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Bounds on the number of generalized partitions and some applications

W.M.B. Dukes

School of Theoretical Physics
Dublin Institute for Advanced Studies
10 Burlington Rd.
Dublin 4
Ireland
dukes@stp.dias.ie

Abstract

We present bounds concerning the number of Hartmanis partitions of a finite set. An application of these inequalities improves the known asymptotic lower bound on the number of linear spaces on n points. We also present an upper bound for a certain class of these partitions which bounds the number of Steiner triple and quadruple systems.

Recent work has extended the known numerical values for the number of linear spaces on n points [1]. Upper and lower bounds of $2^{\binom{n}{3}}$ and 2^n, respectively, were given in [2, 7]. We improve the lower bound by showing an inequality of Knuth (see [5]) to hold for more general structures known as Hartmanis d-partitions. We prove an upper bound for the number of these structures and also give an upper bound for a certain class of these partitions. This last inequality gives asymptotic upper bounds for the number of Steiner triple and quadruple systems.

A linear space is a collection of points and lines such that every pair of distinct points are on a unique line and every line contains at least two points. Let $S_n := \{1, \ldots, n\}$ be a finite set of size n and S_n^d the collection of all d-element subsets of S_n. We say that a collection of subsets $\mathcal{P}$ of S_n is a *d-partition* of S_n (to be more specific, a Hartmanis d-partition, see [4]) if

- for all $X \in \mathcal{P}$, $|X| \geq d$,
- $\bigcup_{X \in \mathcal{P}} X = S_n$,
- every d-element subset of S_n is contained in a unique $X \in \mathcal{P}$.

The one-to-one correspondence between linear spaces on n points and unlabeled 2-partitions of S_n is easily seen by noting that the sets of the 2-partition correspond to the lines. It is also apparent that a Steiner triple system is simply a 2-partition whose sets have cardinality three and a Steiner quadruple system is a 3-partition whose sets have cardinality four. Further details of Steiner systems may be found in [6]. Notice that a 1-partition of S_n is what we normally refer to as a partition of S_n.

Let $p_n(d)$ denote the number of d-partitions of S_n and $p^\star_n(d)$ the corresponding number of unlabeled d-partitions. Let $p_n(d;a)$ denote the number of d-partitions whose sets contain at most a elements and $p^\star_n(d;a)$ the corresponding unlabeled number. From their definitions we have that $p^\star_n(d;a) \leq p^\star_n(d) \leq p_n(d)$ and $p^\star_n(d;a) \leq p_n(d;a) \leq p_n(d)$. The number of linear spaces on n points is given by $p^\star_n(2)$.

Theorem 1 *For all* $0 < d < n$, $p_n(d;d+1) \geq 2^{\binom{n}{d+1}/2n}$ *and* $p^\star_n(d;d+1) \geq \frac{1}{n!}2^{\binom{n}{d+1}/2n}$.

PROOF: Let H be the $n \times k$ matrix whose i^{th} row is the binary representation of i for all $1 \leq i \leq n$ and $k := \lfloor \log_2 n \rfloor + 1$. For any $X \in S_n^{d+1}$, let $\vec{X}$ be its binary representation. Define the partition $\mathcal{U}_j$ of S_n^{d+1} by

$$\mathcal{U}_j := \left\{ X \in S_n^{d+1} \,|\, \vec{X}H = \text{binary representation of } j \right\}$$

for all $0 \leq j < 2^k$. Now notice that if $X, Y \in \mathcal{U}_j$ and $X \neq Y$, then $|X\backslash Y| \geq 2$. Indeed, if $|X\backslash Y| = 1$ then $X = A \cup \{x\}$, $Y = A \cup \{y\}$, with $x \neq y$ and $x, y \notin A$. Hence $(\vec{A} + \vec{\{x\}})H = (\vec{A} + \vec{\{y\}})H$, which is a contradiction since $\vec{\{x\}}H$ is the binary representation of x. Thus for any $X, Y \in \mathcal{U}_j$, $|X \cap Y| \leq d-1$. Since the $\mathcal{U}_j$ partition S_n^{d+1}, there exists some $\mathcal{U}_j$ with at least

