On 3*-connected graphs

Michael Albert, R.E.L. Aldred, Derek Holton

Department of Mathematics and Statistics,
University of Otago

John Sheehan

Department of Mathematical Sciences,
University of Aberdeen

Abstract

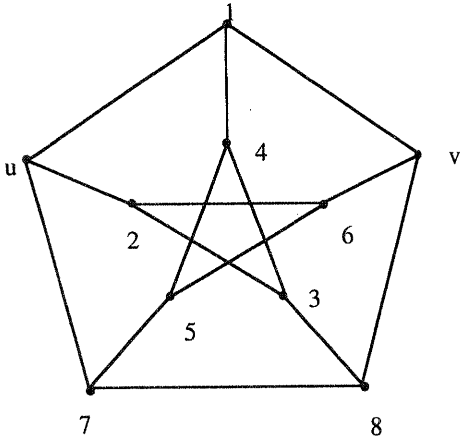
Menger's Theorem states that in a 3-connected graph, any two vertices are joined by three openly disjoint paths. Here we consider 3-connected cubic graphs where two vertices exist so that the three disjoint paths between them contain all of the vertices of the graph (we call these graphs 3*-connected); and also where the latter is true for ALL pairs of vertices (globally 3*-connected). A necessary condition for 3*-connectedness is that the circumference of the graph be at least $2(n+1)/3$ where n is the size of the vertex set.

Global 3*-connectedness is not as strong as it might first appear. Many graphs have this property. In particular the generalized Petersen graphs, $P(n,2)$, are globally 3*-connected if and only if $n \equiv 1, 3 \pmod 6$. The exceptions here are 3*-connected.

1. Introduction

A famous result on connectivity, Menger's Theorem (see [3]), states that in a k-connected graph, there exist k openly disjoint paths between any pair of vertices. We are interested here in graphs where this configuration of the pair of vertices and the openly disjoint paths between them span the graph. Our discussion in this paper will be restricted to the case where $k = 3$ and the graphs are cubic. Hence we say that a 3-connected cubic graph G is **3*-connected** if there exists a pair of vertices $u, v \in V(G)$ such that u, v are the endvertices of three openly disjoint paths P_1, P_2, P_3 such that $V(G) = \bigcup_{i=1}^{3} V(P_i)$. We say that P_1, P_2, P_3 are **covering paths** of G.

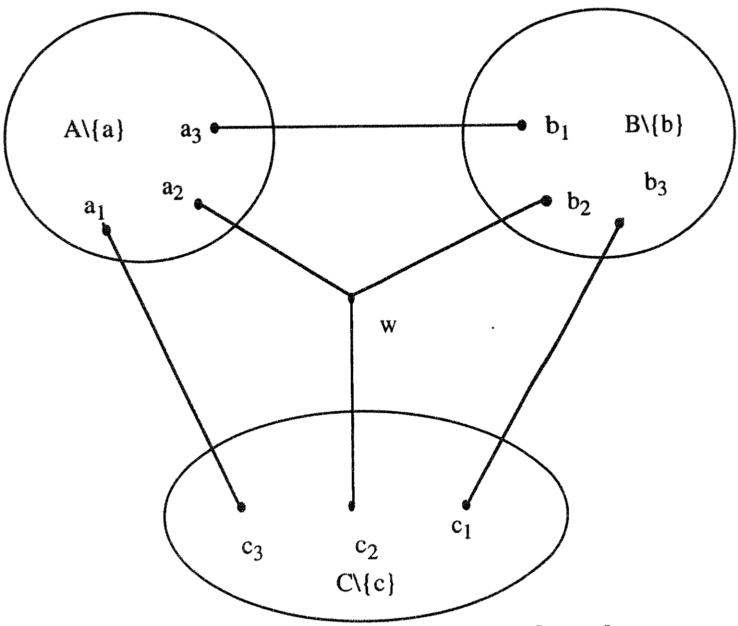
The Petersen graph, P, is an example of a 3*-connected graph. The covering paths are indicated in Figure 1. Many other examples exist. However, not all 3-connected cubic graphs are 3*-connected. In fact we can obtain an infinite class of non 3*-connected graphs by first taking three non-Hamiltonian graphs A, B and C. Let $a \in V(A)$, $b \in V(B)$ and $c \in V(C)$ with neighbours $N(a) = \{a_1, a_2, a_3\}$, $N(b) = \{b_1, b_2, b_3\}$ and $N(c) = \{c_1, c_2, c_3\}$.



$P_1 = (u, 1, v)$
$P_2 = (u, 2, 3, 4, 5, 6, v)$
$P_3 = (u, 7, 8, v)$

Figure 1

Now form the graph Q with $V(Q) = (V(A) \setminus \{a\}) \cup (V(B) \setminus \{b\}) \cup (V(C) \setminus \{c\}) \cup \{w\}$ and $E(Q) = E(A \setminus \{a\}) \cup E(B \setminus \{b\}) \cup E(C \setminus \{c\}) \cup \{wa_2, wb_2, wc_2, a_1c_3, c_1b_3, b_1a_3\}$. The graph is shown schematically in Figure 2.



