have been studied in the literature [9, 13, 16, 17, 18, 19]. In this paper, we are interested in the relaxed game chromatic number of line graphs. Given a graph G, let $L(G)$ be the line graph of G. The d-relaxed game chromatic number of $L(G)$ is called the *d-relaxed game chromatic index* of G and denoted by d-$\chi'_g(G)$.

The d-relaxed game chromatic index of graphs have been studied in [1, 2]. The main focus has been on k-degenerate graphs. A graph $G=(V,E)$ is call *k-degenerate* if there is a linear ordering L of $V(G)$ such that for each vertex x of G, the number of neighbors of x that precede x in L is less than or equal to k. Suppose G is a k-degenerate graph of maximum degree $\Delta(G)$. It was proved in [1] that $\chi'_g(G) \leq \Delta+3k-1$, and proved in [2] that d-$\chi'_g(G) \leq \Delta+k-1$ if $d \geq 2k^2+4k$. Moreover, if T is a forest, i.e., 1-degenerate, then d-$\chi'_g(T) \leq \Delta+1$ if $d \geq 1$ and d-$\chi'_g(T) \leq \Delta$ if $d \geq 3$ in [2].

All the upper bounds for the d-relaxed game chromatic index of a k-degenerate graph are greater than the maximum degree. However, it is natural that when d is large, the d-relaxed game chromatic index of G should be much smaller than $\Delta(G)$. In this paper, we prove an upper bound for d-$\chi'_g(G)$ which improves earlier upper bounds for large d and $\Delta(G)$. Suppose G is a k-degenerate graph. The main result of this paper is that if $d \geq 2k^2+5k-1$, then d-$\chi'_g(G) \leq 2k + \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$.

2 Alice's winning strategy

In this section, we prove the main result:

Theorem 1 *Suppose G is a k-degenerate with maximum degree Δ. If $d \geq 2k^2+5k-1$, then d-$\chi'_g(G) \leq 2k + \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$.*

Let $m = 2k + \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$. To prove Theorem 1, we need to give a winning strategy for Alice in the d-relaxed edge colouring game on G with a colour set of size m. In the following, $G=(V,E)$ is a k-degenerate graph, and X is a colour set with $|X|=m$.

First we define some terms needed in the description and the proof of the correctness of the strategy. Let L be a linear ordering of the vertices of G such that for any vertex $v \in V$, $|N^+(v)| \leq k$. Here $N^+(v) = \{x : x <_L v, x \sim v\}$ is the set of neighbours of v that precede v in the ordering L. The linear ordering L induces an orientation of G: an edge $e = xy$ is oriented from x to y if $y <_L x$. For simplicity, we also use $G = (V, E)$ to denote the orientation of G, but each element of E is an ordered pair (x, y) and is called an *arc* of G. So in the colouring game, the players are colouring the arcs of G. For a vertex u of G, let $N_E^+(u) = \{(u, v) : (u, v) \in E\}$ and let $N_E^-(u) = \{(v, u) : (v, u) \in E\}$ and let $N_E(u) = N_E^+(u) \cup N_E^-(u)$. We extend the linear ordering L to the arc set E as follows: Suppose $e = (x, y)$ and $e' = (x', y')$. We write $e <_L e'$ if either $y <_L y'$ or $y = y'$ and $x <_L x'$.

Suppose the arcs of G are partially coloured. We shall denote by C the set of coloured arcs and denote by U the set of uncoloured arcs. For a coloured arc $e \in C$, we shall denote by $c(e)$ the colour of e. For a set F of arcs, let $c(F) = \{c(e) : e \in F \cap C\}$. For an arc e, $N(e)$ denote the set of arcs adjacent to e (in the line graph of G).

We say a colour α is *permissible* to an arc (x, y) if α is a legal colour to (x, y) and moreover $\alpha \notin c(N_E^+(x) \cup N_E^+(y))$. Alice will colour arcs with permissible colours only. However, Bob may colour an arc with a non-permissible colour (but still Bob can only colour arcs with legal colours).

During the play of the game, Alice will keep record of a set A of *active arcs*. When an arc e is put into A, we say that e is *activated*. Once an arc is activated, it will remain active forever. Initially Alice activates and colours the minimum arc (with respect to the linear order L).

