PAIRWISE BALANCED DESIGNS WITH BLOCK SIZES 4 AND 8

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Abstract:

Let K be a set of positive integers. A pairwise balanced design (PBD) of index unity B(K,1;v) is a pair (X,$\mathcal{B}$) where X is a v-set (of points) and $\mathcal{B}$ is a collection of subsets of X (called blocks) with sizes from K such that every pair of distinct points of X is contained in exactly one block of $\mathcal{B}$. A necessary condition for the existence of a PBD B({4,8},1;v) is $v \equiv 0$ or 1(mod 4). It is shown that this necessary condition is also sufficient for all $v \geq 4$ with 11 exceptions and 25 possible exceptions of which 177 is the largest. We briefly mention some applications to other types of combinatorial structures.

1. Introduction

Let K be a set of positive integers. A pairwise balanced design (PBD) of index unity B(K,1;v) is a pair (X,$\mathcal{B}$) where X is a v-set (of points) and $\mathcal{B}$ is a collection of subsets of X (called blocks) with sizes from K such that every pair of distinct points of X is contained in exactly one block of $\mathcal{B}$. The number $|X| = v$ is called the order of the PBD.

We shall denote by B(K) the set of all integers v for which there exists a PBD B(K,1;v). For convenience, we define $B(k_1,k_2,\ldots,k_r)$ to be the set of all integers v such that there is a PBD $B(\{k_1,k_2,\ldots,k_r\},1;v)$. A set K is said to be PBD-closed if $B(K) = K$.

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Pairwise balanced designs are of fundamental importance in combinatorial theory and have been used extensively in the construction of other types of combinatorial designs. Quite often, one is generally interested in constructing PBDs $B(K,1;v)$ for some specified set K. In this connection, R. M. Wilson's remarkable theory concerning the structure of PBD-closed sets (see [21-23]) often provides us with some form of "asymptotic" results as follows:

<u>**Theorem 1.1**</u> (Wilson's Theorem) Let K be a set of positive integers and define the two parameters:

$$\alpha(K) = g\cdot c\cdot d\cdot\{k-1: k \in K\}, \text{ and}$$

$$\beta(K) = g\cdot c\cdot d\cdot\{k(k-1): k \in K\}.$$

Then there exists a constant C (depending on K) such that, for all integers $v>C$, $v \in B(K)$ if and only if $v-1 \equiv 0 (\text{mod } \alpha(K))$ and $v(v-1) \equiv 0(\text{mod } \beta(K))$.

We wish to remark that, for a given set K, Wilson's theory does not really provide any concrete upper bound on the constant C in Theorem 1.1. In this paper we investigate the spectrum of $B(4,8)$. Wilson's Theorem tells us that there is a constant C such that, for all $v > C$, $v \in B(4,8)$ if and only if $v \equiv 0$ or $1 \pmod 4$. We establish a concrete upper bound on C, namely, we are able to show that $v \in B(4,8)$ for all $v \geq 4$ with 11 exceptions and 25 possible exceptions of which 177 is the largest. The spectrum of $B(4,8)$ is not only of interest in its own right, but it also provides useful applications to the construction of other types of combinatorial structures such as a variety of short conjugate-orthogonal quasigroup identities, orthogonal arrays with interesting conjugacy properties, edge-coloured designs, and Mendelsohn designs. The reader is referred to [1,2,9,13,14,16] for more details.

2. Preliminaries

In this section we shall define some terminology and adapt the notations of earlier papers (see, for example, [3]). For more details on PBDs and related designs, the reader is referred to [4,10,20].

Definition 2.1 Let K and M be sets of positive integers. A group divisible design (GDD) GD(K,1,M;v) is a triple $(\mathbb{X},\mathbb{G},\mathbb{B})$ where

(i) X is a v-set (of points),

(ii) $\mathbb{G}$ is a collection of non-empty subsets of X (called groups) with sizes in M and which partition X,

(iii) $\mathbb{B}$ is a collection of subsets of X (called blocks), each with size at east two in K,

(iv) no block meets a group in more than one point, and

(v) each pairset {x,y} of points not contained in a group is contained in exactly one block.

The group-type (or type) of a GDD $(\mathbb{X},\mathbb{G},\mathbb{B})$ is the multiset $\{|G|\} : G \in \mathbb{G}\ \}$ and we usually use the "exponential" notation for its description: a group-type $1^i2^j3^k$ ... denotes i occurrencies of groups of size 1,j occurrences of groups of size 2, and so on.

Definition 2.2 A transversal design (TD) T(k,1;m) is a GDD with km points, k groups of size m and m^2 blocks of size k, where each block meets every group in precisely one point, that is, each block is a transversal of the collection of groups.

Definition 2.3 Let $(X,\mathbb{B})$ be a PBD B(K,1;v). A parallel class in $(X,\mathbb{B})$ is a collection of disjoint blocks of $\mathbb{B}$, the union of which equals X. $(X,\mathbb{B})$ is called resolvable if the blocks of $\mathbb{B}$ can be partitioned into parallel classes. A GDD GD(K,1,M;v) is resolvable if its associated PBD B(K ∪ M,1;v) is resolvable with M as a parallel class of the resolution.

It is fairly well-known that the existence of a resolvable TD

T(k,1;m) (briefly RT(k,1;m)) is equivalent to the existence of a TD T(k+1,1;m) or equivalently k-1 mutually orthogonal Latin squares (MOLS) of order m. In particular, the following two results can be found in [15].