An infinite family of non 3*-connected graphs
Figure 2

The graph Q is not 3*-connected. To see this assume Q is 3*-connected and $u \in V(A)\backslash\{a\}$ and $v \in V(B)\backslash\{b\}$. Then one of the covering paths of Q has to contain the path a_2wb_2. This forces a Hamiltonian path in $C\backslash\{c\}$ between c_1 and c_3. Hence C has a Hamiltonian cycle, which contradicts the assumption on C. A similar argument applies to any pair of vertices u, v in V(Q) including u = w.

The smallest such non 3*-connected graph is formed by letting A = B = C = P. This graph has order 28.

By **globally 3*-connected** we mean a 3-connected cubic graph G for which **every** distinct pair of vertices in V(G) are the endvertices of some three openly disjoint covering paths. Clearly globally 3*-connected is a stronger concept than 3*-connected. This can be seen from the fact that P is not globally 3*-connected. If it were, then the vertices u, 1 would be endvertices of three openly disjoint paths which span P. However, one of those paths would have to be the edge u1. Then the other two paths would together form a Hamiltonian cycle in P.

This latter example easily generalizes to the following result.

Lemma 1. *If G is globally 3*-connected, then G is Hamiltonian.*

The converse is not true as can be seen from the Hamiltonian graph in Figure 3. For suppose that graph were globally 3*-connected, then there would be three openly disjoint paths between the vertices u and v. Two of those paths would be $P_1 = (u, 1, v)$ and $P_2 = (u, 2, v)$. The third path, starting from u, would need to begin u, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and end 13, v. However, this would omit the vertex 14. Hence, although the graph of Figure 3 is Hamiltonian, it is not globally 3*-connected. Nevertheless, the following lemma does hold.

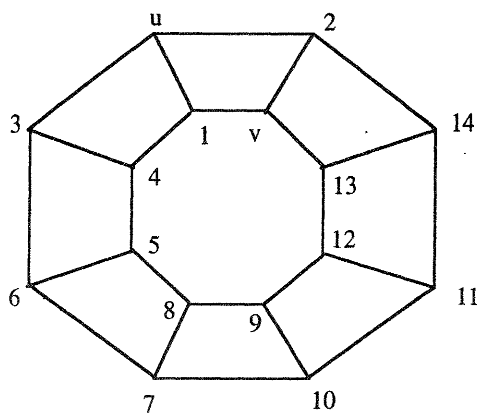


Figure 3

Lemma 2. *If G is 3-connected cubic and Hamiltonian, then G is 3*-connected.*

Proof. Any Hamiltonian cycle H in G has a chord between vertices u,v say. This chord along with the paths from u to v which comprise H, give the three required openly disjoint covering paths. □

Now the reason that the graph of Figure 3 is not globally 3*-connected is that it is bipartite. Before we can prove this result (the Corollary to Lemma 3) we consider a new definition. We say that G is **Hyperhamiltonian** if G is Hamiltonian and $G \setminus \{v\}$ is Hamiltonian for all $v \in VG$.

Lemma 3. *If G is globally 3*-connected, then G is Hyperhamiltonian.*

Proof: G is Hamiltonian by Lemma 1.

Let w be any vertex in G and let (u, w, v) be a path of length 2 in G. Then, since G is globally 3*-connected there exist openly disjoint paths $P_1 = (u, w, v)$, P_2, P_3 such that $\bigcup_{i=1}^{3} V(P_i) = V(G)$. Hence $V(P_2) \cup V(P_3) = V(G) \setminus \{w\}$ and $P_2 \cup P_3$ is a cycle. So $G \setminus \{w\}$ is Hamiltonian for all vertices $w \in VG$ and so G is Hyperhamiltonian.
□

Corollary. *If G is 3-connected cubic bipartite, then G is not globally 3*-connected.*

Proof. Suppose that G is a 3-connected cubic bipartite graph that is globally 3*-connected. Then G is Hyperhamiltonian. But $G \setminus \{v\}$ is non-hamiltonian since it is bipartite and has an odd number of vertices. □

One further elementary result concerns the circumference, c(G), of 3*-connected graphs, G. Recall that the **circumference** of a graph is the length of its longest cycle.

Lemma 4. *If G is 3*-connected, then* $c(G) \geq \left\lceil \frac{2}{3}(n+1) \right\rceil$, *where* $n = |V(G)|$.

Proof. Let $\ell_i = |V(P_i)| - 2$, with $\ell_1 \geq \ell_2 \geq \ell_3$. Then $n = \ell_1 + \ell_2 + \ell_3 + 2$ and

$$c(G) \geq \ell_1 + \ell_2 + 2 = n - \ell_3 \qquad (1)$$

But $$n = \ell_1 + \ell_2 + \ell_3 + 2 \geq 3\ell_3 + 2.$$

Hence from (1),

$$c(G) \geq n - \frac{1}{3}(n-2) = \frac{2}{3}(n+1).$$

The result follows since c(G) is an integer. □

In [2], Bondy and Simonovits constructed an infinite class of 3-connected cubic graphs such that, for $n = |V(G)|$ sufficiently large,

$$c(G) \leq |V(G)|^t, \text{ where } 0 < t < 1.$$

It can be shown that these graphs are not 3*-connected for $n \geq 90$. This provides us with a further set of non 3*-connected graphs in addition to those of Figure 2.

