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Note on competition and phylogeny numbers

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Abstract

Given an acyclic digraph $D = (V, A)$, its *competition graph* $G = K(D)$ is an undirected graph with the same vertex set as D and with an edge $xy \in E(G)$ if and only if there is a vertex a so that $(x, a) \in A$ and $(y, a) \in A$. The *phylogeny graph* $G = P(D)$ of D is an undirected graph with the same vertex set as D and with an edge $xy \in E(G)$ if and only if $(x, y) \in A$ or $(y, x) \in A$ or $(x, a), (y, a) \in A$ for some vertex $a \in V$. If G is any graph, G together with sufficiently many isolated vertices is a competition graph of an acyclic digraph, and the *competition number* of G is the smallest number of such isolated vertices. We shall call the acyclic digraph D a *phylogeny digraph* for G if G is an induced subgraph of $P(D)$ and D has no arcs from vertices outside of G to vertices in G. The *phylogeny number* $p(G)$ is the smallest r such that G has a phylogeny digraph D with $|V(D)| - |V(G)| = r$. In this paper we study the competition and phylogeny numbers for a class of graphs in which each graph G includes at least one not K_2 clique, and any two different maximal cliques of G have at most one common vertex. The results of this paper generalize some results provided by Kim and Roberts, and by Roberts and Sheng.

1 Introduction

The competition graph of a digraph was introduced by Cohen [1] in connection with a problem in ecology. Since the notion of competition graph was introduced, there has been a very large literature on it. Besides ecology, various applications include applications to channel assignments, coding, and modelling of complex economic and energy systems. Some related notions have also been introduced. The phylogeny graph introduced by Roberts and Sheng [6] was one related notion of competition graph.

Let $D = (V, A)$ be a digraph, in which V is the vertex set and A the set of directed arcs. The *competition graph* $K(D)$ is the undirected graph G with the same vertex set as D and with an edge $xy \in E(G)$ if and only if there is a vertex a so that $(x, a) \in A(D)$ and $(y, a) \in A(D)$. We denote by $G \cup I_k$ the graph consisting of G and k isolated vertices. For any graph G, the *competition number* of G, denoted by $k(G)$, is the smallest k so that $G \cup I_k$ is a competition graph of an acyclic digraph.

Given an acyclic digraph $D = (V, A)$, the *phylogeny graph* $P(D)$ of D is an undirected graph G with the same vertex set as D and with an edge $xy \in E(G)$ if and only if $(x, y) \in A$ or $(y, x) \in A$ or $(x, a), (y, a) \in A$ for some vertex $a \in V$. For any graph G, we shall call the acyclic digraph D a *phylogeny digraph* for G if G is an induced subgraph of $P(D)$ and D has no arcs from vertices outside of G to vertices in G. The *phylogeny number* $p(G)$ is the smallest r such that G has a phylogeny digraph D with $|V(D)| - |V(G)| = r$. A phylogeny digraph D for G is called an *optimal phylogeny digraph* for G if $|V(D)| - |V(G)| = p(G)$.

An *edge clique cover* of G is a collection of cliques that covers all edges of G. The *edge clique cover number* of G, denoted by $\theta_e(G)$, is the minimum number of cliques in an edge clique cover of G. Given $G = (V, E)$, let E^- be the subset of E obtained by deleting all the edges of the maximal cliques of G of size at least 3 from E, and let $G^- = (V, E^-)$.

Kim and Roberts [3], and Roberts and Sheng [6] studied the competition numbers and phylogeny numbers for the graphs with one or two triangles respectively. In this paper, as a generalization, we study the competition and phylogeny numbers for a class of graphs in which each graph G includes at least one not K_2 clique, and any two different maximal cliques of G have at most one common vertex. Note that this kind of graph may include many triangles. In Sections 2 and 3 we study the competition and phylogeny numbers of this kind of graph respectively. In fact, we generalize some results provided by Kim and Roberts in [3], and by Roberts and Sheng in [6]. Let $[1, n]$ denote $\{1, 2, \ldots, n\}$, and $\omega(G)$ denote the number of connected components of G. For all undefined notation and terminology, see [3,6].

2 Competition numbers

We first introduce some lemmas.

Lemma 2.1 (Harary et al. [2]). *Let $D = (V, A)$ be a digraph. Then D is acyclic if and only if there exists an ordering of vertices, $\sigma = [v_1, v_2, \ldots, v_n]$, such that one of the following two conditions holds:*

(1) For all $i, j \in [1, n]$, $(v_i, v_j) \in A$ implies that $i < j$.

(2) For all $i, j \in [1, n]$, $(v_i, v_j) \in A$ implies that $i > j$.