Suppose Bob has just coloured an arc (z, x) with colour α. Let

$$\begin{aligned}
Y &= \{y \in N^+(x) : (x, y) \in C \text{ and } c((x, y)) = \alpha\} \\
Y' &= \{y \in Y : N_E^+(y) \subset C \text{ or } \alpha \in c(N_E^+(y))\} \\
Y^* &= \{y \in Y : \alpha \notin c(N_E^+(y)) \text{ and } \exists e \in N_E^-(y) \cap U \\
&\qquad\qquad \text{such that } \alpha \text{ is permissible to } e\}.
\end{aligned}$$

Alice will first activate arc (z, x) if it is not active yet, i.e., let $A := A \cup \{(z, x)\}$. Then starts from (z, x), Alice searches the arc to be coloured.

1. If $N_E^+(x) \cap U = \emptyset$ and $Y' = Y$ and $Y^* \neq \emptyset$, then let y^* be an arbitrary vertex of Y^*. Let h be an arc in $N_E^-(y^*) \cap U$ whose permissible colours contain α. Colour h with colour α and activate h if it is not active yet.

2. If $N_E^+(x) \cap U = \emptyset$ and $Y' = Y$ and $Y^* = \emptyset$, then colour the minimum uncoloured arc h with an arbitrary permissible colour and activate h if it is not active yet.

3. If $N_E^+(x) \cap U \neq \emptyset$, then Alice *jump to* the minimum uncoloured arc, say e, of $N_E^+(x)$ and go to the recursive stage. If $N_E^+(x) \cap U = \emptyset$ and $Y \neq Y'$, then let $y = \min Y \setminus Y'$ and Alice jumps to the minimum uncoloured arc, say e, of

$N_E^+(y)$ and go to the recursive stage. In the former case, we say (z,x) made a contribution to e, and e received one contribution from (z,x); in the latter case we say (x,y) made a contribution to e, and e received one contribution from (x,y).

Recursive Stage

- Assume Alice arrived at an arc (a,b). If (a,b) is active, or (a,b) is not active but $N_E^+(b) \cap U = \emptyset$, then colour (a,b) with an arbitrary permissible colour (and activate it if it is not active yet). If (a,b) is not active and $N_E^+(b) \cap U \neq \emptyset$, then activate (a,b) and jumps to the minimum uncoloured arc, say e, of $N_E^+(b)$, return to the recursive stage. In this case we say (a,b) made a contribution to e.

This completes the description of Alice's strategy. Now we show that this is a winning strategy for Alice. For this purpose, it suffices to show that any uncoloured arc has a permissible colour. Assume to the contrary that $e = (a,b)$ is an uncoloured arc and e has no permissible colour. Let $X' = X \setminus (c(N_E^+(a) \cup N_E^+(b)))$. A colour $\alpha \in X'$ is not permissible to e only if either there are $d+1$ arcs adjacent to e that are coloured by α (i.e., $|c^{-1}(\alpha) \cap (N_E(a) \cup N_E(b))| \geq d+1$), or there is an arc e' adjacent to e such that e' is coloured by colour α and moreover, e' is adjacent to d other arcs of colour α. Let

$$\begin{aligned} X_1' &= \{\alpha \in X' : |c^{-1}(\alpha) \cap N(e)| \geq d+1\} \\ X_2' &= \{\alpha \in X' : \exists e' \in N_E^-(b), c(e') = \alpha, |c^{-1}(\alpha) \cap N(e')| \geq d\}. \\ X_3' &= \{\alpha \in X' : \exists e' \in N_E^-(a), c(e') = \alpha, |c^{-1}(\alpha) \cap N(e')| \geq d\}. \end{aligned}$$

Since all the colours of X' are not permissible to e, we have $X' = X_1' \cup X_2' \cup X_3'$. Since $|X| = m = 2k + \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$ and $|c(N_E^+(a) \cup N_E^+(b))| \leq 2k-1$ (note that $(a,b) \in N_E^+(a)$ and (a,b) is uncoloured), to derive a contradiction, it suffices to show that $|X'| \leq \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$.