On the other hand, there are also infinite sets of globally 3*-connected graphs. The graph of Figure 3 is an even prism. We know that none of these are globally 3*-connected because they are bipartite. However, the odd prisms are all globally 3*-connected. We note that G is a **prism** if $V(G) = \{a_i, b_i : 1, 2, \ldots, n\}$ and $E(G) = \{a_ia_{i+1}, b_ib_{i+1}, a_ib_i, i = 1, 2, \ldots, n\}$ where $a_na_{n+1} = a_na_1$ and $b_nb_{n+1} = b_nb_1$. A prism is **odd** if $|V(G)| = 2n$, where n is odd.

Lemma 5. *The odd prisms are globally 3*-connected.*

Proof

Case 1: Let $u = a_i$ and $v = a_j$, where $j > i$. Then, since n is odd, we may assume without loss of generality, that $j - i$ is odd. The three covering paths are then

$(a_i, a_{i-1}, a_{i-2}, \ldots, a_1, a_n, \ldots, a_{j+1}, a_j)$,
$(a_i, a_{i+1}, b_{i+1}, b_{i+2}, a_{i+2}, a_{i+3}, b_{i+3}, b_{i+4}, a_{i+4}, \ldots, a_j)$,
$(a_i, b_i, b_{i-1}, \ldots, b_{j+1}, b_j, a_j)$.

Case 2: Let $u = a_i$ and $v = b_j$. If $j = i$, then the covering paths are

(a_i, b_i),
$(a_i, a_{i-1}, b_{i-1}, b_i)$,
$(a_i, a_{i+1}, \ldots, a_{i-2}, b_{i-2}, b_{i-3}, \ldots, b_{i+1}, b_i)$.

If $j \neq i$, then we may assume that $j - i$ is odd. In this case, covering paths are

$(a_i, a_{i+1}, \ldots, a_j, b_j)$,
$(a_i, b_i, ,b_{i+1}, \ldots, b_{j-1}, b_j)$,
$(a_i, a_{i-1}, b_{i-1}, b_{i-2}, a_{i-2}, \ldots, a_{j+3}, b_{j+3}, b_{j+2}, a_{j+2}, a_{j+1}, b_{j+1}, b_j)$. □

The ladders form another infinite set of globally 3*-connected graphs. A graph G is a **ladder** if $V(G) = \{a, b, a_i, b_i ; i = 1, 2, \ldots, n\}$ and $E(G) = \{aa_1, ab_1, ba_n, bb_n, ab\} \cup \{a_i a_{i+1}, b_ib_{i+1}, a_ib_i; i = 1, 2, \ldots, n-1\}$.

Lemma 6. *The ladders are globally 3*-connected.*

Proof: We use a number of cases to find the covering paths P_i, $i = 1, 2, 3$ with endvertices u and v.

Case 1: $u = a$.

Case 1.1: $v = b$. Then $P_1 = (u, v)$, $P_2 = (u, a_1, a_2, \ldots, a_n, v)$ and $P_3 = (u, b_1, b_2, \ldots, b_n, v)$.

Case 1.2: $v = a_i$. Then $P_1 = (u, a_1, a_2, \ldots, v)$ and $P_2 = (u, b_1, b_2, \ldots, b_i, v)$. If $n - i$ is even, then $P_3 = (u, b, a_n, b_n, b_{n-1}, a_{n-1}, a_{n-2}, b_{n-2}, b_{n-3}, a_{n-3}, \ldots, b_{i+2}, b_{i+1}, a_{i+1}, v)$. If $n - i$ is odd, then $P_3 = (u, b, b_n, a_n, a_{n-1}, b_{n-1}, b_{n-2}, a_{n-2}, a_{n-3}, b_{n-3}, \ldots, b_{i+2}, b_{i+1}, a_{i+1}, v)$.

(The case where $u = b$ follows by symmetry.)

Case 2: $u = a_i$, $v = a_j$, for $j > i$.

Here let $P_1 = (u, a_{i+1}, a_{i+2}, \ldots, a_{j-1}, v)$ and $P_2 = (u, b_i, b_{i+1}, \ldots, b_j, v)$. Now let $Q(a_j, b)$ be one of the paths P_3 from Case 1.2 and $Q'(a_i, a)$ be the corresponding path from a_i to a. Then here $P_3 = Q'(a_i, a) \cup (a, b) \cup Q(b, a_j)$, where $Q(b, a_j)$ is the path $Q(a_j, b)$ traversed in the other direction.

Case 3: $u = a_i$, $v = b_j$, where $j \geq i$ without loss of generality. Here let $P_1 = (u, a_{i+1}, a_{i+2}, \ldots, a_j, v)$ and $P_2 = (u, b_i, b_{i+1}, \ldots, b_{j-1}, v)$. The path P_3 is the same as P_3 from Case 2, except that that part of P_3 from b to v is reflected about the line of symmetry ab. □

Note that in the above proof, P_1 is always a **geodesic** (a path of shortest distance between its end vertices). Hence we could say that the ladders are geodesically globally 3*-connected. On the other hand, the odd prisms are not geodesically globally 3*-connected. This can be seen by considering the two vertices a_i, a_{i+3}.