<u>Theorem 2.4</u> For every prime power q, there exists a T(q+1,1;q).

<u>Theorem 2.5</u> Let $m = p_1^{k_1} \; p_2^{k_2} \ldots p_r^{k_r}$ be the factorization of m into powers of distinct primes p_i, then a T(k,1;m) exists where $k \le 1 + \min \{p_i^{k_i}\}$.

The following result will be quite useful (see [7]).

<u>Theorem 2.6</u> A T(8,1;m) exists for all integers m > 76.

We need to establish some more notations. We shall simply write B(k,1;v) for B({k},1;v) and similarly GD(k,1,m;v) for GD({k},1,{m};v). We observe that a PBD B(k,1;v) is essentially a <u>balanced incomplete block design</u> (BIBD) with parameters v, k and $\lambda = 1$. If $k \notin K$, then $B(K \cup \{k^*\},1;v)$ denotes a PBD $B(K \cup \{k^*\},1;v)$ which contains a unique block of size k and if $k \in K$, then a $B(K \cup \{k^*\},1;v)$ is a PBD B(K,1;v) containing at least one block of size k. We shall sometimes refer to a GDD $(X,\mathcal{G},\mathcal{B})$ as a K-GDD if $|B| \in K$ for every block $B \in \mathcal{B}$.

For some of our recursive constructions of PBDs and GDDs, we shall make use of Wilson's "Fundamental Construction" (see [20]). We define a <u>weighting</u> of a GDD $(X,\mathcal{G},\mathcal{B})$ to be any mapping $w: X \to Z^+ \cup \{0\}$. We present a brief description of Wilson's construction relating to GDDs below.

<u>Construction 2.7</u> (Fundamental Construction) Suppose that $(X,\mathcal{G},\mathcal{B})$ is a "master" GDD and let $w: X \to Z^+ \cup \{0\}$ be a weighting of the GDD. For every $x \in X$, let S_x be w(x) "copies" of x. Suppose that for each block $B \in \mathcal{B}$, a GDD $(\cup_{x \in B} S_x, \{S_x : x \in B\}, \mathcal{A}_B)$ is given. Let $X^* = \cup_{x \in X} S_x$, $\mathcal{G}^* = \{\cup_{x \in G} S_x : G \in \mathcal{G}\}$, and $\mathcal{B}^* = \cup_{B \in \mathcal{B}} \mathcal{A}_B$. Then $(X^*, \mathcal{G}^*, \mathcal{B}^*)$ is a GDD.

3. Useful Known Results

In most of what follows, we shall make use of some well-known results which we state below. The interested reader may wish to consult the references cited for more details.

Theorem 3.1 (see [10]). A $B(4,1;v)$ exists if and only if $v \equiv 1$ or 4 (mod 12).

Theorem 3.2 (see [11]). A resolvable $B(4,1;v)$ exists if and only if $v \equiv 4 \pmod{12}$.

Theorem 3.3 (see [6]). A $B(\{4,7^*\},1;v)$ exists if and only if $v \equiv 7$ or $10 \pmod{12}$, $v \neq 10, 19$.

Theorem 3.4 (see [8]). If $v \equiv 2 \pmod 6$ and $v \geq 14$, then there exists a $\{4\}$-GDD of group-type $2^{v/2}$.

Theorem 3.5 (see [5]). Suppose q is a prime power and $0 < t < q^2 - q + 1$. Then $t(q^2 + q + 1) \in B(t, q + t)$.

Theorem 3.6 (see [18]). For any positive integer n, there is a resolvable BIBD with parameters $(2^{2n-1}-2^{n-1}, 2^{2n}-1, 2^n+1, 2^{n-1},1)$.

4. Basic Lemmas

The following lemma (see [3, Lemma 2.14]) employs the technique of adding a set of fixed ("infinite") points to a GDD (see also [17] for other generalizations). This lemma will be used in conjunction with Construction 2.7 in some of our constructions.

Lemma 4.1 Let K be a set of positive integers and $s \geq 0$. Suppose there exists a K-GDD of group-type $T = (m_1, m_2, \ldots, m_n)$.

(a) If a PBD $B(K \cup \{s^*\},1;m_i+s)$ exists for $1 \leq i \leq n$, then, for each i, $v + s \in B(K \cup \{(m_i+s)^*\})$ where $v = \Sigma_{1 \leq i \leq n} m_i$.

(b) If a PBD $B(K \cup \{s^*\},1;m_i+s)$ exists for $1 \leq i \leq n-1$, then $v+s \in B(K \cup \{(m_n+s)^*\})$ where $v = \Sigma_{1 \leq i \leq n} m_i$.

In order to establish our main lemmas we shall need some "small" input designs.

Lemma 4.2 There exist {4}-GDDs of the following group-types:

(a) 3^8, (b) $3^8 6^1$, (c) 3^9, (d) $3^9 6^1$, (e) 4^4, (f) 4^7, (g) $4^6 1^1$, (h) 7^4.

Proof: For group-types (a) and (c), we take a B(4,1;v) where $v \in \{25, 28\}$ and delete one point from a block. For group-types (b) and (d), we take a B({4,7*},1;v) where $v \in \{31,34\}$ and delete one point from the block of size 7. For group-types (e) and (f) we take a parallel class of blocks as groups in a resolvable B(4,1;v) where $v \in \{16, 28\}$. For group-type (g), we make use of the existence of a resolvable {4}-GDD of type 3^8 (see, for example, [12]) and adjoin one infinite point to the groups. We then take a parallel class of blocks and the infinite point as groups of the resulting design. For group-type (h), we have a T(4,1;7) from Theorem 2.5. This completes the proof.

