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Total k-subdominating functions on graphs

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Abstract

A two-valued function f defined on the vertices of a graph $G = (V, E)$, $f: V \to \{-1, 1\}$, is an opinion function. For each vertex v of G, the vote of v is the sum of the function values of f over the open neighborhood of v. A total k-subdominating function (TkSF) of a graph G is an opinion function for which at least k of the vertices have a vote value of at least one. The total k-subdomination number, $\gamma^t_{ks}(G)$, of G is the minimum value of $f(V)$ over all TkSFs of G where $f(V)$ denotes the sum of the function values assigned to the vertices under f. We give a lower bound on $\gamma^t_{ks}(G)$ in terms of the minimum degree, maximum degree and the order of G. A lower bound on $\gamma^t_{ks}(G)$ in terms of the degree sequence of G is given. Lower and upper bounds on $\gamma^t_{ks}(G)$ for a tree G are presented.

1 Introduction

In this paper we consider a model of situations in which a network of people must make a global yes/no decision. We assume that each individual has one vote and that

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each individual has an initial opinion. We also assume, however, that an individual's vote is affected by the opinions of neighboring individuals. In particular, each individual gives equal weight to the opinions of neighboring individuals (thus individuals of high degree have greater "influence"). A global decision requires at least k of the votes. Similar topics are considered in [1, 2, 5, 6, 13], but in those papers the assumption is that each individual gives equal weight to his/her own opinion and to the opinions of neighboring individuals.

Our mathematical model is a finite, simple graph $G = (V, E)$ with vertex set V and edge set E of order $n(G) = |V|$ and size $m(G) = |E|$. The *open neighborhood* of a vertex v is $N(v) = \{u \mid uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. An *opinion function* on G is a function $f: V \to \{-1, +1\}$; $f(v)$ is the *opinion* of the vertex v. The *weight* $w(f)$ of an opinion function f is the sum of its values, i.e., $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. For a vertex v in V, we denote $f(N(v))$ by $f[v]$ for notational convenience. For $S \subseteq V$, we denote the subgraph induced by S in the graph G by $G[S]$.

For a vertex v in a rooted tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. The set of leaves in T is denoted by $L(T)$. We define a *branch vertex* as a vertex of degree at least 3. The set of branch vertices of T is denoted by $B(T)$.

For a positive integer k, Cockayne and Mynhardt [5] define a *k-subdominating function* of G as an opinion function such that the sum of the function values, taken over closed neighborhoods of vertices, is at least one for at least k vertices of G. The minimum weight of such a function is defined as the *k-subdomination number* of G. In the special case where $k = |V|$, we have the *signed domination number* which is studied in [5, 6, 7, 8, 9, 10, 11, 14, 15, 17, 19] and elsewhere. When $k = \lceil |V|/2 \rceil$, we have the *weak majority number* (also called the majority domination number) studied in [1, 2] and elsewhere. When $k = \lceil (|V| + 1)/2 \rceil$, we have the *strict majority function* studied in [13] and elsewhere.

In this paper, we develop an analogous theory for total k-subdominating functions that arise when we simply change "closed" neighborhood in the definition of a k-subdominating function to "open" neighborhood. Here we define the *vote* of a vertex v as the sum of the opinions in $N(v)$, the open neighborhood of v. When the vote is positive, we say that v votes *aye*; otherwise, v votes *nay*. A *total k-subdominating function* (TkSF) of a graph G is an opinion function for which at least k of the vertices vote aye. The *weight* of an opinion function is the sum of its values. The *total k-subdomination number* of G, denoted by $\gamma^t_{ks}(G)$, is the minimum weight of a total k-subdominating function of G. The weight of a total k-subdominating function is small when, in our original scenario, the number of individuals with positive opinions needed to produce a global positive decision is small. A TkSF of G of weight $\gamma^t_{ks}(G)$ is called a $\gamma^t_{ks}(G)$-function. If f is a TkSF of a graph G, we let $C_f(G) = \{v \in V \mid f[v] \geq 1\}$, and when the graph G is clear from context, we denote $C_f(G)$ simply by C_f. In the special case where $k = |V|$, the total k-subdomination number is the *signed total domination number* $\gamma_t^s(G)$ which is studied in [12, 18].

Unless stated otherwise, the order of a graph $G = (V, E)$ is denoted by n, the size by m, minimum and maximum degrees by δ and Δ and for k-subdomination it is assumed that $1 \le k \le n$.

2 Lower Bounds

Our aim in this section is to give lower bounds on the total k-subdomination number of a graph. We first establish such a lower bound in terms of its minimum degree, maximum degree and its order. The second lower bound is in terms of the degree sequence of the graph. We begin with the following observation. (Recall that for a vertex $v \in V$, we denote $f(N(v))$ by $f[v]$.)