Given any two globally 3*-connected graphs it is possible to find a new globally 3*-connected graph using the following construction. Let G, H be 3-connected cubic graphs and let $x \in V(G)$ and $y \in V(H)$. Suppose $N_G(x) = \{x_1, x_2, x_3\}$ and $N_H(y) = \{y_1, y_2, y_3\}$. We define the **xy-join of G and H** to be the graph $G*H$ with vertex set $(V(G) \setminus \{x\}) \cup V(H \setminus \{y\})$ and edge set $E(G \setminus \{x\}) \cup E(H \setminus \{y\}) \cup \{x_i y_i : i = 1, 2, 3\}$.

Lemma 7. *If G, H are globally 3*-connected, then so is G*H.*

Proof: Let $u, v \in V(G*H) \cap V(G)$. Let $P_1 P_2, P_3$ be paths with endvertices u, v, such that $\bigcup_{i=1}^{3} V(P_i) = V(G)$. Suppose, without loss of generality, that $xx_1, xx_2 \in E(P_3)$. Since H is globally 3*-connected it contains a Hamiltonian cycle C avoiding the edge yy_3. Then $P_1' = P_1$, $P_2' = P_2$ and P_3' derived from $(P_3 \setminus \{xx_1, xx_2\}) \cup (C \setminus \{yy_1, yy_2\}) \cup \{x_1y_1, x_2y_2\}$, are the appropriate covering paths in $G*H$.

Let $u \in V(G^*H) \cap V(G)$ and $v \in V(G^*H) \cap V(H)$. Then there exist paths P_1, P_2, P_3 with endvertices, u and x which cover G and paths Q_1, Q_2, Q_3 with endvertices y and v which cover H. Without loss of generality assume that $xx_i \in E(P_i)$, $yy_i \in E(Q_i)$, $i = 1, 2, 3$. Then P_i' derived from $(P_i \setminus \{xx_i\}) \cup \{x_i y_i\} \cup (Q_i \setminus \{yy_i\})$, $i = 1, 2, 3$, are the appropriate covering paths in G*H. □

On the other hand, if the 3-connected cubic graph K contains a cyclic edge cut of size 3, then K can be considered as isomorphic to G*H for some G and H. (A **cyclic edge cut** is an edge cut whose removal breaks the graph into two components, each of which contains a cycle.) If K is globally 3*-connected, then so are G and H. The proof follows in a similar way to that of Lemma 7.

Corollary: *Suppose $K = G^*H$. Then K is globally 3*-connected if and only if both G and H are globally 3*-connected.*

This Corollary suggests that we should concentrate our investigation on cyclically 4-edge connected cubic graphs. These are graphs with cyclic edge cuts of size greater than 3. As a step in this direction, in the next section, we look at the generalized Petersen graphs P(n, 2).

2. Generalized Petersen Graphs

For the remainder of this paper we consider the 3*-connected properties of the generalized Petersen graph P(n, 2). The generalized Petersen graph P(n, k) for $n \geq 5$, has vertex set $\{i, i' : i = 1, 2, \ldots, n\}$ and edge set

$$\{i(i+1), i\,i' : i = 1, 2, \ldots, n\} \cup \{i'(i+k)' : i = 1, 2, \ldots, n\},$$

where the subscripts larger than n are to be read modulo n.

We first show that P(n, 2) is Hyperhamiltonian if and only if $n \equiv 1, 3 \pmod 6$.

Lemma 8. *Let $n \equiv 1, 3 \pmod 6$, $n \geq 5$. Then P(n, 2) is Hyperhamiltonian.*

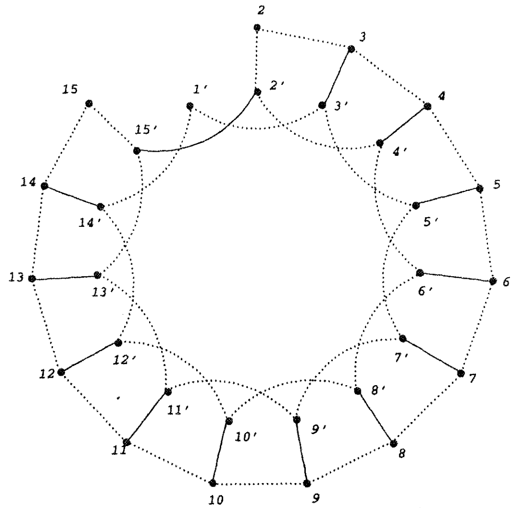
Proof: Suppose $n \equiv 1, 3 \pmod 6$. From Bondy [1] we know that P(n, 2) is Hamiltonian since $n \not\equiv 5 \pmod 6$.