Lemma 4.3 There exist {4,8}-GDDs of the following group-types:

(a) $3^7 7^1$, (b) $3^8 7^1$, (c) 4^8, (d) $4^7 1^1$.

Proof: For group-type (a), we start with a T(4,1;7) and adjoin an infinite point, say ∞, to the groups. We then delete from the resulting design a point $x \neq \infty$ to form a {4,8}-GDD of type $3^7 7^1$. For group-type (b), we delete one point from a T(4,1;8). For group-type (c), we take a parallel class of blocks as groups in an RT(4,1;8). For group-type (d), we adjoin an infinite point to the groups of an RT(4,1;7) and then use a parallel class of blocks and the infinite point as groups in the resulting design. This completes the proof.

We are now able to establish the following important lemmas, using input designs from Lemmas 4.2 and 4.3.

Lemma 4.4 If $v \equiv 2 \pmod 6$ and $v \geq 14$, then there exists a {4}-GDD of group-type $8^{v/2}$ and $4v \in B(4,8)$.

Proof: If $v \equiv 2 (\mathrm{mod}\ 6)$ and $v \geq 14$, then Theorem 3.4 guarantees the existence of a {4}-GDD of group-type $2^{v/2}$. We give each point of this GDD weight 4. The result follows directly.

Lemma 4.5 Suppose there is a $T(8,1;m)$ and $0 \le x \le m$. Then there exsits a {4,8}-GDD of group-type $(4m)^7 (4x)^1$. Moreover, the following hold:

(a) If $\{4m,4x\} \subseteq B(4,8)$, then $28m + 4x \in B(4,8)$.

(b) If $\{4m + 1, 4x + 1\} \in B(4,8)$, then $28m + 4x + 1 \in B(4,8)$.

Proof: In all groups but one of a $T(8,1;m)$, we give the points weight 4. In the last group, we give x points weight 4 and the remaining points weight 0. We require {4,8}-GDDs of types 4^7 and 4^8, which come from Lemmas 4.2 and 4.3, and thus obtain a {4,8}-GDD of type $(4m)^7(4x)^1$. The result (a) follows directly, and (b) follows by adjoining an infinite point to the groups of our resulting GDD.

Lemma 4.6 Suppose there is a $T(8,1;m)$ and $0 \le x,y,z \le m$, where $x + y = m$. Then there is a {4,8}-GDD of group-type $(4m)^6(4x+y)^1(4z)^1$. Moreover, the following hold:

(a) If $\{4m, 4x + y, 4z\} \subseteq B(4,8)$, then $24m + 4x + y + 4z \in B(4,8)$.

(b) If $\{4m +1, 4x + y +1, 4z + 1\} \subseteq B(4,8)$, then $24m + 4x + y + 4z +1 \in B(4,8)$.

Proof: In all groups but two of a $T(8,1;m)$, give the points weight 4. In the second last group, give x points weight 4 and y points weight 1 such that $x+y = m$. In the last group, give z points weight 4 and give the remaining points weight 0. The resulting design is a {4,8}-GDD of group-type $(4m)^6(4x+y)^1(4z)^1$, using {4,8}-GDDs of types $4^6 1^1, 4^7, 4^7 1^1, 4^8$ which come from Lemmas 4.2 and 4.3. The result (a) follows directly and (b) is obtained by adjoining an infinite point to the groups of our resulting GDD.

Lemma 4.7 Suppose there is a $T(9,1;m)$ and $0 \le x,y,z \le m$, where $x + y + z \le m$. Then there is a {4,8}-GDD of group-type

$(3m)^8(3x + 6y + 7z)^1$. Moreover the following hold:

(a) If $\{3m, 3x + 6y + 7z\} \subseteq B(4,8)$, then $24m + 3x + 6y + 7z \in B(4,8)$.

(b) If $\{3m + 1, 3x + 6y + 7z + 1\} \subseteq B(4,8)$, then $24m + 3x + 6y + 7z + 1 \in B(4,8)$.

Proof: In all groups but one of a $T(9,1;m)$, give the points weight 3. In the last group, give x points weight 3, y points weight 6, and z points weight 7, and give the remaining points weight 0. The resulting design is a $\{4,8\}$-GDD of type $(3m)^8(3x + 6y + 7z)^1$. We require $\{4,8\}$-GDDs of types 3^8, 3^9, $3^8 6^1$, $3^8 7^1$, which come from Lemmas 4.2 and 4.3 The results (a) and (b) follow easily.

Lemma 4.8 Suppose there is a $T(9,1;m)$ and $0 \le x, y, z \le m$, where $x + y = m$. Then there is a $\{4,8\}$-GDD of group-type $(3m)^7(3x + 7y)^1(3z)^1$.

(a) If $\{3m, 3x + 7y, 3z\} \subseteq B(4,8)$, then $21m + 3x + 7y + 3z \in B(4,8)$.

(b) If $\{3m+1, 3x+7y+1, 3z+1\} \subseteq B(4,8)$, then $21m+3x+7y+3z+1 \in B(4,8)$.

Proof: In all groups but two of a $T(9,1;m)$, give the points weight 3. In the second last group, give x points weight 3 and y points weight 7 such that $x + y = m$. In the last group, give z points weight 3 and give the remaining points weight 0. We require $\{4,8\}$-GDDs of types $3^7 7^1$, 3^8, $3^8 7^1$, 3^9, which come from Lemmas 4.2 and 4.3, and thus obtain the $\{4,8\}$-GDD of type $(3m)^7(3x+7y)^1(3z)^1$. The results (a) and (b) follow easily.