Observation 1 *Let f be a* TkSF *of G and let $v \in C_f$. If* $\deg v$ *is even, then $f[v] \ge 2$, while if* $\deg v$ *is odd, then $f[v] \ge 1$.*

Theorem 2 *Let G be a graph, let f be a $\gamma_{ks}^t(G)$-function, and let ℓ denote the number of vertices with even degree in C_f. Then,*

$$\gamma_{ks}^t(G) \ge \frac{2k(1+\Delta) + \delta n - 3n\Delta + 2\ell}{\Delta + \delta}.$$

Proof. We consider the sum $N = \sum\sum f(u)$, where the outer sum is over all $v \in V$ and the inner sum is over all $u \in N(v)$. This sum counts the value $f(u)$ exactly $\deg u$ times for each $u \in V$, so $N = \sum(\deg u) \cdot f(u)$, over all $u \in V$. Let V_{even} denote the set of all vertices with even degree in C_f. Then, by Observation 1, $N = \sum f[v]$ over all $v \in V$ satisfies

$$\begin{aligned} N &= \sum_{v \in V_{even}} f[v] + \sum_{v \in C_f - V_{even}} f[v] + \sum_{v \notin C_f} f[v] \\ &\ge 2\ell + |C_f| - \ell + (n - |C_f|)(-\Delta) \\ &= \ell + |C_f|(1+\Delta) - n\Delta \\ &\ge \ell + k(1+\Delta) - n\Delta. \qquad (1) \end{aligned}$$

Let P and M be the sets of those vertices in G which are assigned under f the values $+1$ and -1, respectively. Then, $\gamma_{ks}^t(G) = f(V) = |P| - |M| = n - 2|M|$. We now write V as the disjoint union of at most six sets. Let $P = P_\Delta \cup P_\delta \cup P_\lambda$ where P_Δ and P_δ are the sets of all vertices of P with degree equal to Δ and δ, respectively, and P_λ contains all other vertices in P, if any. Let $M = M_\Delta \cup M_\delta \cup M_\lambda$ where M_Δ, M_δ, and M_λ are defined similarly. Further, for $i \in \{\Delta, \delta, \lambda\}$, let V_i be defined by $V_i = P_i \cup M_i$. Thus, $n = |V_\Delta| + |V_\delta| + |V_\lambda|$.

If $u \in V_\lambda$, then $\delta + 1 \le \deg u \le \Delta - 1$. Therefore, writing the sum in (1) as the sum of six summations and replacing $f(u)$ with the corresponding value of 1 or -1 yields

$$\sum_{u\in P_\Delta} \Delta + \sum_{x\in P_\delta} \delta + \sum_{x\in P_\lambda} (\Delta - 1) - \sum_{x\in M_\Delta} \Delta - \sum_{x\in M_\delta} \delta - \sum_{x\in M_\lambda} (\delta + 1) \ge \ell + k(1+\Delta) - n\Delta.$$

Replacing $|P_i|$ with $|V_i| - |M_i|$ for $i \in \{\Delta, \delta, \lambda\}$, yields

$$\begin{aligned} \Delta|V_\Delta| + \delta|V_\delta| + (\Delta - 1)|V_\lambda| - 2\Delta|M_\Delta| - 2\delta|M_\delta| - (\Delta+\delta)|M_\lambda| \\ \ge \ell + k(1+\Delta) - n\Delta. \end{aligned} \tag{2}$$

We now simplify the left hand side of (2) as follows. Replacing $|V_\delta|$ with $|P_\delta|+|M_\delta|$, and $|M_\delta| + |M_\lambda|$ with $|M| - |M_\Delta|$, we have

$$\delta|V_\delta| - 2\delta|M_\delta| - \delta|M_\lambda| = \delta|P_\delta| - \delta|M_\delta| - \delta|M_\lambda| = \delta|P_\delta| - \delta|M| + \delta|M_\Delta|. \tag{3}$$

Further, replacing $|V_\Delta|$ with $n - |V_\delta| - |V_\lambda|$, we have

$$\begin{aligned} & \Delta|V_\Delta| + \Delta|V_\lambda| - 2\Delta|M_\Delta| - \Delta|M_\lambda| \\ = \; & n\Delta - \Delta|V_\delta| - 2\Delta|M_\Delta| - \Delta|M_\lambda| \\ = \; & n\Delta - \Delta|P_\delta| - \Delta|M| - \Delta|M_\Delta|. \end{aligned} \tag{4}$$

Using (3) and (4), the left hand side of (2) can be written as

$$n\Delta - |V_\lambda| - (\Delta - \delta)|P_\delta| - (\Delta+\delta)|M| - (\Delta-\delta)|M_\Delta|.$$

Thus (2) becomes

$$\begin{aligned} 2n\Delta - k(1+\Delta) - \ell \;&\ge\; |V_\lambda| + (\Delta-\delta)|P_\delta| + (\Delta+\delta)|M| + (\Delta-\delta)|M_\Delta| \\ &\ge\; (\Delta+\delta)|M|. \end{aligned} \tag{5}$$

Hence, since $\gamma^t_{ks}(G) = n - 2|M|$, it follows from (5) that

$$\gamma^t_{ks}(G) \ge n - 2\left(\frac{2n\Delta - k(1+\Delta) - \ell}{\Delta+\delta}\right) = \frac{2k(1+\Delta) + \delta n - 3n\Delta + 2\ell}{\Delta+\delta},$$

as desired. □

The next result gives a lower bound on the total k-subdomination number of a graph in terms of its degree sequence.