Now the automorphism group of P(n, 2) has at most two orbits. These are $\{i : i = 1, 2, \ldots, n\}$, the outer rim vertices, and $\{i' : i = 1, 2, \ldots, n\}$, the inner rim vertices. Hence we only have to show that $P(n, 2) \setminus \{1\}$ and $P(n, 2) \setminus \{1'\}$ are Hamiltonian. This follows since

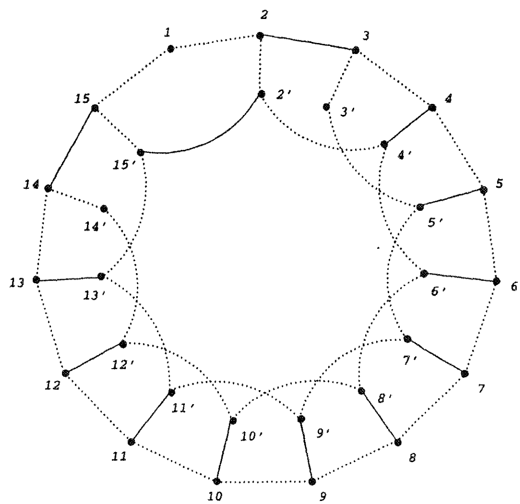
$(2, 3, \ldots, n, n', (n-2)', (n-4)', \ldots, 5', 3', 1', (n-1)', \ldots, 6', 4', 2', 2)$ and

(1, 2, 2′, 4′, 6′, 8′, 10′, …, (n - 1) ′, n - 1, n - 2, …, 3, 3′, 5′, 7′, …, n′, n, 1)

are the required Hamiltonian cycles. (This is illustrated in Figures 4 and 5 for n = 15. Actually these cycles exist in P(n, 2) less the appropriate vertex for all odd n.) □



a Hamiltonian cycle in P(15, 2)\1
Figure 4



a Hamiltonian cycle in P(15, 2)\1′
Figure 5

In the proof of the following lemma we omit some of the tedious details.

Lemma 9. *Let $n \not\equiv 1, 3 \pmod 6$ and $n \geq 5$. Then $P(n, 2)$ is not Hyperhamiltonian.*

Proof. From Bondy [1], we see that $P(n, 2)$ is not Hamiltonian if and only if $n \equiv 5 \pmod 6$. So we now suppose that $n \equiv 0, 2, 4 \pmod 6$. We know that $P(n, 2)$ is Hamiltonian for these values of n so we will concentrate on $G = P(n, 2) \setminus \{1\}$ and $G' = P(n, 2) \setminus \{1'\}$. We will show that for $n \equiv 0, 4 \pmod 6$, G is not Hamiltonian and for $n \equiv 2 \pmod 6$, G' is not Hamiltonian. In each case the proof is by contradiction. In the style of Bondy [1] we make a series of observations (Ai, Bj), which can be confirmed by reference to specific cases.

Case 1: $n \equiv 0, 4 \pmod 6$. Suppose that G has a Hamiltonian cycle C. Then the edges of C naturally partition the outer rim of G into paths $P_1, P_2, \ldots, P_k$ of lengths $n_1 - 1, n_2 - 1, \ldots, n_k - 1$, respectively, where the paths are sequentially labelled so that $2 \in V(P_1)$. The following general observations can be deduced.

(A1) $\sum_{i=1}^{k} n_i = n - 1 \equiv 1 \pmod 2$;

(A2) $k \geq 2$;

(A3) $n_i \geq 2$;

(A4) $n_i \equiv 1 \pmod 2$ implies $n_i = 3$;

(A5) $n_1 + n_k > 4$;

(A6) $n_1 \geq 4$ implies $n_2 = 2$;

(A7) $n_k \geq 4$ implies $n_{k-1} = 2$;

(A8) $n_i \geq 4$ $(1 < i < k)$ implies $n_{i-1} = n_{i+1} = 2$;

(A9) $n_i \geq 4$ $(1 \leq i \leq k)$ implies $n_j \neq 3$ $(1 \leq j \leq k)$;

(A10) $2 \leq n_i \leq 3$.

Observation A1 follows since in G there are only an odd number of outer rim vertices remaining. If $k = 1$, then C is not Hamiltonian and so A2 follows. Clearly, A3, $n_i \neq 1$, since C is a cycle. If n_i is odd and $n_i > 3$, then C contains a sub-cycle $(k, k + 1, k + n_i - 1, (k + n_i - 1)', (k + n_i - 3)', \ldots, k', k)$ of order $\frac{(3n_i + 1)}{2}$. Hence A4 is true.

Consider A5. Suppose $n_1 = n_k = 2$. Then $2'n' \notin E(C)$ since C is a Hamiltonian cycle.

Hence $n_2 = n_{k-1} = 2$. By A1 there must be an odd n_i and by A4 such an $n_i = 3$. Because of the pattern developed in producing C we may assume that this odd path includes $6', 6, 7$ or $8', 8, 9$. In each case we contradict the fact that C is Hamiltonian.

Observations A6, A7 and A8 quickly follow from A4 and drawing any special case. For A9 we may as well suppose $n_{i-2} = 3$, because otherwise the pattern of C continues until an odd n_j is reached. But then we have a 5-cycle in C. By A9, if $n_i \geq 4$ there is no j with $n_j = 3$ and so A1 is contradicted. Hence A10 is proved.