Lemma 4.9 Suppose there is a $T(10,1;m)$ and $0 \le x \le m$. Then there exists a $\{4\}$-GDD of group-type $(3m)^9(6x)^1$. Moreover, the following hold:

(a) If $\{3m, 6x\} \subseteq B(4,8)$, then $27m + 6x \in B(4,8)$.

(b) If $\{3m + 1, 6x + 1\} \subseteq B(4,8)$, then $27m + 6x + 1 \in B(4,8)$.

Proof: In all groups but one of a $T(10,1;m)$, give the points weight 3. In the last group, give x points weight 6 and give the remaining points weight 0. We need {4}-GDDs of types 3^9, $3^9 6^1$, which come from Lemma 4.2, and thus we obtain a {4}-GDD of type $(3m)^9(6x)^1$. The results (a) and (b) readily follow.

<u>Lemma 4.10</u> For all integers $n \geq 0$, the following hold:

(a) $24n + 8 \in B(4,8)$,

(b) $84n + 8 \in B(4,8)$,

(c) $84n + 29 \in B(4,8)$,

(d) $96n + 29 \in B(4,8)$.

<u>Proof</u>: First of all, the existence of a $T(4,1;8)$ implies $32 \in B(4,8)$. Consequently, $\{8,32\} \subseteq B(4,8)$ and the result (a) follows from Lemma 4.4. Next, $29 \in B(4,8)$ from adjoining an infinite point to the groups of a $T(4,1;7)$. For the proofs of (b), (c) and (d), we therefore consider $n \geq 1$. For the proof of (b), we take a {4}-GDD of group-type of type 1^{12n+1}, which exists from Theorem 3.1, and give all points weight 7 to obtain a {4}-GDD of type 7^{12n+1}. We need a {4}-GDD of type 7^4, which exists from Lemma 4.2. By adjoining an infinite point to the groups of our resulting GDD, we obtain the result (b). The result (c) follows in a similar manner by starting with a {4}-GDD of type 1^{12n+4} and giving all points weight 7. For the proof of (d), we take a $T(4, 1; 24n + 7)$ and adjoin an infinite point to the groups to obtain $96n + 29 \in B(4,24n + 8) \subseteq B(4,8)$, since $24n + 8 \in B(4,8)$ from (a). This completes the proof of the lemma.

We shall make use of the following lemma, which is a consequence of Theorem 2.6.

<u>Lemma 4.11</u> There exists a sequence $M = \{m_i : i = 1,2,3,...\} = \{7, 13, 16, 19, 25, 31, 37, 43, 49, 61, 64, 67, 70, 73, 79, 82, 85, 88, ...\}$ such that $m_i \equiv 1 \pmod 3$, $m_{i+1} - m_i \leq 12$, and a $T(8,1;m_i)$ exists for all $i = 1,2,....$

<u>Proof</u>: First of all, it is known (see, for example, [7]) that a

$T(8,1;m)$ exists for all integers $m > 76$ and also for $m < 76$ listed in M. It is very easy check that the conditions $m_i \equiv 1 \pmod 3$ and $m_{i+1} - m_i \leq 12$ hold.

The combination of Lemmas 4.6 and 4.11 will be used extensively to investigate the spectrum of $B(4,8)$. In view of Theorem 3.1, we need only focus our attention on members $v \in B(4,8)$ for which $v \equiv 0, 5, 8, 9 \pmod{12}$. It will be convenient to consider these cases separately.

5. Members of B(4,8) congruent to 8 modulo 12.

Lemma 5.1 Let M be as defined in Lemma 4.11. If $m \in M$ and m_o is the integer in $\{m, m + 3, m + 6, m + 9\}$ such that $m_o \equiv 4 \pmod{12}$, then $v \in B(4,8)$ for all $v \equiv 8 \pmod{12}$ in the interval $24m + m_o + 4 \leq v \leq 32m$.

Proof. We shall apply Lemma 4.6 with $m \in M$ so we have $4m \in B(4)$. Since $m \equiv 1 \pmod 3$, we can choose $4x + y \equiv 4 \pmod{12}$ where $0 \leq x, y \leq m$, $x + y = m$, and $m_o \leq 4x + y \leq 4m$. We choose $4z \equiv 4 \pmod{12}$, where $4 \leq 4z \leq 4m$. Note that $\{4x + y, 4z\} \subseteq B(4)$. Let $v = 24m + 4x + y + 4z$. Then it readily follows that $v \in B(4,8)$, and consequently, we can obtain $v \in B(4,8)$ for all values of $v \equiv 8 \pmod{12}$ in the interval $24m + m_o + 4 \leq v \leq 32m$.

Lemma 5.2 If $v \equiv 8 \pmod{12}$, then $v \in B(4,8)$ holds for all $v \geq 188$, where $\notin \{236, 248, 260, 272, 284, 296, 308, 320, 620\}$.

Proof: We shall apply Lemma 5.1. If $m = 7, 13, 16, 19, 25$, then we obtain $v \in B(4,8)$ for all values of $v \equiv 8 \pmod{12}$ in the interval $188 \leq v \leq 800$, apart from the exceptions listed in the statement of the lemma. For $m \geq 25$, if we apply Lemma 5.1 repeatedly, then we find that the intervals for v overlap and we obtain $v \in B(4,8)$ holds for all $v \equiv 8 \pmod{12}$ where $v \geq 632$. This completes the proof of the lemma.