$$|\mathcal{U}_j| \;\geq\; \binom{n}{d+1}/2^k \;\geq\; \binom{n}{d+1}/2n$$

sets. This particular $\mathcal{U}_j$ (and any collection of subsets of it), along with all the d-sets not contained in any member of $\mathcal{U}_j$, defines a d-partition. Thus there are at least $2^{|\mathcal{U}_j|} \geq 2^{\binom{n}{d+1}/2n}$ such d-partitions of S_n. Note that the fraction of d-element subsets covered by the largest $\mathcal{U}_j$ is $(n-d)/2n$ as $\binom{d+1}{d}\binom{n}{d+1}/2n = ((n-d)/2n)\binom{n}{d}$. The second inequality holds by dividing this number by $n!$ to rule out any isomorphisms. $\square$

The construction of the matrix H is indicative of Hamming codes and indeed Knuth [5] elucidates this point in his particular $d = [n/2] - 1$ case. In our case it is equivalent to finding the a collection of binary code words of length n with $d+1$ 1's which is single error-correcting.

Numerous computer computations with $d = 2$ and $10 \leq n \leq 30$ showed the largest of the $\mathcal{U}$ families, although only marginal, was always $\mathcal{U}_0$. For a special case of $d = 2$ we may improve the bound in the previous theorem to $2^{n(n-1)/6}$ by explicitly evaluating $|\mathcal{U}_0|$.

Theorem 2 *If* $d = 2$ *and* $n = 2^m - 1$ *for some* $m > 1$*, then* $|\mathcal{U}_0| = \binom{n}{2}/3$.

PROOF: If $n = 2^m - 1$ then the rows of the matrix H will consist of all non-zero binary vectors of length m (so that $k = m$.) Let $\vec{r_i}$ be the vector representing the i^{th} row of H. Since $d = 2$ we have

$$\begin{aligned} |\mathcal{U}_0| &= \#\left\{X \in S_n^3 \,\middle|\, \vec{X}H = \vec{0} \mod 2\right\} \\ &= \#\left\{ \{i,j,l\} \subseteq S_n \,\middle|\, \vec{r_i} + \vec{r_j} + \vec{r_l} = \vec{0} \mod 2\right\}. \end{aligned}$$

Notice that if we have i, j, l such that $\vec{r_i} + \vec{r_j} + \vec{r_l} = \vec{0} \mod 2$ then l is uniquely determined by i and j as $\vec{r_l} = \vec{r_i} + \vec{r_j} \mod 2$. Similarly i can be determined from j and l, and j from i and l. Thus $|\mathcal{U}_0|$ will be the number of pairs in S_n, scaled down by a factor of 3. Hence $|\mathcal{U}_0| = \binom{n}{2}/3$. □

Note that the above theorem holds for general d, the cardinality of $|\mathcal{U}_0|$ will be $\binom{n}{d}/(d+1)$ by using the same argument. We now give a short proof of an upper bound on the number of d-partitions. The proof for $d = 2$ can be found in [2].

Theorem 3 *For all* $0 < d < n$, $p_n(d) \leq 2^{\binom{n}{d+1}}$.

PROOF: Let $\mathcal{P}$ be a d-partition of S_n and exclude from $\mathcal{P}$ any sets of size d. Define $f(\mathcal{P}) := \{X \in S_n^{d+1} | X \subseteq P \in \mathcal{P}\}$. The map f is injective and we may easily construct the inverse as follows: Let $\mathcal{P}' = f(\mathcal{P})$. If $X, Y \in \mathcal{P}'$ and $|X \cap Y| \geq d$ then replace X and Y in $\mathcal{P}'$ by $X \cup Y$. Iterate this step until $|X \cap Y| < d$ for all $X, Y \in \mathcal{P}'$. Insert into $\mathcal{P}'$ all d-element subsets of S_n not contained in members of $\mathcal{P}'$. The collection $\mathcal{P}'$ is now the original collection $\mathcal{P}$. Thus for each d partition $\mathcal{P}$ we have a unique collection $f(\mathcal{P}) \subseteq S_n^{d+1}$. The number of such collections is bounded above by $2^{\binom{n}{d+1}}$. □