Now suppose that $2'n' \notin E(C)$. Then $n_1 \neq 3$, $n_k \neq 3$ because otherwise C contains a 5-cycle including 2 or n. By A10, $n_1 = n_k = 2$. But this contradicts A4. Hence suppose that $2'n' \in E(C)$. From A5 and A10 there is no loss of generality in assuming that $n_1 = 3$. If $n_i = 3$ for each i then C contains the smaller cycle $(2, 3, 4, 4', 6', 8', 8, 9\ 10, \ldots, (n-2)', n-2, n-1, n, n', 2', 2)$ which is impossible. Therefore we may assume, using A10, that there exists $1 \leq t < k$ such that $n_1 = n_2 = \ldots = n_t = 3$ and $n_{t+1} = 2$. This easily implies that $n_{t+2} = 2$. To avoid a further "closed" sequence in C, it now follows that $n_j = 3$, $t + 3 \leq j \leq k$. Hence, $\sum_{\ell=1}^{k} n_\ell = 3k - 2 \equiv 1 \pmod 2$ from A1. So $k \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 6$. We have therefore proved that P(n, 2) is not Hyperhamiltonian when $n \equiv 0, 4 \pmod 6$.

Case 2: $n \equiv 2 \pmod 6$. Suppose that G′ has a Hamiltonian cycle C. Then the edges of C naturally partition the outer rim of G′ into paths P_i of length $n_i - 1$ where, as above, the paths are sequentially labelled but in this case $1 \in V(P_1)$. As above, we now make a number of observations.

(B1) $\sum_{i=1}^{k} n_i = n \equiv 2 \pmod 6$;

(B2) $k \geq 2$;

(B3) $n_i \geq 2$;

(B4) $3 \leq n_1 \leq 5$;

(B5) $n_1 = 3$ implies $n_2 \neq 3, n_k \neq 3$;

(B6) $n_1 = 4$ implies $\{n_2, n_k\} = \{2, 3\}$;

(B7) $n_1 = 5$ implies $2 \leq n_2, n_k \leq 3$, $n_2 + n_k > 4$;

(B8) $n_i \equiv 1 \pmod 2$, $i \neq 1$ implies $n_i = 3$;

(B9) $\quad n_2 \geq 4$ implies $n_1 = 3, n_3 = 2$;

(B10) $\quad n_k \geq 4$ implies $n_1 = 3,\ n_{k-1} = 2$;

(B11) $\quad n_i \geq 4\ (3 \leq i < k)$ implies $n_{i-1} = n_{i+1} = 2$;

(B12) $\quad n_i \geq 4\ (i \neq 1)$ implies $n_j \neq 3$ for all $j \neq i,\ (2 \leq j \leq k)$.

Now suppose that $n_i \geq 4\ (i \neq 1)$. If $n_1 = 5$, then B7 and B12 give a contradiction. If $n_1 = 4$ then B6 and B12 give a contradiction. Finally if $n_1 = 3$ then B8 together with B12 imply $n \equiv 1 \pmod 2$ which contradicts B1. Hence

(B13) $\quad 2 \leq n_i \leq 3\ (i \neq 1)$;

(B14) $\quad n_i = 3\ (i \neq 1)$ implies $n_{i-1} + n_{i+1} \geq 5$ (integers modulo k);

(B15) $\quad n_{i-1} = 2\ (i \neq 1)$ implies $n_i \neq 3$.

We prove B15 as follows. Suppose that $n_i = 3$. Then it follows easily that $n_{i+1} = n_{i+2} = \ldots = n_k = 3$. Since the vertex $n - 1$ cannot belong to a P_i with $n_i = 3$, it follows that $P_k = (n-4,\ n-3, n-2)$ and

(i) $\quad P_1 = (n-1, n, 1, 2)$ or $\qquad$ (ii) $\quad P_1 = (n-1, n, 1, 2, 3)$.

If (i) is true $n_2 = 2,\ n_3 = 3$ and $n = 3(k-2) + 4 + 2 = 3k$. This contradicts B1. So we may suppose (ii) to be the case. It then follows that all of the n_i's equal 3 apart from n_1 which is 5 and exactly one or two of the n_i's which equal 2. In both cases B1 is contradicted. Similarly

(B16) $\quad n_i = 2\ (i \neq 1)$ implies $n_{i-1} \neq 3$;

(B17) $\quad n_i = 2\ (i \neq 1)$.

We prove B17 as follows. Suppose that $n_i = 3$ for some $1 < i \leq k$. Then, from B15 and B16, $n_i = 3$ for $1 \leq i \leq k$. Hence from B5, B6 and B7, $n_1 = 5$ and in this case C contains the closed sequence:

$$(1, 2, 3, 3', 5', 7', 7, 8, 9, 9', \ldots, n-7, n-6, n-5, (n-5)', (n-3)', (n-1)', n-1, n, 1)$$

which is impossible.

Finally, from B5, B6 and B17, $n_1 = 3$ and hence $n \equiv 1 \pmod 2$ which contradicts B1.