Lemma 5.3 If $v \in \{8, 32, 56, 80, 92, 104, 128, 152, 176, 248, 260, 272, 296, 320\}$, then $v \in B(4,8)$.

Proof: We apply Lemma 4.10 for the result.

Lemma 5.4 $116 \in B(4,8)$.

Proof: The result follows from the existence of a $T(4,1;29)$ and $29 \in B(4,8)$.

Lemma 5.5 $140 \in B(4,8)$.

Proof: There exists a $\{5\}$-GDD of group-type 8^6 (see, for example, [19]). Starting with this GDD, we delete 4 points from one block to form a $GD(\{4,5\},1,\{7,8\};44)$. In this GDD we give each point weight 3 to obtain a $GD(\{4\},1,\{21,24\};132)$, using $\{4\}$-GDDs of types 3^4 and 3^5. We then apply Lemma 4.1 to adjoin 8 infinite points to the resulting GDD, using the fact that $\{29,32\} \subseteq B(4,8)$, and we thus obtain $140 \in B(4,8)$.

Lemma 5.6 $236 \in B(4,8)$.

Proof: We apply Lemma 4.8(b) with $m = 9$, $x = 8$, $y = 1$ and $z = 5$, using the fact that $\{28,32,16\} \subseteq B(4,8)$.

Lemma 5.7 $284 \in B(4,8)$.

Proof: We first adjoin 8 infinite points to an $RT(7,1;9)$ so as to form a $\{7,8,10\}$-GDD of group-type $7^9 8^1$, where one of the infinite points is adjoined to the groups and the remaining seven are adjoined one each to seven parallel classes of blocks. In our resulting GDD, we give each point weight 4 to obtain a $\{4,8\}$-GDD of type $(28)^9(32)^1$, using $\{4,8\}$-GDDs of types 4^7, 4^8 and 4^{10}. Then $284 \in B(4,8)$ follows from the fact that $\{28,32\} \subseteq B(4,8)$.

Lemma 5.8 $308 \in B(4,8)$.

Proof: Take a $T(4,1;76)$ and adjoin 4 infinite points to the groups by applying Lemma 4.1 with the fact that $80 \in B(4,8)$. Consequently, $308 \in B(4,8)$.

Lemma 5.9 $620 \in B(4,8)$.

Proof: Take a $T(7,1;11)$ and give each point weight 8 to obtain a $\{4,8\}$-GDD of type $(88)^7$, using a $\{4\}$-GDD of type 8^7 from Lemma 4.4 We then apply Lemma 4.1 to adjoin 4 infinite points to the resulting GDD, using $92 \in B(4,8)$, and thus obtain $620 \in B(4,8)$.

Combining the results of Lemmas 5.2 - 5.9, we have proved the following theorem:

<u>Theorem 5.10</u> If $v \equiv 8 \pmod{12}$, then $v \in B(4,8)$ holds for all $v \geq 8$, where $v \neq 20, 44, 68, 164$.

6. <u>Members of B(4,8) congruent to 5 modulo 12</u>.

<u>Lemma 6.1</u> Let M be as defined in Lemma 4.11. If $m \in M$ and m_o is the integer in $\{m, m + 3, m + 6, m + 9\}$ such that $m_o \equiv 1 \pmod{12}$, then $v \in B(4,8)$ holds for all $v \equiv 9 \pmod{12}$ in the interval $24m + m_o + 4 \leq v \leq 32m - 3$.

<u>Proof</u>: The proof is similar to that of Lemma 5.1. Here we also apply Lemma 4.6 with $m \in M$ so that $4m \in B(4)$. We can choose $4x + y \equiv 1 \pmod{12}$ such that the conditions $0 \leq x, y \leq m$, $x+y = m$, $m_o \leq 4x + y \leq 4m - 3$ all hold. We choose $4z \equiv 4 \pmod{12}$, where $4 \leq 4z \leq 4m$. Then $\{4x + y, 4z\} \subseteq B(4)$. Let $v = 24m + 4x + y + 4z$. Then $v \in B(4,8)$ holds for all $v \equiv 5 \pmod{12}$ in the interval $24m + m_o + 4 \leq v \leq 32m - 3$.

<u>Lemma 6.2</u> If $v \equiv 5 \pmod{12}$, then $v \in B(4,8)$ holds for all $v \geq 185$ where $v \notin \{233, 245, 257, 269, 281, 293, 305, 317, 617\}$.

<u>Proof</u>: We apply Lemma 6.1 repeatedly. If $m = 7, 13, 16, 19, 25$, then we obtain $v \in B(4,8)$ for all $v \equiv 5 \pmod{12}$ in the interval $185 \leq v \leq 797$, apart from the exceptions listed in the lemma. For $m \geq 25$, the intervals of v overlap and we obtain $v \in B(4,8)$ for all $v \equiv 5 \pmod{12}$ where $v \geq 629$. This completes the proof of the lemma.

<u>Lemma 6.3</u> If $v \in \{29, 113, 125, 197, 281, 317, 617\}$, then $v \in B(4,8)$.

<u>Proof</u>: The result is an immediate consequence of Lemma 4.10.

<u>Lemma 6.4</u> $\{137, 149\} \subseteq B(4,8)$.