By this lemma, if D is an acyclic digraph, we can find a vertex labelling $\pi : V \to \{1, 2, \ldots, |V|\}$ so that whenever (x, y) is in A, $\pi(y) < \pi(x)$. We call π an acyclic labelling of D. Conversely, if D is a digraph with an acyclic labelling, then D is acyclic. In the following, let $n = |V(G)|$ and $e = |E(G)|$.

Lemma 2.2 (Roberts [5]). *If G is a connected graph with no triangles, then $k(G) = e - n + 2$.*

Lemma 2.3 (Opsut [4]). *For any graph G, $\theta_e(G) \le k(G) + n - 2$.*

The following theorem is the main result in this section.

Theorem 2.4. *Suppose $K_{n_1}, K_{n_2}, \ldots, K_{n_t}$ are the maximal cliques of size at least 3 of a connected graph $G = (V, E)$, where $t \ge 1$ and $n_i \ge 3$ for $i = 1, 2, \ldots, t$. If any two different maximal cliques of G have at most one common vertex, then*

$$e - \sum_{i=1}^{t} \binom{n_i}{2} - n + t + 2 \le k(G) \le e - \sum_{i=1}^{t} \binom{n_i - 1}{2} - n + 2.$$

Proof. Suppose any two different maximal cliques of G have at most one common vertex. Since by the conditions of the theorem, each clique $K_{n_i}, i = 1, 2, \ldots, t$, has $\binom{n_i}{2}$ edges, and the other cliques of G are all K_2, then

$$\theta_e(G) = e - \sum_{i=1}^{t} \binom{n_i}{2} + t.$$

By Lemma 2.3,

$$k(G) \ge \theta_e(G) - n + 2 = (e - \sum_{i=1}^{t} \binom{n_i}{2} + t) - n + 2 = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + t + 2. \quad (1)$$

In order to prove the upper bound, let $\{v_1^s, v_2^s, \ldots, v_{n_s}^s\}$ be the vertex set of K_{n_s}, where $s = 1, 2, \ldots, t$. Deleting the edges $v_i^s v_j^s$ from G, where $2 \le i < j \le n_s$ and $s = 1, 2, \ldots, t$, the resulting graph

$$G' = G - \bigcup_{s=1}^{t} (\bigcup_{2 \le i < j \le n_s} \{v_i^s v_j^s\})$$

is connected and triangle-free. Lemma 2.2 implies that

$$k(G') = (e - \sum_{i=1}^{t} \binom{n_i - 1}{2}) - n + 2 = e - \sum_{i=1}^{t} \binom{n_i - 1}{2} - n + 2.$$

Let D' be an acyclic digraph whose competition graph is

$$G' \cup I_{e-\sum\limits_{i=1}^{t}\binom{n_i-1}{2}-n+2}.$$

Since $\bigcup\limits_{s=1}^{t}\{v_1^s v_2^s, v_1^s v_3^s, \ldots, v_1^s v_{n_s}^s\}$ is the subset of $E(G')$, there are arcs (v_1^s, a_2^s), (v_2^s, a_2^s), (v_1^s, a_3^s), (v_3^s, a_3^s), (v_1^s, a_4^s), (v_4^s, a_4^s), ..., $(v_1^s, a_{n_s}^s)$, $(v_{n_s}^s, a_{n_s}^s)$ in D' for vertices $a_2^s, a_3^s, \ldots, a_{n_s}^s$ of D', where $s = 1, 2, \ldots, t$. For any $i, j \in [2, n_s]$, if $i \neq j$, then $a_i^s \neq a_j^s$ since $v_i^s v_j^s \notin E(G')$, where $s \in \{1, 2, \ldots, t\}$. By Lemma 2.1, there is an acyclic labelling π of D' such that whenever (x, y) is an arc of D', $\pi(y) < \pi(x)$. So $\pi(a_i^s) < \pi(v_i^s)$, where $2 \leq i \leq n_s$ and $s = 1, 2, \ldots, t$. Without loss of generality, we may assume that $\pi(a_2^s) < \pi(a_j^s)$, where $3 \leq j \leq n_s$ and $s = 1, 2, \ldots, t$. So $\pi(a_2^s) < \pi(v_j^s)$, where $2 \leq j \leq n_s$ and $s = 1, 2, \ldots, t$. Let D be the digraph obtained from D' by adding arcs $(v_3^s, a_2^s), (v_4^s, a_2^s), \ldots, (v_{n_s}^s, a_2^s)$ to D', where $s = 1, 2, \ldots, t$. It is easy to see that D is acyclic. Therefore, the competition graph of D is

$$G \cup I_{e-\sum\limits_{i=1}^{t}\binom{n_i-1}{2}-n+2},$$

so

$$k(G) \leq e - \sum_{i=1}^{t}\binom{n_i - 1}{2} - n + 2. \tag{2}$$

Combining (1) and (2), the proof is complete. □

The following corollary is the special case of the above theorem when $t = 1$ and $n_1 = m$.