Lemma 1 *If $N_E^+(x)$ contains uncoloured arcs, then $|N_E^-(x) \cap C| \leq 2k$.*

Proof. As $N_E^+(x)$ contains uncoloured arcs, when an arc $e \in N_E^-(x)$ is activated, it will make a contribution to an arc in $N_E^+(x)$. Each arc can receive at most 2 contributions: the first time an arc receives a contribution, it is activated, the second time an arc receives a contribution, then it is coloured, and coloured arcs will not receive any contribution. As $|N_E^+(x)| \leq k$ and $N_E^+(x)$ contains uncoloured arcs, the arcs in $N_E^+(x)$ received at most $2k$ contributions. As each coloured arc is active, $N_E^-(x)$ contains at most $2k$ coloured arcs. ∎

By Lemma 1, we have $N_E^-(a)$ contains at most $2k$ coloured arcs. To derive an upper bound for the cardinality of X_2' and X_3', a more careful analysis is needed.

Lemma 2 *Suppose (w', w) is an arc coloured by colour α. If $N_E^-(w')$ contains $2k^2 + 2k$ arcs coloured by colour α, then all the arcs in $N_E^+(w)$ are coloured.*

Proof. By Lemma 1, when the first $2k$ arcs of $N_E^-(w')$ are activated, all the arcs of $N^+(w')$ are coloured. After all the arcs of $N^+(w')$ are coloured, each time another arc of $N_E^-(w')$ is coloured by colour α, one of the arcs in $N_E^+(v)$ is activated or coloured, where $v \in N^+(w')$ and (w', v) is coloured by colour α. Since the number of such arcs in the set $\cup_{v \in N^+(w'), c((w',v))=\alpha} N_E^+(v)$ is at most k^2, it follows that when $2k^2 + 2k$ arcs of $N_E^-(w')$ are coloured by colour α, then after Alice finish her move, all the arcs of $N_E^+(v)$ are coloured for all $v \in N^+(w')$ with $c((w', v)) = \alpha$. In particular, all the arcs in $N_E^+(w)$ are coloured. ∎

Corollary 1 *If $\gamma \in X_3'$, then $N_E^-(a)$ contains at least $d - 2k^2 - 2k + 1$ arcs of colour γ.*

Proof. Assume $\gamma \in X_3'$. Then there is an arc $e' = (w, a) \in N_E^-(a)$ such that $c(e') = \gamma$, $|c^{-1}(\gamma) \cap N(e')| \geq d$. By Lemma 2, $|N_E^-(w) \cap c^{-1}(\gamma)| \leq 2k^2 + 2k - 1$ (as the arc (a, b) is uncoloured). Therefore $N_E^-(a)$ contains at least $d - 2k^2 - 2k + 1$ arcs of colour γ. ∎

Suppose $\alpha \in X$ is a colour, u is vertex of G. An arc (u, v) is called a *α-saturated arc* (respectively, *nearlyα-saturated arc* if the following hold:

1. $c((u, v)) = \alpha$ and (u, v) is adjacent to d (respectively, $d - 1$) arcs of colour α.
2. $N_E^+(v) \cap c^{-1}(\alpha) = \emptyset$.
3. There exists $w \in N^-(v)$ such that (w, v) is uncoloured and $N_E^+(w) \cap c^{-1}(\alpha) = \emptyset$.

Lemma 3 *Assume (u, v) is a nearly α-saturated arc and $N^-(v)$ contains no α-saturated arc. Then $N_E^-(v)$ contains at least $\lceil \frac{d-2k^2-4k+2}{k+1} \rceil$ arcs of colour α.*

Proof. Let $v_1, v_2, \cdots, v_t$ be all the out neighbours of u such that $c((u, v_i)) = \alpha$, where $v = v_1$. By definition, there is an arc $(w, v) \in N_E^+(w)$ which is uncoloured. By Lemma 2 that $|N_E^-(w) \cap C| \leq 2k$. Moreover, if (w', w) is an arc in $N_E^-(w)$ coloured by colour α, then by Lemma 2, $N_E^-(w')$ contains at most $2k^2 + 2k - 1$ arcs coloured by colour α. As $N^+(w')$ contains at most $k - 1$ arcs other than (w', w), we conclude that (w', w) is adjacent to at most $2k^2 + 5k - 2 < d$ arcs of colour α. As (u, v) is a nearlyα-saturated arc and $N^-(v)$ contains no α-saturated arc, it follows that no arc in $N^-(v)$ of colour α is adjacent to d arcs of colour α. This implies that α is a permissible colour for (w, v). So after all the arcs of $\cup_{j=1}^t N^+(v_j)$ are coloured, whenever an arc $e \in N_E^-(u)$ is coloured by colour α, Alice applies choice (1) in her strategy. Hence some arcs in $\cup_{j=1}^t N_E^-(v_j)$ will be coloured by colour α. Assume $N_E^-(u)$ contains p arcs coloured by colour α, and for $j = 1, 2, \cdots, t$, let q_j be the number of arcs in $N^-(v_j)$ that are coloured by colour α. By Lemma 2, when the first $2k^2+2k$ arcs in $N^-(u)$ are coloured by colour α, all the arcs in $\cup_{j=1}^t N^+(v_j)$ are coloured. Therefore, $\sum_{j=1}^t q_j \geq p - 2k^2 - 2k$.