<u>Proof</u>: For the case $v = 137$, we start with a $\{5\}$-GDD of group-type 8^6 and delete 5 points from a block to obtain a

GD({4,5},1,{7,8};43). We then give each point of this GDD weight 3 to obtain a GD({4},1,{21,24};129). Finally, we apply Lemma 4.1 to adjoin 8 infinite points to this GDD and obtain 137 $\in$ B(4,8), using {29,32} $\subseteq$ B(4,8). For the case v = 149, the construction is similar. We first delete one point from a {5}-GDD of type 8^6 to obtain a GD({4, 5}, 1, {7, 8}; 47) and then give each point weight 3 to obtain a GD({4}, 1, {21, 24}; 141). Finally, we adjoin 8 infinite points to this GDD and get 149 $\in$ B(4,8).

<u>Lemma 6.5</u> 233 $\in$ B(4,8).

<u>Proof</u>: Start with a T(9,1;9) and delete one point to form a GD({9},1,{8};80). We further delete 5 points from a block of this GDD to form a GD({4, 8, 9}, 1, {7,8}; 75). In this resulting GDD, we give each point weight 3 to get a GD({4}, 1, {21, 24}; 225) to which we then adjoin 8 infinite points and thus obtain 233 $\in$ B(4,8).

<u>Lemma 6.6</u> {245, 257, 269} $\subseteq$ B(4,8).

<u>Proof</u>: We apply Lemma 4.8(b) with m = 9, x = 2, y = 7 and z $\in$ {0, 4, 8}. We need the fact that 56 $\in$ B(4, 8) and {13, 25, 28} $\subseteq$ B(4).

<u>Lemma 6.7</u> 293 $\in$ B(4,8).

<u>Proof</u>: We first adjoin 7 infinite points to an RT(8,1;11) so as to form a {8, 9, 12}-GDD of type $8^{11}7^1$, where one of the infinite points is adjoined to the groups. In the resulting GDD, we give each point weight 3 to form a {4}-GDD of type $(24)^{11}(21)^1$, using {4}-GDDs of types $3^8, 3^9, 3^{12}$. Finally, we adjoin 8 infinite points to this GDD using Lemma 4.1 to obtain the desired result with {29, 32} $\subseteq$ B(4, 8).

<u>Lemma 6.8</u> 305 $\in$ B(4, 8).

<u>Proof</u>: Take a T(8,1;11) and delete one block entirely to get a {7,8}-GDD of type 10^8. In all but one of the groups, we give weight 4 to each point. In the last group, give weight 1 to five

points and weight 4 to the remaining five points. This gives a $\{4,8\}$-GDD of type $(40)^7(25)^1$, using $\{4,8\}$-GDDs of types $4^7, 4^8, 4^7 1^1$ which come from Lemmas 4.2 and 4.3. It follows that $305 \in B(4,8)$.

Combining Lemmas 6.2 - 6.8, we have essentially proved the following result.

<u>Theorem 6.9</u> If $v \equiv 5 \pmod{12}$, then $v \in B(4,8)$ holds for all $v \geq 5$, where $v \notin \{5, 17, 41, 53, 65, 77, 89, 101, 161, 173\}$.

7. <u>Members of B(4,8) congruent to 0 modulo 12</u>.

<u>Lemma 7.1</u> Let M be as defined in Lemma 4.11. Let $m \in M$ and let m_0 be the integer in $\{m, m+3, m+6, m+9\}$ such that $m_0 \equiv 4 \pmod{12}$. Then the following hold:

(a) If $m \notin \{7,13,19,43\}$, then $v \in B(4,8)$ holds for all $v \equiv 0 \pmod{12}$ in the interval $24m + m_0 + 8 \leq v \leq 32m - 8$.

(b) If $m \in \{7,13,19,43\}$, then $v \in B((4,8)$ holds for all $v \equiv 0 \pmod{12}$ in the interval $24m + m_0 + 8 \leq v \leq 32m-20$.

<u>Proof</u>: We apply Lemma 4.6 with $m \in M$ so that $4m \in B(4)$. In each of (a) and (b), we take $4x+y \equiv 4 \pmod{12}$, where $0 \leq x, y \leq m$, $x + y = m$, and $m_0 \leq 4x + y \leq 4m$. For (a), we can choose $4z \equiv 8 \pmod{12}$ such that $8 \leq 4z \leq 4m - 8$ and (b), choose $4z \equiv 8 \pmod{12}$ such that $8 \leq 4z \leq 4m - 20$, where $4z \in B(4,8)$ from Theorem 5.10. Let $v = 24m + 4x + y + 4z$. Then it is readily checked that the results (a) and (b) follow. Note that the gap between consecutive values of v for which $v \in B(4,8)$ in Theorem 5.10 is at most 24.

<u>Lemma 7.2</u> If $v \equiv 0 \pmod{12}$, then $v \in B(4,8)$ holds for all $v \geq 192$, where $v \notin \{216, 228, 240, 252, 264, 276, 288, 300, 312, 324, 600, 612, 624\}$.

<u>Proof</u>: We shall apply Lemma 7.1. If we put $m = 7, 13, 16, 19, 25$ in Lemma 7.1, then we obtain $v \in B(4,8)$ for all $v \equiv 0 \pmod{12}$ in the interval $192 \leq v \leq 792$, apart from the exceptions cited in the lemma. If we choose $m \geq 25$ and apply Lemma 7.1 repeatedly, then

the intervals for v overlap and we obtain $v \in B(4,8)$ holds for all $v \equiv 0 \pmod{12}$ where $v \geq 636$.

Lemma 7.3 $36 \in B(4,8)$.