Theorem 3 *Let G be a graph, the degrees d_i of whose vertices satisfy $d_1 \le d_2 \le \cdots \le d_n$, let f be a $\gamma_{ks}^t(G)$-function, and let ℓ denote the number of vertices of even degree in C_f. Then,*

$$\gamma_{ks}^t(G) \ge \left(\frac{\ell + k + \sum_{i=1}^k d_i}{d_n}\right) - n.$$

Proof. Let f be a $\gamma_{ks}^t(G)$-function. Let V_{even} denote the set of all vertices with even degree in C_f. Let $g: V \to \{0,1\}$ be the function defined by $g(v) = (f(v)+1)/2$ for all vertices $v \in V$. We consider the sum $N = \sum\sum g(u)$, where the outer sum is over all $v \in C_f$ and the inner sum is over all $u \in N(v)$. Then,

$$\begin{aligned} N &= \sum_{v\in C_f}\sum_{u\in N(v)}\frac{1}{2}(f(u)+1) = \sum_{v\in C_f}\frac{1}{2}(f[v]+\deg v) = \frac{1}{2}(\sum_{v\in C_f} f[v] + \sum_{v\in C_f}\deg v) \\ &\ge \frac{1}{2}(\sum_{i=1}^k d_i + \sum_{v\in V_{even}}\deg v + \sum_{v\in C_f - V_{even}}\deg v) \ge \frac{1}{2}(2\ell + |C_f| - \ell + \sum_{i=1}^k d_i) \\ &\ge \frac{1}{2}(\ell + k + \sum_{i=1}^k d_i). \end{aligned}$$

On the other hand,

$$N \le \sum_{v\in V}\sum_{u\in N(v)} g(u) = \sum_{v\in V}(\deg v)\cdot g(v) \le d_n\, g(V),$$

and so

$$g(V) \ge \frac{(\ell + k + \sum_{i=1}^k d_i)}{2d_n}$$

The desired result now follows since $\gamma_{ks}^t(G) = f(V) = 2g(V) - n$. □

As an immediate consequence of Theorem 2 or Theorem 3, we have the following result.

Corollary 4 *For $r \ge 1$, if G is an r-regular graph, then*

$$\gamma_{ks}^t(G) \ge \begin{cases} k\left(\dfrac{r+1}{r}\right) - n & \text{if } r \text{ is odd} \\[2ex] k\left(\dfrac{r+2}{r}\right) - n & \text{if } r \text{ is even} \end{cases}$$

Corollary 5 *If G is a graph, then*

$$\gamma_{ks}^t(G) \ge k - 2n + \frac{k+2m}{\Delta}.$$

Proof. Let the degrees d_i of the vertices of G satisfy $d_1 \le d_2 \le \cdots \le d_n = \Delta$. It follows from Theorem 3 that

$$\begin{aligned}\gamma^t_{ks}(G) &\ge \frac{1}{\Delta}\left(k + \sum_{i=1}^{k} d_i\right) - n \\ &= \frac{1}{\Delta}\left(k + 2m - \sum_{i=k+1}^{n} d_i\right) - n \\ &\ge \frac{1}{\Delta}\left(k + 2m - (n-k)\Delta\right) - n \\ &= k - 2n + \frac{k+2m}{\Delta}. \qquad \square\end{aligned}$$

3 Cycles

Our aim in this section is to determine the total k-subdomination number of a cycle. As a special case of Corollary 4, we have that $\gamma^t_{ks}(C_n) \ge 2k - n$. If $k \in \{n/2, n\}$, we show this lower bound is sharp. We shall prove:

Proposition 6 *For $n \ge 3$ and $1 \le k \le n$,*

$$\gamma^t_{ks}(C_n) = \begin{cases} 2k - n & \text{if } k \in \{n/2, n\} \\ 2k + 2 - n & \text{otherwise.} \end{cases}$$

Proof. We show first that $\gamma^t_{ks}(C_n) \ge 2k + 2 - n$ except when $k = n/2$ or $k = n$, in which case $\gamma^t_{ks}(C_n) = 2k - n$. Let f be a $\gamma^t_{ks}(C_n)$-function. Let $M = \{v \in V(C_n) \mid f(v) = -1\}$ and $P = \{v \in V(C_n) \mid f(v) = +1\}$. Note that, since $k \ge 1$, $P \ne \emptyset$. Let $M_c = C_f \cap M$, $P_c = C_f \cap P$, $M_{uc} = M - M_c$ and $P_{uc} = P - P_c$. Let $H = G[M_c \cup P]$, i.e., H is the subgraph of G induced by $M_c \cup P$. The two vertices adjacent to a vertex in M_c are in P_{uc}, while the two vertices adjacent to a vertex in P_c are in P. It follows that

$$2m(G[P]) = \sum_{v \in P} \deg_{G[P]} v \ge \sum_{v \in P_c} \deg_{G[P]} v = 2|P_c|,$$

whence $m(G[P]) \ge |P_c|$. Thus $m(H) = 2|M_c| + m(G[P]) \ge 2|M_c| + |P_c|$. Further if $m(G[P]) = |P_c|$, then $\deg_{G[P]}(v) = 0$ for all $v \in P_{uc}$ and, since C_n is connected and none of the vertices in P_c is adjacent to any of the vertices of $M \cup P_{uc}$, either $V = P_c$ or $P_c = \emptyset$. So, if $m(G[P]) = |P_c|$, either $V = P_c$ or $P = P_{uc}$ and $m(G[P]) = 0$.