The number of arcs of colour α adjacent to (u,v) is $p+t-1+q_1-1$ (note that (u,v) contributes 1 to the quantity q_1). Since (u,v) is adjacent to $d-1$ arcs of colour α, we have $p+q_1+t-1=d$. Similarly, we have $p+q_j+t-1 \leq d+1$ for all j (for otherwise (u,v_j) is adjacent to more than d arcs of colour α). Thus we have $q_1+1 \geq q_j$. Hence

$$\begin{aligned} q_1+1 &\geq \frac{1+\sum_{j=1}^t q_j}{t} \geq \frac{p-2k^2-2k+1}{t} \\ &= \frac{d-q_1-t+1-2k^2-2k+1}{t} \geq \frac{d-q_1-2k^2-3k+2}{k}. \end{aligned}$$

This implies that $q_1 \geq \frac{d-2k^2-4k+2}{k+1}$. ∎

Corollary 2 *If $\alpha \in X_2'$, then $N_E^-(b)$ contains at least $\lceil \frac{d-2k^2-4k+2}{k+1} \rceil$ arcs of colour α.*

Proof. Assume $\alpha \in X_2'$. It follows from the definition that there is an arc (w,b) which is an α-saturated arc. Assume (w,b) is the first arc in $N^-(b)$ that becomes α-saturated. Before the last arc adjacent to (w,b) is coloured by α, (w,b) was a nearlyα-saturated arc. Hence the conclusion follows from Lemma 3. ∎

Now we can derive an upper bound for $|X'|$.

Corollary 3 $|X'| \leq \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$.

Proof. For each colour $\alpha \in X_1'$, it follows from the definition that there are at least $d+1$ arcs in $N_E^-(a) \cup N_E^-(b)$ that are coloured by colour α. For each colour $\beta \in X_2'$, by Corollary 2, there are at least $\frac{d-2k^2-4k+2}{k+1}$ arcs in $N_E^-(b)$ coloured by colour β. For each colour $\gamma \in X_3'$, by Corollary 1, there are at least $d-2k^2-2k+1$ arcs in $N_E^-(b)$ coloured by colour γ. As $N_E^-(b)$ contains at most $\Delta - |N_E^+(b)| - 1$ coloured arcs and $N_E^-(a)$ contains at most $2k$ coloured arcs, we conclude that

$$\begin{aligned} &\Delta - |N_E^+(b)| - 1 + 2k \\ &\geq |X_1'|(d+1) + |X_2'|\frac{d-2k^2-4k+2}{k+1} + |X_3'|(d-2k^2-2k+1) \\ &\geq (|X_1'|+|X_2'|+|X_3'|)\frac{d-2k^2-4k+2}{k+1}. \end{aligned}$$

Since $|N_E^+(b)| \leq k$, we have $|X'| \leq \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$. ∎

Thus we have proved that the strategy described at the beginning of this section is a winning strategy for Alice. This completes the proof of Theorem 1.

How good is the upper bound in Theorem 1? Our next result shows that if $k=1$, then the upper bound is not too far from the lower bound.

Theorem 2 *For any positive integer $d \leq \Delta - 2$, there is a tree T with maximum degree Δ for which d-$\chi'_g(T) \geq \frac{2\Delta}{d+3}$.*

Proof. Before proving this result, we observe that if $d \geq 6$, then it follows from Theorem 1 that d-$\chi'_g(T) \leq 2 + \frac{2\Delta}{d-4}$. So the upper bound and the lower bound are not too far away.