Proof: We present a direct construction of a resolvable $B(\{4,8\},1;36)$ with 9 parallel classes, each containing seven 4-blocks and one 8-block with an automorphism of order 9. The design is based on the set $X = Z_9 \times Z_4$ where, for convenience, we write ij for the point (i,j) of X with i between 0 and 8 and j between 0 and 3 inclusive. One parallel class of the design is

$$P = \{(00,80,01,71,02,62,03,53), (10,31,41,70), (11,22,42,61), (12,23,33,52), (13,20,43,60), (63,21,83,51), (72,82,30,50), (81,73,32,40)\}.$$

All of the parallel classes are obtained by developing P modulo 9 in the first coordinate while keeping the second coordinate fixed. If $\mathcal{B}$ is the union of these 9 parallel classes, then it is readily checked that $(X,\mathcal{B})$ is a resolvable $B(\{4,8,\},1;36)$ and the result follows.

Lemma 7.4 $\{84, 120, 324\} \subseteq B(4,8)$.

Proof: For $v = 84$, we apply Theorem 3.5 with $t = 4$ and $q = 4$ to obtain $84 \in B(4,8)$. Applying Theorem 3.6 with $n = 4$, we obtain $120 \in B(8)$. Finally, for $324 \in B(4,8)$, we adjoin 4 infinite points to a $T(4,1;80)$ by applying Lemma 4.1 and using the fact $84 \in B(4,8)$.

Lemma 7.5 If $v \in \{216, 228, 240, 252, 288, 300, 312, 600, 612, 624\}$, then $v \in B(4,8)$.

Proof: For $v \in \{216, 228, 240, 252\}$, we apply Lemma 4.6 (a) with $m = 8$, $4x + y \in \{8, 32\}$, and $z \in \{1,4,7\}$ to get $v = 24m + 4x + y + 4z \in B(4,8)$. For $v \in \{288, 300, 312\}$, we apply Lemma 4.8 with $m = 11$, $x = 6$, $y = 5$ and $z \in \{0,4,8\}$ to first obtain a $\{4,8\}$-GDD of type $(33)^7(53)^1(3z)^1$. We then apply Lemma 4.1(b) to adjoin 4 infinite points to this GDD, using the fact that $\{37,3z+4\}$

$\subseteq B(4)$ and $57 \in B(8)$. For $v \in \{600, 612, 624\}$, we apply Lemma 4.6(a) with $m = 23$, $x = 3$, $y = 20$, and $z \in \{4, 7, 10\}$. Then we have $\{4m, 4x + y, 4z\} \subseteq B(4, 8)$ and we readily obtain $v \in B(4, 8)$.

<u>Lemma 7.6</u> If $v \in \{132,144,180,264,276\}$, then $v \in B(4,8)$.

<u>Proof</u>: For $v = 132$, we start with a $T(4,1;32))$ and adjoin 4 infinite points to the groups by applying Lemma 4.1 and the fact that $36 \in B(4,8)$ from Lemma 7.3. For $v \in \{144,180\}$, we make use of $\{4\}$ - GDDs of group-types $(36)^4$ and $(36)^5$ (see, for example, [8]) and the fact that $36 \in B(4,8,)$ to easily obtain $\{144,180\} \subseteq B(4,8)$. For $v \in \{264,276\}$, we first apply Lemma 4.6 with $m=9$, $4x+y \in \{12,24\}$, and $z=8$ to get a $\{4,8\}$ - GDD of group-type $(36)^6 (4x+y)^1 (32)^1$. We then apply Lemma 4.1 to adjoin 4 infinite points to this GDD, using the fact that $\{16,28,36,40\} \subseteq B(4,8)$, and thus obtain $\{264,276\} \subseteq B(4,8)$.

Combining Lemmas 7.2 - 7.6, we have proved the following:

<u>Theorem 7.7</u> If $v \equiv 0 \pmod{12}$, then $v \in B(4,8)$ holds for all $v \geq 12$, where $v \neq 12, 24, 48, 60, 72, 96, 108, 156, 168$.

8. <u>Members of B(4,8) congruent to 9 modulo 12</u>.

<u>Lemma 8.1</u> Let M be as defined in Lemma 4.11. Let $m \in M$ and let m_0 be the integer in $\{m, m + 3, m + 6, m + 9\}$ such that $m_0 \equiv 1 \pmod{12}$. Then the following hold:

(a) If $m \notin \{7, 13, 19, 43\}$, then $v \in B(4, 8)$ holds for all $v \equiv 9 \pmod{12}$ in the interval $24m + m_0 + 8 \leq v \leq 32m - 11$.

(b) If $m \in \{7, 13, 19, 43\}$, then $v \in B(4, 8)$ holds for all $v \equiv 9 \pmod{12}$ in the interval $24m + m_0 + 8 \leq v \leq 32m - 23$.

<u>Proof</u>: We apply Lemma 4.6 with $m \in M$ so that we have $4m \in B(4)$. In each of (a) and (b), we take $4x + y \equiv 1 \pmod{12}$ such that the conditions $0 \leq x$, $y \leq m$, $x + y = m$, $m_0 \leq 4x + y \leq 4m - 3$ hold. Note that $4x + y \in B(4)$. For (a) we choose $4z \equiv 8 \pmod{12}$ such that $8 \leq 4z \leq 4m - 8$ and for (b), choose $4z \equiv 8 \pmod{12}$ such that

$8 \le 4z \le 4m - 20$, where $4z \in B(4,8)$ from Theorem 5.10. The gap is at most 24 between consecutive values of v for which $v \in B(4, 8)$ in Theorem 5.10. If we put $v = 24m + 4x + y + 4z$, then it is not difficult to check that we obtain $v \in B(4, 8)$ for all $v \equiv 9 \pmod{12}$ in the specified intervals.