□

Theorem 10 *P(n, 2) is globally 3*-connected if and only if $n \equiv 1, 3 \pmod 6$.*

To help simplify the proof we adopt the following notation for paths. Each square bracket represents a repeating pattern.

$$[i, j] = (i, i + 1, i + 2, \ldots, j);$$

$$[i', j']_2 = (i', (i + 2)', (i + 4)', \ldots, j');$$

$$[i, j]^* = (i, i + 1, (i + 1)', (i + 3)', i + 3, i + 2, (i + 2)', (i + 4)', i + 4, i + 5, \ldots, j);$$

$$[i, j]_3 = (i, i + 1, i + 2, (i + 2)', (i + 4)', (i + 6)', i + 6, i + 7, i + 8, \ldots, j);$$

$[u, v]_\alpha^-$ is any of the paths above traced in an anti-clockwise direction in the sense that $[i, j]^{-1} = (j, j + 1, j + 2, \ldots, i - 1, i)$.

The paths $[i, j]$ and $[i', j']_2$ consist solely of vertices in the outer and inner rims, respectively. This is not the case for paths $[i, j]^*$ and $[i, j]_3$. It turns out that we often require a subpath of these paths which may start and end on outer or inner vertices. For instance

$$[i', j]^* = (i', (1 + 2)', \ i + 2, \ i + 1, \ (i + 1)', (i + 3)', i + 4, \ldots, j), \text{ and}$$
$$[i', j']_3 = (i', (i + 2)', (i + 4)', i + 4, i + 5, i + 6, (i + 6)', (i + 8)', (i + 10)', (i + 10), \ldots, j').$$

No matter what the end vertices of these subpaths, the pattern of the path is followed.

Proof: By Lemma 9, P(n, 2) is not Hyperhamiltonian for $n \not\equiv 1, 3 \pmod 6$. So if $n \not\equiv 1,3 \pmod 6$, P(n, 2) is not globally 3*-connected. We may therefore assume that $n \equiv 1, 3 \pmod 6$.

Since P(n, 2) has two orbits, the inner and outer rim vertices, the proof only requires us to consider the three cases of covering paths between (1) a fixed vertex, u, on the outer rim and any vertex, v, on the outer rim; (2) a fixed vertex on the outer rim, u, and any vertex, v, on the inner rim; and (3) a fixed vertex, u, on the inner rim and any vertex, v, on the inner rim. By Lemma 8 we know that P(n, 2) is Hamiltonian. Since there are at most three different kinds of edges in P(n, 2) and at least one of every kind must be absent from a Hamiltonian cycle in P(n, 2), then there are three covering paths joining two neighbouring vertices. If u and v are a distance two apart, let w be the vertex such that (u, w, v) is a path in P(n, 2). Then by Lemma 8 there is a Hamiltonian cycle in P(n, 2) \ {w}. Hence we may take two of the covering paths from the Hamiltonian cycle and the other is (u, w, v). Hence in what follows we do not have to consider neighbouring vertices or vertices which are a distance two apart.

We also note here that the permutation σ defined by $\sigma(1) = 1$, $\sigma(1') = 1'$, $\sigma(i) = n - i + 2$ and $\sigma(i') = (n - i + 2)'$ is an automorphism of G. Since n is odd, if k is even, then $\sigma(k)$ and $\sigma(k')$ are odd. So if we find covering paths for 1 and k, 1 and k′ or 1′ and k′ for k even, the corresponding paths for k odd follow by applying σ. Hence in the three cases below may assume that k is even.

Case 1. Here we consider covering paths between the vertex u = 1 and the vertex v= k.

If $k \equiv 0 \pmod 6$, then $P_1 = [1, k]^-$, $P_2 = (1, [1', k']_2^-, k)$ and $P_3 = (1, 2, 3, [3', (k+1)']_3, [(k+3)', 4']_2, [4, k]_3)$.

We show these paths in Figure 6, for n = 25 and k = 12.

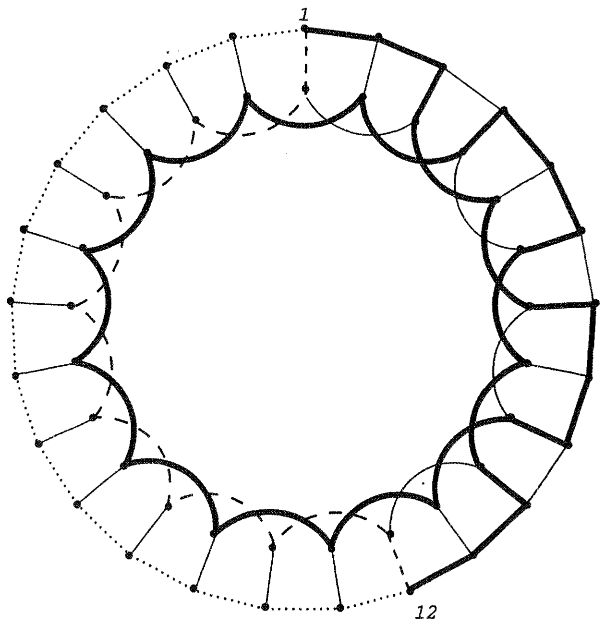


Figure 6

If $k \equiv 2 \pmod 6$ and $n \equiv 1 \pmod 6$, then $P_1 = [1, k]$, $P_2 = (1, [1', (k+1)']_2, k+1, k)$ and $P_3 = ([1, k+4]_3^-, (k+4)', (k+2)', k+2, k+3, [(k+3)', 2']_3, [4', k']_2, k)$.
If $k \equiv 2 \pmod 6$ and $n \equiv 3 \pmod 6$, then $P_1 = [1, k]_3^-$, $P_2 = (1, [1', (k-3)']_2, k-3, k-2, (k-2)', k', k)$ and $P_3 = ([1, k-4], [(k-4)', 4']_2^-, [2', (k-1)']_3^-, k-1, k)$.