Case 1. $M_{uc} = \emptyset$. Then $H \cong C_n$, so $|M_c| + |P| = m(H) \ge 2|M_c| + |P_c|$. Thus, $|P| \ge |M_c| + |P_c| = |C_f| \ge k$ and so $\gamma^t_{ks}(C_n) \ge 2k - n$. If we have strict inequality in any of the above inequalities or if $|C_f| \ge k+1$, then $|P| \ge k+1$, and $\gamma^t_{ks}(C_n) = 2|P| - n \ge 2(k+1) - n = 2k+2-n$. Hence, suppose we have equality throughout in the above inequalities and $|M_c| + |P_c| = k$. Then, by our remarks above, either $V = P_c$, in which case $|P_c| = k = n$, or $P_c = \emptyset$, in which case $|M_c| = k$ and $n = m(H) = 2|M_c| = 2k$ and so $k = n/2$.

Case 2. $M_{uc} \neq \emptyset$. In this case H consists of a disjoint union of $\ell \ge 1$ paths. Then, $|M_c| + |P| - \ell = m(H) \ge 2|M_c| + |P_c|$. Thus, $|P| \ge |M_c| + |P_c| + \ell \ge |C_f| + 1 \ge k+1$, and so $\gamma^t_{ks}(C_n) \ge 2(k+1) - n = 2k+2-n$.

We have shown that $\gamma^t_{ks}(C_n) \ge 2(k+1) - n = 2k+2-n$ except when $k = n/2$ or $k = n$, in which case $\gamma^t_{ks}(C_n) \ge 2k - n$. We now show that $\gamma^t_{ks}(C_n) \le 2k - n$ if $k = n/2$ or $k = n$ and that $\gamma^t_{ks}(C_n) \le 2k+2-n$ otherwise. For this purpose, we denote the vertex set of the cycle C_n by $\{0, 1, \ldots, n-1\}$. We now define a function $f(V(C_n)) \to \{-1, 1\}$ as follows:

For $1 \le k < n/2$, let $f(v_i) = 1$ if $i \in \{0, 2, \ldots, 2k\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2(k+1) - n$, and $\{v_1, v_3, \ldots, v_{2k-1}\} \subseteq C_f$, so that $|C_f| \ge k$.

For $k = n/2$, let $f(v_i) = 1$ if i is even and $f(v_i) = -1$ otherwise. Then, $f(V) = 0$ and $\{v_1, v_3, \ldots, v_{n-1}\} \subseteq C_f$, so that $|C_f| \ge k$.

For $(n+2)/2 \le k \le n-1$ and n even, let $f(v_i) = 1$ if i is even or $i \in \{1, 3, \ldots, 2k-n+1\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2|P| - n = 2k - n + 2$, and $\{v_1, v_3, \ldots, v_{n-1}\} \cup \{v_2, v_4, \ldots, v_{2k-n}\} \subseteq C_f$ so that $|C_f| \ge n/2 + (k - n/2) = k$.

For $(n+1)/2 \le k \le n-1$ and n odd, let $f(v_i) = 1$ if i is even or $i \in \{1, 3, \ldots, 2k-n\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2|P| - n = 2k - n + 2$, and $\{v_1, v_3, \ldots, v_{n-2}\} \cup \{v_0, v_2, \ldots, v_{2k-n-1}\} \subseteq C_f$ so that $|C_f| \ge (n-1)/2 + (2k - n + 1)/2 = k$.

For $k = n$, the function that assigns 1 to every vertex of the cycle is the desired function.

In all the above cases, f is a TkSF of C_n. Thus, $\gamma^t_{ks}(C_n) \le f(V) = 2k - n$ if $k = n/2$, while $\gamma^t_{ks}(C_n) \le f(V) = 2k+2-n$ otherwise. □

4 Graphs with equal total k- and ℓ-subdomination numbers

Our aim in this section is to give a characterization of graphs G with equal total k-subdomination and total ℓ-subdomination numbers where $1 \le k < \ell \le |V(G)|$. Our proof is along similar lines to that presented in [16].

Theorem 7 *Let G be a graph. Then $\gamma^t_{ks}(G) = \gamma^t_{\ell s}(G)$ if and only if there exists a partition (P, M) of V for which*

1. $|N(x) \cap P| - |N(x) \cap M| \ge 1$ *for at least ℓ of the vertices of G, and*
2. *for any $P' \subseteq P$ and any $M' \subseteq M$ satisfying $|P'| > |M'|$, we have* $|\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \ge |N(x) \cap P| - |N(x) \cap M|\}| > n - k$.