Let $n = \Delta$ and let T be the rooted tree with root vertex v_0 which has n sons, $u_1, u_2, \cdots, u_n$, and each of u_i $(i = 1, 2, \cdots, n)$ has $n - 1$ sons.

Let $m = \lceil \frac{2n}{d+3} \rceil - 1$ and suppose the game is played on T with colour set $X = \{1, 2, \cdots, m\}$. Bob's strategy is to make sure that the following is true:

(*): For each colour $i \in X$, there is a vertex $b_i \in \{u_1, u_2, \cdots, u_n\}$ such that (b_i, v_0) is coloured by colour i and $|N_E^-(b_i) \cap c^{-1}(i)| \geq |N_E^-(v_0) \cap c^{-1}(i)| - 2$.

Assume (*) holds. Let i_0 be the colour used the most number of times on the arcs $N_E^-(v_0)$. Then $|N_E^-(v_0) \cap c^{-1}(i_0)| \geq n/m$. By (*), this implies that (b_{i_0}, v_0) is coloured by colour i_0, and $|N_E^-(b_{i_0}) \cap c^{-1}(i_0)| \geq n/m - 2$. Thus (b_{i_0}, v_0) is coloured by colour i_0 and adjacent to at least $(n/m - 1) + (n/m - 2) = 2n/m - 3$ edges of colour i_0. As the colouring is a d-relaxed edge colouring of T, we conclude that $d \geq 2n/m - 3$, and hence $m \geq 2n/(d+3)$, which is a contradiction.

To ensure that (*) always holds, Bob needs to find, for each $i \in \{1, 2, \cdots, m\}$, a vertex $b_i \in \{u_1, u_2, \cdots, u_n\}$ for which (*) holds. Let $I \subseteq \{1, 2, \cdots, m\}$ be the set of indices such that b_i has been found already. Initially $I = \emptyset$, and eventually I will be the whole set $\{1, 2, \cdots, m\}$.

Suppose Alice has just coloured (x, y) with colour j and it is Bob's turn. Let $I' = \{i \in I : |N_E^-(b_i) \cap c^{-1}(i)| + |N_E^-(v_0) \cap c^{-1}(i)| < d + 1\}$.

1. If $j \notin I$ and $(x, y) = (u_r, v_0) \in N_E^-(v_0)$ for some r, then Bob colours an uncoloured arc of $N_E^-(u_r)$ with colour j and let $b_j := u_r$.

2. If $j \in I'$ and $(x, y) \in N_E^-(v_0)$, then Bob colours an uncoloured arc of $N_E^-(b_j)$ with colour j.

3. If $(x, y) \in N_E^-(b_i)$ for some $i \in I'$, then Bob colours an uncoloured arc of $N_E^-(b_i)$ with colour i.

4. Assume (1), (2) and (3) do not apply. If there is an index $i \notin I$, then let $R = \{r : (u_r, v_0) \in U\}$ and choose $r \in R$ such that $N_E^-(u_r)$ contains the least number of coloured arcs and colour (u_r, v_0) with colour i and let $b_i := u_r$.

5. Assume (1), (2), (3) and (4) do not apply. If $I' \neq \emptyset$, then choose $i \in I'$ and colours an uncoloured arc of $N_E^-(b_i)$ with colour i.

6. Assume all the above rules do not apply. Then arbitrarily colours an uncoloured arc with a legal colour, if possible.

With this strategy, for each i, (b_i, v_0) is the first edge of $N_E^-(v_0)$ that is coloured by colour i. At the time b_i is defined, $|N_E^-(b_i) \cap c^{-1}(i)| \geq |N_E^-(v_0) \cap c^{-1}(i)| - 1$. After b_i is defined, whenever Alice colours an edge of $N_E^-(v_0)$ with colour i, Bob immediately coloured an edge of $N_E^-(b_i)$ with colour i, provided that $|N_E^-(b_i) \cap c^{-1}(i)| + |N_E^-(v_0) \cap c^{-1}(i)| < d$. This implies that (*) always hold. This completes the proof of Theorem 2. ∎