If $k \equiv 4 \pmod 6$, then $P_1 = [1, k]^-$, $P_2 = (1, [1', k']_2^-, k)$ and $P_3 = (1, 2, [2', (k+3)']_2^-$, $[(k+1)', 5']_3^-$, $3', 3, 4, [4', k]_3)$. (Note here that $[4', k]_3$ is a proper subpath of $[4', k']_3$ which has endvertices $4'$ and k. This path is empty if $k = 4$. Similarly, if $k = 4$, $[(k+1)', 5']_3^-$ is the single vertex $5'$. Below we will not comment on similar redundancies when k is small.)

Case 2. Here we consider covering paths between the vertex $u = 1$ and the vertex $v = k'$.

If $k \equiv 0 \pmod 6$ and $n \equiv 1 \pmod 6$, then $P_1 = (1, 2, [2', k']_2)$, $P_2 = (1, 1', (n-1)'$, $n-1, n-2, n-3, [(n-3)', k']_3^-)$ and $P_3 = (1, n, [n', (k-3)']_3^-, [(k-5)', 3]_2^-, [3, k], k')$.

If $k \equiv 0 \pmod 6$ and $n \equiv 3 \pmod 6$, then $P_1 = ([1, k], k')$, $P_2 = (1, n, [n', k']_2)$ and $P_3 = (1, [1', (k-5)']_2, [(k-3)', (n-2)']_3, n-2, n-1, (n-1)', (n-3)', [(n-3), k']_3^-)$.

If $k \equiv 2 \pmod 6$, then $P_1 = ([1, k]^-, k')$, $P_2 = (1, [1', k']_2^-)$ and $P_3 = ([1, k-5]_3, (k-5)'$, $(k-3)', k-3, k-4, [(k-4)', n']_3^-, [(n-2)', (k-1)']_2^-, k-1, k-2, (k-2)', k')$.

If $k \equiv 4 \pmod 6$, then $P_1 = ([1, k]^-, k')$, $P_2 = (1, [1', k']_2^-)$ and $P_3 = ([1, k-1]_3$, $[(k-1)', (n-2)']_2, [n', k']_3)$.

Case 3. Here we consider covering paths between the vertex $u = 1'$ and the vertex $v = k'$.

If $k \equiv 0 \pmod 6$, then $P_1 = (1', [1, k]^-, k')$, $P_2 = [1', k']_2^-$ and $P_3 = (1', 3', 3, 2$, $[2', (k+3)']_2^-, [(k+3)', 5']_3^-, 5, 4, 4', 6', [6, k']_3)$.

If $k \equiv 2 \pmod 6$, then $P_1 = (1', [1, k]^-, k')$, $P_2 = [1', k']_2^-$ and $P_3 = ([1', (k+3)']_3$, $[(k+3)', 2']_2, [2, k']_3)$.

If $k \equiv 4 \pmod{12}$ and $n \equiv 1 \pmod 6$, then $P_1 = (1', [1, k], k')$, $P_2 = [1', k']_3$ and $P_3 = ([1', (k-3)']_2, [(k-3)', n']_3, [n', k']_2)$.

If $k \equiv 4 \pmod{12}$ and $n \equiv 3 \pmod 6$, then $P_1 = (1', 3', 3, 2, 2', 4', [4, k']^*)$, $P_2 = (1', 1$, $n, [n', (k+1)']_3^-, k+1, k, k')$ and $P_3 = (1', (n-1)', [(n-1), (k+2)']_3^-, k')$.

If $k \equiv 10 \pmod{12}$ and $n \equiv 1 \pmod 6$, then $P_1 = (1', 3', [3, k']^*)$, $P_2 = [1', k']_3^-$ and $P_3 = (1', 1, 2, 2', [n', (k+3)']_3^-, [k+3, k+1]^-, (k+1)', (k-1)', k-1, k, k')$.

If $k \equiv 10 \pmod{12}$ and $n \equiv 3 \pmod 6$, then $P_1 = (1', 3', 5', [5, 2]^-, 2', 4', 6', [6, k']^*)$, $P_2 = (1', 1, n, [n', (k+1)']_3^-, k+1, k, k')$ and $P_3 = (1', (n-1)', [(n-1), k']_3^-$. □

When we first considered the concept of globally 3*-connected cubic graphs, we anticipated that the set would be quite restricted and that a simple characterization might be obtained. To date we have only shown that hyperhamiltonicity, non-biparticity and a circumference condition are necessary conditions. The very large number of small graphs which are globally 3*-connected suggests that a simple characterization may not be possible.

References

[1] J.A. Bondy, Variations on the hamiltonian theme, *Canad. Math Bulletin*, 15, 1972, 57-62.

[2] J.A. Bondy and M. Simonovits, Longest cycles in 3-connected cubic graphs, *Canad. J. Math*, 32, 1980, 987-992.

[3] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.*, 10, 1927, 95-115.

(Received 21/7/2000)