Proof. Suppose $\gamma^t_{ks}(G) = \gamma^t_{\ell s}(G)$. Let f be a TℓSF of G such that $f(V) = \gamma^t_{ks}(G) = \gamma^t_{\ell s}(G)$. Let $P = \{x \in V \mid f(x) = 1\}$ and $M = \{x \in V \mid f(x) = -1\}$. Then (P, M) constitutes a partition of V. For each $x \in C_f$, we have $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$. Since $|C_f| \geq \ell$, Condition (1) holds.

To verify that Condition (2) holds, consider any $P' \subseteq P$ and $M' \subseteq M$ such that $|P'| > |M'|$. Let $X = (P \setminus P') \cup M'$ and $Y = (M \setminus M') \cup P'$. Define a function $g : V \to \{-1, 1\}$ as follows: $g(x) = 1$ for every $x \in X$ and $g(x) = -1$ for every $x \in Y$. Then $g(V) = |X| - |Y| = (|P| - |P'| + |M'|) - (|M| - |M'| + |P'|) = |P| - |M| - 2(|P'| - |M'|) < |P| - |M| = f(V) = \gamma^t_{ks}(G)$. Thus, g is not a TkSF of G, and so $|C_g| < k$. Consequently,

$$|\{x \in V \mid g[x] \leq 0\}| = |V - C_g| = n - |C_g| > n - k. \tag{6}$$

Note that

$$\begin{aligned}
g[x] &= |N(x) \cap X| - |N(x) \cap Y| \\
&= |N(x) \cap ((P \setminus P') \cup M')| - |N(x) \cap ((M \setminus M') \cup P')| \\
&= |N(x) \cap (P \setminus P')| + |N(x) \cap M'| - |N(x) \cap (M \setminus M')| \\
&\qquad -|N(x) \cap P'| \\
&= |N(x) \cap P| - |N(x) \cap P'| + |N(x) \cap M'| - |N(x) \cap M| \\
&\qquad +|N(x) \cap M'| - |N(x) \cap P'| \\
&= |N(x) \cap P| - |N(x) \cap M| - 2(|N(x) \cap P'| - |N(x) \cap M'|). \qquad (7)
\end{aligned}$$

Combining (7) and (8), we obtain Condition 2.

For the sufficiency, suppose there is a partition (P, M) of V such that Conditions (1) and (2) hold. Define a function $f : V \to \{-1, 1\}$ as follows: $f(x) = 1$ for every $x \in P$ and $f(x) = -1$ for every $x \in M$. Then $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$ for at least ℓ vertices of G (by Condition (1)). Thus, f is a TℓSF of G, so that $\gamma^t_{\ell s}(G) \leq |P| - |M|$.

We now show that $\gamma^t_{ks}(G) \geq |P| - |M|$: Suppose, to the contrary, $\gamma^t_{ks}(G) < |P| - |M|$. Let g be a TkSF of G such that $\gamma^t_{ks}(G) = g(V)$. Let $X = \{x \in V \mid g(x) = 1\}$ and $Y = \{x \in V \mid g(x) = -1\}$. Let $P' = P \setminus X$ and $M' = M \setminus Y$. Then $P' \subseteq P$, $M' \subseteq M$, $X = (P \setminus P') \cup M'$ and $Y = (M \setminus M') \cup P'$. Moreover, $|P| - |M| + 2(|M'| - |P'|) = |P| - |P'| + |M'| - |M| + |M'| - |P'| = |X| - |Y| = \gamma^t_{ks}(G) < |P| - |M|$, so that $|P'| > |M'|$. By Condition (2), $|V - C_g| = |\{x \in V \mid g[x] \leq 0\}| = |\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k$. Thus, $|C_g| < k$, contradicting the fact that g is TkSF of G. Hence, $\gamma^t_{ks}(G) \geq |P| - |M|$.

We conclude that $|P| - |M| \leq \gamma^t_{ks}(G) \leq \gamma^t_{\ell s}(G) \leq |P| - |M|$, so that $\gamma^t_{ks}(G) = \gamma^t_{\ell s}(G)$. □

Theorem 8 *Let G be a graph. Then $\gamma^t_{ks}(G) = a$ if and only if there exists a partition (P, M) of V for which*

1. $|N(x) \cap P| - |N(x) \cap M| \geq 1$ *for at least* k *of the vertices of* G,

2. $|P| - |M| = a$, *and*

3. *for any* $P' \subseteq P$ *and any* $M' \subseteq M$ *satisfying* $|P'| > |M'|$, *we have* $|\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k$.

Proof. Suppose $\gamma_{ks}^t(G) = a$. Let f be a TkSF of G such that $f(V) = \gamma_{ks}^t(G) = a$. Let $P = \{x \in V \mid f(x) = 1\}$ and $M = \{x \in V \mid f(x) = -1\}$. Conditions (1) and (3) follows as in the proof of Theorem 7. Moreover, $f(V) = |P| - |M|$, so Condition (2) holds.