Corollary 2.5. *If a connected graph $G = (V, E)$ has a clique K_m and every triangle of G is included in the K_m, where $3 \leq m \leq n$, then*

$$e - \binom{m}{2} - n + 3 \leq k(G) \leq e - \binom{m-1}{2} - n + 2.$$

By Corollary 2.5, as $m = 3$ the following corollary follows.

Corollary 2.6 (Kim and Roberts [3]). *If G is connected and has exactly one triangle, then $k(G) = e - n$ or $e - n + 1$.*

3 Phylogeny numbers

In order to prove the main result in this section, we cite some lemmas from [6]. Given graph $G = (V, E)$, let $D = (V \cup I_m, A)$ be a phylogeny digraph for G and let $uv \in E$. We say that G is *triangulated* if it contains no chordless cycle of four or more vertices. We say that uv is *taken care of* by a vertex $a \in V \cup I_m$ if $(u, a) \in A$

and $(v, a) \in A$. If $E_1 \subseteq E$, we will let $\overline{E}_1 = E - E_1$. A triangle is a *mixed triangle* of G relative to E_1 if some edge of the triangle is in E_1 and some edge of the triangle is in $\overline{E}_1$.

Lemma 3.1 (Roberts and Sheng [6]). *For any graph $G = (V, E)$, $p(G) \geq \theta_e(G) - n + 1$.*

Lemma 3.2 (Roberts and Sheng [6]). *If $G = (V, E)$ is a triangulated graph, then $p(G) = 0$.*

Lemma 3.3 (Roberts and Sheng [6]). *If $G = (V, E)$ is a connected graph with no triangles, then $p(G) = e - n + 1$.*

Lemma 3.4 (Roberts and Sheng [6]). *If $G = (V, E)$ is a graph with no triangles and with k connected components, then $p(G) = e - n + k$. Moreover, if x is an arbitrary vertex of G, there is an optimal phylogeny digraph for G with x a source vertex.*

Lemma 3.5 (Roberts and Sheng [6]). *Suppose $G = (V, E)$, let $E_1 \subseteq E$ and $G_1 = (V, E_1)$. Let $D = (V \cup I_m, A)$ be an optimal phylogeny digraph for G and let $I_{\overline{m}_1}$ be the set of $\overline{m}_1$ vertices in I_m that only take care of edges in $\overline{E}_1$. If G has no mixed triangle relative to E_1, then $p(G) \geq p(G_1) + \overline{m}_1 \geq p(G_1) + p(\overline{G}_1)$.*

Theorem 3.6. *Suppose $K_{n_1}, K_{n_2}, \ldots, K_{n_t}$ be the maximal cliques of size at least 3 of connected graph $G = (V, E)$, where $t \geq 1$ and $n_i \geq 3$ for $i = 1, 2, \ldots, t$, and any two different maximal cliques of G have at most one common vertex.*

(1) *If G^- is connected, then $p(G) = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + t + 1$.*

(2) *If G^- has a connected component including exactly one vertex for some clique K_{n_i}, $i \in \{1, 2, \ldots, t\}$, then*

$$e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) \leq p(G) \leq e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) + t - 1.$$

Otherwise,

$$e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) \leq p(G) \leq e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) + t.$$

Proof. Let $\{v_1^s, v_2^s, \ldots, v_{n_s}^s\}$ be the vertex set of K_{n_s}, where $s = 1, 2, \ldots, t$.

Case 1. Suppose that G^- is connected. By the proof of Theorem 2.4,

$$\theta_e(G) = e - \sum_{i=1}^{t} \binom{n_i}{2} + t.$$

It follows from Lemma 3.1 that

$$p(G) \geq \theta_e(G) - n + 1 = e - \sum_{i=1}^{t} \binom{n_i}{2} + t - n + 1. \tag{3}$$

By Lemma 3.3,

$$p(G^-) = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + 1.$$

We may assume that D^- is an optimal phylogeny digraph of G^-. Let D be the digraph obtained from D^- by adding vertices $a_1, a_2, \ldots, a_t$ to $V(D^-)$ and adding the arcs $(v_1^s, a_s), (v_2^s, a_s), \ldots, (v_{n_s}^s, a_s)$ to $A(D^-)$, where $s = 1, 2, \ldots, t$. It is easy to see that D is acyclic and a phylogeny digraph for G. So

$$\begin{aligned} p(G) &\leq |V(D)| - |V(G)| = |V(D^-)| + t - |V(G^-)| \\ &= p(G^-) + t = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + t + 1. \end{aligned} \tag{4}$$

Inequalities (3) and (4) imply that $p(G) = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + t + 1$.