3 Colouring k-degenerate graphs with 2 colours

In the previous section, the number of colours needed in the edge colouring game for k-degenerated graphs is always greater than $2k$. On the other hand, it is natural that if d is close to the maximum degree, the the d-relaxed game chromatic index will be close to 2. In this section, we consider the problem how big should be d so that d-$\chi'_g(G) = 2$. For this purpose, we consider a slightly different game: Suppose G is a graph and k is a positive integer. Alice and Bob take turns colouring the edges of G with colours from a set X of k colours. After all the edges are coloured, the *deficiency* $\delta(e)$ of an edge e is the number of edges that are coloured the same colour as e and are adjacent to e. The *score* of the game s is defined to be the maximum deficiency of all edges. In other words, the score s is the minimum d such that the resulting graph is a d-relaxed edge colouring of G. Alice's goal is to minimize the score of the game and Bob's goal is to maximize it. Let $\mu_k(G)$ be the minimum s for which Alice has a strategy to ensure that the score of the game is at most s.

Lemma 4 *If $\mu_k(G) = s$, then for $d \geq s$, d-$\chi'_g(G) \leq k$.*

Proof. Alice simply uses her winning strategy in the later colouring game which ensures the score is at most d. This is also a winning strategy for the former d-relaxed colouring game on G with k colours. ∎

The converse of Lemma 4 is not true (see example at the end of the paper) . A winning strategy for Alice to win the d-relaxed colouring game on G with k colours is not necessarily a winning strategy for her to win the later colouring game, because Bob may have chances to increase the deficiency of an edge in the later game, which is illegal in the d-relaxed colouring game.

Theorem 3 *Let G be k-degenerate with maximum degree Δ. Then $\mu_2(G) \leq 2k + 2\lfloor \frac{\Delta - k}{2} \rfloor + 1$.*

Proof. Let $X = \{1, 2\}$ and L be the linear ordering of $V(G)$ in Theorem 1. In her first turn, Alice colours an arbitrary edge. Assume Bob colours (x, y). If $N_E^-(y) \cap U \neq \emptyset$, Alice chooses a colour i such that $|N_E^-(y) \cap c^{-1}(i)| \leq |N_E^-(y) \cap c^{-1}(3 - i)|$ and colours an arc $(z, y) \in N_E^-(y) \cap U$ by colour i. Otherwise Alice arbitrarily choose a vertex z for which $N_E^-(z) \cap U \neq \emptyset$, and choose a colour i for which $|N_E^-(z) \cap c^{-1}(i)| \leq$

$|N_E^-(z) \cap c^{-1}(3-i)|$, and colours an arc $(w,z) \in N_E^-(z) \cap U$ by colour i. It is easy to see that using this strategy, for any vertex z, $-2 \leq |N_E^-(z) \cap c^{-1}(i)| - |N_E^-(z) \cap c^{-1}(3-i)| \leq 2$ at any time. Let $|N_E^+(x)| = k_1$ and $|N_E^+(y)| = k_2$. To show that the resulting colouring is legal, we observe that for any arc (x,y) of colour i, $N_E^+(x)$ and $N_E^+(y)$ contains, respectively, at most $\lfloor \frac{\Delta-k_1}{2} \rfloor + 1$ and $\lfloor \frac{\Delta-k_2}{2} \rfloor + 1$ arcs that are coloured by colour i. Together with the arcs in $N_E^+(x) \cup N_E^+(y)$, there are at most $k_1 + k_2 + \lfloor \frac{\Delta-k_1}{2} \rfloor + \lfloor \frac{\Delta-k_2}{2} \rfloor + 1$ arc adjacent to (x,y) that are coloured the same colour as (x,y). Since $k_1, k_2 \leq k$. We have $k_1 + k_2 + \lfloor \frac{\Delta-k_1}{2} \rfloor + \lfloor \frac{\Delta-k_2}{2} \rfloor \leq 2k + 2\lfloor \frac{\Delta-k}{2} \rfloor + 1$. So the resulting colouring is indeed a d-relaxed edge colouring of G. ∎

Corollary 4 *Let G be k-degenerate with maximum degree Δ. If $d \geq 2k+2\lfloor \frac{\Delta-k}{2} \rfloor + 1$, then d-$\chi'_g(G) \leq 2$.*

It can be proved that for the tree T constructed in the proof of Theorem 2, we have $\mu_k(T) \geq n-1$ for $k < n/2$. But d-$\chi'_g(T)$ is roughly $2n/d$. This shows that the converse of Lemma 4 is not true.

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