For the sufficiency, suppose there is a partition (P, M) of V such that Conditions (1), (2) and (3) hold. Define a function $f: V \to \{-1, 1\}$ as follows: $f(x) = 1$ for every $x \in P$ and $f(x) = -1$ for every $x \in M$. Then $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$ for at least k vertices of G (by Condition (1)). Thus, f is a TkSF of G, so that $\gamma_{ks}(G) \leq |P| - |M| = a$ (by Condition (2)). As in the proof of Theorem 7, $\gamma_{ks}(G) \geq |P| - |M|$. Hence, $|P| - |M| \leq \gamma_{ks}(G) \leq |P| - |M| = a$, so that $\gamma_{ks}(G) = a$. □

5 Trees

We have two immediate aims: first to show that the total k-subdomination number of a tree can be arbitrarily large negative if k is less than the order of the tree, and secondly to determine an upper bound on the total k-subdomination number of a tree and characterize trees attaining this bound.

As pointed out earlier, when $k = n$, the total k-subdomination number is the signed total domination number. In [12], lower and upper bounds on the signed total domination number of a tree in terms of its order are given and the trees attaining these bounds are characterized.

Theorem 9 ([12]) *If* T *is a tree of order* $n \geq 2$, *then*

$$2 \leq \gamma_t^s(T) \leq n.$$

Furthermore, $\gamma_t^s(T) = 2$ *if and only if every vertex* $v \in V(T) - L(T)$ *has odd degree and is adjacent to at least* $(\deg v - 1)/2$ *leaves, while* $\gamma_t^s(T) = n$ *if and only if every vertex of* T *is a support vertex or is adjacent to a vertex of degree* 2.

By giving a positive opinion to the center of a star of order $n \geq 3$ and negative opinions to all the leaves we obtain a TkSF of the star. Thus

Proposition 10 *For* $n \geq 3$ *and* $1 \leq k < n$, $\gamma_{ks}^t(K_{1,n-1}) = 2 - n$.

Hence the total k-subdomination number of a tree can be arbitrarily large negative if k is less than the order of the tree. Next we establish the total k-subdomination number of a path. We begin with the following lemma.

Lemma 11 *For $n \geq 3$ and $1 \leq k < n$, there exists a $\gamma^t_{ks}(P_n)$-function that assigns to one of its leaves a negative opinion and to its neighbour a positive opinion.*

Proof. Let T be the path $v_1, v_2, \ldots, v_n$ and let f be a $\gamma^t_{ks}(T)$-function. Let i be the smallest subscript such that $f(v_i) = -1$. If $i \geq 2$, then the function obtained from f by interchanging the values of v_1 and v_i is an opinion function having the same weight as f and with at least as many vertices voting aye as under f. Hence, we can choose f so that $f(v_1) = -1$. Now let j be the smallest subscript such that $f(v_j) = 1$. If $j \geq 3$, then the function obtained from f by interchanging the values of v_2 and v_j is an opinion function having the same weight as f and with at least as many vertices voting aye as under f. Hence, we can choose f so that $f(v_2) = 1$. □

Proposition 12 *For $n \geq 2$,*

$$\gamma^t_{ks}(P_n) = \begin{cases} -1 & \text{if } k = \frac{1}{2}(n+1) \\ 2k - n & \text{otherwise,} \end{cases}$$

Proof. We proceed by induction on the order $n \geq 2$ of a path P_n. If $n = 2$, then $\gamma^t_{ks}(P_2) = 2k - n$ for $k = 1$ or $k = 2$. Suppose $n = 3$. If $k = 3$, then $\gamma^t_{ks}(P_3) = 3 = 2k - n$, while for $1 \leq k \leq 2$, $\gamma^t_{ks}(P_3) = -1$ by Proposition 10 and the desired result follows. This proves the base cases when $n = 2$ or $n = 3$.

Suppose that $n \geq 4$ and that for every nontrivial path $P_{n'}$ of order $n' < n$, and any integer k' with $1 \leq k' \leq n'$, $\gamma^t_{ks}(P_{n'}) \leq -1$ if $k' = (n'+1)/2$ and $\gamma^t_{ks}(P_{n'}) \leq 2k' - n'$ otherwise. Let T be a path P_n of order n. Let u be a leaf of T and let v be the vertex adjacent to u.

If $k = 1$, then giving a positive opinion to v and negative opinions to all other vertices of T we obtain a TkSF of T of weight $2 - n$. Since $\gamma^t_{ks}(G) \geq 2 - n$ for all graphs G with no isolated vertex, $\gamma^t_{ks}(P_n) = 2 - n = 2k - n$. Hence we may assume $k \geq 2$. Furthermore, if $k = n$, then the result follows from Theorem 9. Hence we may assume that $k < n$. Let $T' = T - u - v$. Then, T' is a path of order $n' = n - 2$. Let $k' = k - 1$. Since $2 \leq k \leq n - 1$, it follows that $1 \leq k' \leq n'$.

Let f' be a $\gamma^t_{k's}(T')$-function. Let $f: V(T) \to \{-1, 1\}$ be the function defined by $f(x) = f'(x)$ if $x \in V(T')$, $f(v) = 1$ and $f(u) = -1$. Every vertex that votes aye in T' also votes aye in T, while u votes aye in T. Hence at least $k' + 1 = k$ vertices of T vote aye, and so f is a TkSF of T. Thus, $\gamma^t_{ks}(T) \leq w(f) = w(f') = \gamma^t_{k's}(T')$. On the other hand, by Lemma 11 there exists a $\gamma^t_{ks}(T)$-function g that assigns to u a negative opinion and to v a positive opinion. Let g' be the restriction of g to $V(T')$. Then, g' is a Tk'sF of T'. Thus, $\gamma^t_{k's}(T') \leq w(g') = w(g) = \gamma^t_{ks}(T)$. Consequently, $\gamma^t_{ks}(T) = \gamma^t_{k's}(T')$.