Case 2. Suppose that $C_1, C_2, \ldots, C_{\omega(G^-)}$ are the connected components of G^-, where $\omega(G^-) > 1$. Lemma 3.2 implies $p(\bigcup_{i=1}^{t} K_{n_i}) = 0$, and Lemma 3.4 implies

$$p(G^-) = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-).$$

So by Lemma 3.5,

$$\begin{aligned} p(G) &\geq p(G^-) + p(\bigcup_{i=1}^{t} K_{n_i})) \\ &= e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) + 0 = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-). \end{aligned} \tag{5}$$

Suppose that G^- has a connected component including exactly one vertex for some clique K_{n_i}, $i \in \{1, 2, \ldots, t\}$. Without loss of generality, we may assume that C_1 includes exactly one vertex of K_{n_1}, say v_1^1. By Lemma 3.4, there is an optimal phylogeny digraph D^- for G^- with v_1^1 a source vertex. Let D be the digraph obtained from D^- by adding vertices $a_2, a_3, \ldots, a_t$ to $V(D^-)$ and adding the arcs (v_2^1, v_1^1), (v_3^1, v_1^1), $\ldots, (v_{n_1}^1, v_1^1)$, (v_1^2, a_2), $(v_2^2, a_2), \ldots, (v_{n_2}^2, a_2)$, $\ldots, (v_1^t, a_t), \ldots, (v_{n_t}^t, a_t)$ to $A(D^-)$. It is easy to see that D is acyclic and a phylogeny digraph for G. So

$$\begin{aligned} p(G) &\leq |V(D)| - |V(G)| = |V(D^-)| + (t-1) - |V(G^-)| \\ &= p(G^-) + (t-1) = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) + t - 1. \end{aligned} \tag{6}$$

Combining (5) and (6), we have

$$e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) \leq p(G) \leq e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) + t - 1.$$

Suppose for any clique K_{n_s}, $s \in \{1, 2, \ldots, t\}$, each component of G^- includes either at least two vertices of K_{n_s} or none. By Lemma 3.4, $p(G^-) = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-)$, and suppose that D^- is an optimal phylogeny digraph for G^-. Let D be the digraph obtained from D^- by adding vertices $a_1, a_2, \ldots, a_t$ to $V(D^-)$ and adding the arcs $(v_1^1, a_1), (v_2^1, a_1), \ldots, (v_{n_1}^1, a_1), (v_1^2, a_2), (v_2^2, a_2), \ldots, (v_{n_2}^2, a_2), \ldots, (v_1^t, a_t), \ldots, (v_{n_t}^t, a_t)$ to $A(D^-)$. It is easy to see that D is acyclic and a phylogeny digraph for G. So

$$\begin{aligned} p(G) &\leq |V(D)| - |V(G)| = |V(D^-)| + t - |V(G^-)| \\ &= p(G^-) + t = e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) + t. \end{aligned} \tag{7}$$

By (5) and (7),

$$e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) \leq p(G) \leq e - \sum_{i=1}^{t} \binom{n_i}{2} - n + \omega(G^-) + t.$$

Combining all the cases above, the proof is complete. □

The following corollary is the special case of the above theorem when $t = 1$ and $n_1 = m$.

Corollary 3.7. *Suppose the connected graph $G = (V, E)$ has a clique K_m and any triangle of G is included in the K_m, where $3 \leq m \leq n$.*

(1) *If G^- is connected, then $p(G) = e - \binom{m}{2} - n + 2$.*

(2) *If G^- has a connected component including exactly one vertex of K_m, then $p(G) = e - \binom{m}{2} - n + \omega(G^-)$. Otherwise, $p(G) = e - \binom{m}{2} - n + \omega(G^-)$ or $p(G) = e - \binom{m}{2} - n + \omega(G^-) + 1$.*

By Corollary 3.7, as $m = 3$ the following corollary follows.

Corollary 3.8 (Roberts and Sheng [3]). *Let $G = (V, E)$ be a connected graph with exactly one triangle. Then*

$$p(G) = \begin{cases} e - n & \text{if } G^- \text{ has three components,} \\ e - n - 1 & \text{if } G^- \text{ has one or two components.} \end{cases}$$

4 Concluding remarks

Corollary 2.6 and Corollary 3.8 show that Theorem 5 in [3] is the special case of our Theorem 2.4, and Theorem 17 in [6] is the special case of our Theorem 3.6. So this paper generalizes some results provided by Kim and Roberts in [3], and by Roberts and Sheng in [6].

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