Theorem 4 *For all* $1 < d < n$, $p_n(d; d+1) < 2^{n+1+(n+1)^d(\log_2 e + \log_2(n-d))}$.

PROOF: Let $\mathcal{P} = \{H_1, \ldots, H_p\}$ be a d-partition of S_n with d-element sets removed and such that $|H| = d+1$ for all $H \in \mathcal{P}$. The $(d+1)p$ sets $\{X | X \subset H \in \mathcal{P}$ and $|X| = d\}$ are unique. Thus

$$(d+1)p \leq \binom{n}{d} \quad \Leftrightarrow \quad p \leq \frac{1}{n+1}\binom{n+1}{d+1}$$

Let $N(n,d) := \binom{n+1}{d+1}/(n+1)$. Since $\binom{n}{k} < (\frac{en}{k})^k$ for $n \geq k \geq 1$ (see p. 1077 of [3]), the number of such d-partitions is bounded by

$$\begin{aligned}\sum_{i=0}^{N(n,d)} \binom{\binom{n}{d+1}}{i} &< (N(n,d)+1)\binom{\binom{n}{d+1}}{N(n,d)} \\ &< 2^{n+1}\left(\frac{e\binom{n}{d+1}}{N(n,d)}\right)^{N(n,d)} \\ &< 2^{n+1}(e(n-d))^{N(n,d)}\end{aligned}$$

and using $N(n,d) < (n+1)^d$ for $d > 1$,

$$\begin{aligned} &< 2^{n+1}(e(n-d))^{(n+1)^d} \\ &= 2^{n+1+(n+1)^d(\log_2 e+\log_2(n-d))}.\end{aligned}$$

□

For n large, it is clear that the upper bound in the previous theorem can be given by $2^{(n+1)^d(\log_2 e+\log_2 n)}$ by absorbing $n+1$ into the $\log_2(n-d)$ term. However, attempting to use this technique to bound $p_n(d)$ yields $\log_2 p_n(d) = O(n^{d+1})$ which is already apparent from Theorem 3. Inserting $d = 2$ in Theorems 1 and 4 yields the following bounds on the number of linear spaces

$$\frac{1}{n!}2^{(n-1)(n-2)/12} \leq p_n^\star(2) \leq 2^{\binom{n}{3}}$$

and the number of linear spaces whose lines contain at most three points is bounded by

$$p_n(2;3) < 2^{n+1+(n+1)^2(\log_2 e+\log(n-2))}.$$

Theorem 4 is interesting from the point that it serves as an upper bound for the number of Steiner triple/quadruple systems ($d = 2, 3$). Recall that $f(n) = O(g(n))$ (resp. $\Omega(g(n))$) if there exist numbers C, n_0 such that $f(n) \leq Cg(n)$ (resp. $\geq$) for all $n \geq n_0$. The results in this paper may be summarized asymptotically (each is readily apparent from the exponents of the bounds in Theorems 1–4):

$$\begin{aligned}\log_2 p_n(d;d+1) &= \Omega(n^d) \\ \log_2 p_n^\star(d;d+1) &= \Omega(n^d - n\log n) \\ \log_2 p_n(d;d+1) &= O(n^d \log n) \\ \log_2 p_n(d) &= O(n^{d+1}).\end{aligned}$$

Achieving better asymptotics for the numbers $p_n(d)$ seems a difficult problem. Attempts at constructing classes containing all d-partitions on S_n resulted in $\log_2 p_n(d) = O(n^{d+1})$.

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