Suppose $k' = (n'+1)/2$. Then, $k = (n+1)/2$ and by the inductive hypothesis, $w(f') = -1$, and so $\gamma^t_{ks}(T) = w(f') = -1$. Suppose $k' \neq (n'+1)/2$. Then, $k \neq (n+1)/2$ and by the inductive hypothesis, $w(f') = 2k' - n' = 2k - n$, and so $\gamma^t_{ks}(T) = w(f') = 2k - n$. □

Next we present an upper bound on the total k-subdomination number of a tree.

Theorem 13 *For any tree T of order $n \geq 2$,*

$$\gamma^t_{ks}(T) \leq \begin{cases} -1 & \text{if } k = \frac{1}{2}(n+1) \\ 2k - n & \text{otherwise,} \end{cases}$$

and these bounds are sharp.

Proof. We proceed by induction on the order $n \geq 2$ of a tree T. If $n \in \{2,3\}$, then $T = P_n$ and the result follows from Proposition 12. This proves the base cases when $n = 2$ or $n = 3$.

Suppose that $n \geq 4$ and that for every nontrivial tree T' of order $n' < n$, and any integer k' with $1 \leq k' \leq n'-1$, $\gamma^t_{ks}(T') \leq -1$ if $k' = (n'+1)/2$ and $\gamma^t_{ks}(T') \leq 2k' - n'$ otherwise. Let T be a tree of order n.

If T is a star, then, by Proposition 10, $\gamma^t_{ks}(T) = 2-n < -1$. Thus, $\gamma^t_{ks}(T) = 2k-n$ if $k = 1$, while $\gamma^t_{ks}(T) < 2k - n$ if $2 \leq k \leq n$. Hence the desired result follows if T is a star. Thus we may assume that $\operatorname{diam} T \geq 3$.

If $k = n$, then, by Theorem 9, $\gamma^t_{ks}(T) \leq n = 2k-n$. Hence we may assume $k < n$.

Let T be rooted at a leaf r of a longest path. Let v be a vertex at distance $\operatorname{diam} T - 1$ from r on a longest path starting at r, and let w be the parent of v. Let $|C(v)| = m$. Then, $m \geq 1$. If $k \leq m$, then giving a positive opinion to v and negative opinions to all the other vertices we obtain a TkSF of T of weight $2 - n$, and the desired result follows. Hence we may assume $k > m$.

Let $T' = T - V(T_v)$. Then, T' has order $n' = n - m - 1$. Since $\operatorname{diam} T \geq 3$, $n' \geq 2$. Let $k' = k - m$. Since $m + 1 \leq k \leq n - 1$, we have $1 \leq k' \leq n'$. Let f' be a $\gamma^t_{k's}(T')$-function. Let $f: V(T) \to \{-1, 1\}$ be the function defined by $f(x) = f'(x)$ if $x \in V(T')$, $f(v) = 1$ and $f(u) = -1$ for every child of v. Every vertex that votes aye in T' also votes aye in T, while each child of v votes aye in T. Hence at least $k'+m = k$ vertices of T vote aye, and so f is a TkSF of T. Thus, $\gamma^t_{ks}(T) \leq w(f) = w(f')+1-m$.

Suppose $k' = (n'+1)/2$. Then, $k = (n+m)/2$. By the inductive hypothesis, $\gamma^t_{ks}(T') \leq -1$, and so $\gamma^t_{ks}(T) \leq -m$. Thus if $m = 1$, then $k = (n+1)/2$ and $\gamma^t_{ks}(T) \leq -1$, while if $m \geq 2$, then $k \geq (n+2)/2$ and $\gamma^t_{ks}(T) \leq -2 < 2 \leq 2k - n$. In any event, the result follows.

On the other hand, suppose $k' \neq (n'+1)/2$. By the inductive hypothesis, $\gamma^t_{ks}(T') \leq 2k' - n' = 2k - n + 1 - m$, and so $\gamma^t_{ks}(T) \leq 2k - n + 2(1 - m)$. Suppose $k = (n+1)/2$. Then, $k' = (n' - m + 2)/2$. Since $k' \neq (n'+1)/2$, it follows that $m \geq 2$, and so $\gamma^t_{ks}(T) \leq 2k - n + 2(1-m) \leq -1$. Suppose $k \neq (n+1)/2$. Then, since $m \geq 1$, $\gamma^t_{ks}(T) \leq 2k - n$. Once again, the desired result follows.

That the bounds are sharp, follows from Proposition 12. □

As an immediate consequence of Theorem 8 we have the following result.

Corollary 14 *Let T be a tree. Then, $\gamma_{ks}(T) = 2k - n$ if and only if there exists a partition (P, M) of V for which*

1. $|N(x) \cap P| - |N(x) \cap M| \geq 1$ *for at least k of the vertices of T,*
2. $|P| - |M| = 2k - n$, *and*
3. *for any $P' \subseteq P$ and any $M' \subseteq M$ satisfying $|P'| > |M'|$, we have* $|\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k$.

We show next that when T is a nontrivial tree of even order n, and $k = n/2 + i$ for some integer i with $0 \leq i \leq 3$, then the result of Theorem 13 can be improved slightly.

Theorem 15 *For any tree T of even order $n \geq 2$, and any integer k with $n/2 \leq k \leq n$ and $k = n/2 + i$ where $0 \leq i \leq 3$,*

$$\gamma^t_{ks}(T) \leq 2(k-1) - n,$$

unless T is a path, in which case $\gamma^t_{ks}(T) = 2k - n$.

Proof. We proceed by induction on the order n of a tree T, where $n \geq 2$ is even. If $n = 2$, then $T = P_2$ and $k \in \{1, 2\}$ and the result follows from Proposition 12. If $n = 4$, then $k \in \{2, 3, 4\}$ and either $T = P_4$, in which case the result follows from Proposition 12, or $T = K_{1,3}$, in which case the result follows from Proposition 12 (if $k \in \{2, 3\}$) or Theorem 9 (if $k = 4$). This proves the base cases when $n = 2$ and $n = 4$.

Suppose that $n \geq 4$ is even and that for every nontrivial tree T' of even order $n' < n$, and any integer k' with $n'/2 \leq k' \leq n'$ and $k' = n'/2 + i$ where $0 \leq i \leq 3$, $\gamma^t_{k's}(T') \leq 2(k'-1) - n'$, unless T' is a path, in which case $\gamma^t_{k's}(T') = 2k' - n$. Let T be a tree of order n.

If $k = n$, then since $k \leq (n+6)/2$, $n = k = 6$, and so it follows from Theorem 9 that $\gamma^t_{ks}(T) \leq 4$ unless $T = P_6$, in which case $\gamma^t_{ks}(T) = 6$. Hence the desired result follows if $k = n$. Thus we may assume $k < n$. In particular, if $k = n/2 + 3$, then $n \geq 8$.

Following the notation used in paragraph 5 and 6 of the proof of Theorem 13, $\gamma^t_{ks}(T) \leq w(f) = w(f') + 1 - m$. If $k' = (n'+1)/2$, then $k = (n+m)/2$. By Theorem 13, $\gamma^t_{k's}(T') \leq -1$, and so $\gamma^t_{ks}(T) \leq -m$. Hence we may assume $k' \neq (n'+1)/2$. By Theorem 13, $\gamma^t_{k's}(T') \leq 2k' - n' = 2k - n + 1 - m$, and so $\gamma^t_{ks}(T) \leq 2k - n + 2(1-m)$. If $m \geq 2$, then $\gamma^t_{ks}(T) \leq 2k - n + 2(1-m) \leq 2(k-1) - n$, as desired. Hence we may assume $m = 1$, and so $k' = k - m = k - 1$, $n' = n - 2$ and $\gamma^t_{ks}(T) \leq w(f') = \gamma^t_{k's}(T')$. Furthermore, n' is even and $n'/2 \leq k' \leq n'$ and $k' = n'/2 + i$ where $0 \leq i \leq 3$. Applying the inductive hypothesis to T', $\gamma^t_{k's}(T') \leq 2(k'-1) - n'$, unless T' is a path, in which case $\gamma^t_{k's}(T') = 2k' - n$. Let u denote the child of v.

If $\gamma^t_{k's}(T') \leq 2(k'-1) - n'$, then $\gamma^t_{ks}(T) \leq 2(k-1) - n$, as desired. Hence we may assume that T' is a path.

Suppose w is neither a leaf nor a support vertex of T'. Let v' be the child of w different from v, and let u' be the child of v'. Assign a positive opinion to w and its two children and to all vertices of degree 2 at even distance from w. Assign a negative opinion to all remaining vertices. If $k = n/2$, reassign to w a negative opinion. If $k = n/2 + 2$, reassign to u a positive opinion, while if $k = n/2 + 3$, reassign to each of u and u' a positive opinion. In all cases, this produces a TkSF of weight $2(k-1) - n$, and so $\gamma^t_{ks}(T) \leq 2(k-1) - n$, as desired. Note that here T is not a path.

Suppose that w is a support vertex of T'. Let v' be the child of w different from v. Assign a positive opinion to w and to all vertices different from v' whose distance from w in T is odd. Assign a negative opinion to all remaining vertices. If $k = n/2$, reassign to w a negative opinion. If $k = n/2 + 2$, reassign to u a positive opinion, while if $k = n/2 + 3$, reassign to each of the two vertices at distance 2 from w positive opinion. In all cases, this produces a TkSF of weight $2(k-1) - n$, and so $\gamma^t_{ks}(T) \leq 2(k-1) - n$, as desired. Note that here T is not a path.

Finally, if w is a leaf in T', then T is a path of even order and so, by Proposition 12, $\gamma^t_{ks}(T) = 2k - n$. □

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