<u>Lemma 8.2</u> If $v \equiv 9 \pmod{12}$, then $v \in B(4, 8)$ for all $v \ge 189$ where $v \notin \{213, 225, 237, 249, 261, 273, 285, 297, 309, 321, 405, 597, 609, 621\}$.

<u>Proof</u>: We apply Lemma 8.1. If $m = 7, 13, 16, 19, 25$, then we obtain $v \in B(4,8)$ for all $v \equiv 9 \pmod{12}$ in the interval $189 \le v \le 789$, apart from the exceptions cited in the lemma. If we choose $m \in M$, $m \ge 25$, and apply Lemma 8.1 repeatedly, then it is readily checked that the intervals for v overlap and we obtain $v \in B(4,8)$ for all $v \equiv 9 \pmod{12}$ where $v \ge 633$.

<u>Lemma 8.3</u> $\{57, 141, 285, 597\} \subseteq B(4,8)$.

<u>Proof</u>: We adjoin an infinite point to the groups of a $T(8,1;7)$ to obtain $57 \in B(8)$. For $v = 141$, we adjoin an infinite point to the groups of a $T(4,1;35)$ and use the fact $36 \in B(4,8)$. For $v = 285$, we start with a $T(5,1;19)$ and give each point weight 3 to obtain $285 \in B(4, 57) \subseteq B(4, 8)$. For $v = 597$, we apply Lemma 4.9 (a) with $m = 19$, $x = 14$ to obtain $597 \in B(4, 8)$, using $\{57, 84\} \subseteq B(4, 8)$.

<u>Lemma 8.4</u> If $v \in \{225, 237, 249, 273, 297, 309, 321, 405, 609, 621\}$, then $v \in B(4, 8)$.

<u>Proof</u>: For $v \in \{225, 237, 249\}$, we apply Lemma 4.6(a) with $m = 8$, $x = 7$, $y = 1$, and $4z \in \{4, 16, 28\}$, using the fact $\{29, 32, 4z\} \subseteq B(4,8)$. For $v = 273$, we apply Lemma 4.7(b) with $m = 9$, $x = 0$, $y = 0$, $z = 8$, using $\{28, 57\} \subseteq B(4,8)$. If $v \in \{297, 309, 321\}$, then we apply Lemma 4.8 with $m = 11$, $x = 6$, $y = 5$, and $z \in \{3, 7, 11\}$ to first obtain a $\{4, 8\}$-GDD of type $(33)^7(53)^1(3z)^1$. We then apply Lemma 4.1(b) to adjoin 4 infinite points to this GDD, using the fact that $\{37, 3z + 4\} \subseteq B(4)$ and $57 \in B(8)$, and thus

obtain $v \in B(4,8)$. For $v = 405$, we apply Lemma 4.8 (b) with $m = 16$, $x = 14$, $y = 2$, and $z = 4$. Finally, for $v \in \{609, 621\}$, we apply Lemma 4.8(b) with $m = 25$, $x = 23$, $y = 2$, and $z \in \{0,3\}$, using the fact that $\{13,76\} \subseteq B(4)$ and $84 \in B(4, 8)$, and thus obtain $v \in B(4, 8)$. This completes the proof of the lemma.

<u>Lemma 8.5</u>: If $v \in \{213,261\}$, then $v \in B(4,8)$.

<u>Proof</u>: For $v = 213$ we apply Lemma 4.8 with $m = 8$, $x = 6$, $y = 2$ and $z = 3$ to get a $\{4,8\}$ - GDD of group-type $(24)^7 (32)^1 (9)^1$. To this GDD we adjoin 4 infinite points by applying Lemma 4.1 and the fact that $\{13, 28, 36\} \subseteq B(4,8)$. This gives us $213 \in B(4,8)$. For $v = 261$, we apply Lemma 4.6 with $m = 9$, $x = 0$, $y = 9$ and $z = 8$ to obtain a $\{4,8\}$ - GDD of group-type $(36)^6 (9)^1 (32)^1$. We then apply Lemma 4.1 to adjoin 4 infinite points to this GDD, using the fact that $\{13, 36, 40\} \subseteq B(4,8)$, and obtain $261 \in B(4,8)$.

Combining Lemmas 8.2 - 8.5, we have proved the following:

<u>Theorem 8.6</u> If $v \equiv 9 \pmod{12}$, then $v \in B(4,8)$ holds for all $v \geq 9$, where $v \neq 9, 21, 33, 45, 69, 81, 93, 105, 117, 129, 153, 165, 177$.

9. <u>The spectrum of B(4,8) and applications</u>.

Before stating the main result regarding the spectrum of $B(4,8)$, we shall deal with some impossible cases. First of all, it is not difficult to establish the following lemma.

<u>Lemma 9.1</u> Let $(X, \mathcal{B})$ be a PBD $B(K,1;v)$ whose smallest block size is at least m and which contains a block of size k which is the largest block size in K. Then $v \geq k(m-1)+1$.

<u>Proof</u>: Let A be a block of size k and x be a point not on A. The number of blocks containing x and a point of A is k and consequently, v must be at least $k(m-1)+1$.

<u>Lemma 9.2</u> If $v \in \{5, 9, 12, 17, 20, 21, 24\}$, then there does not exist a PBD $B(\{4,8\},1;v)$.

<u>Proof</u>: If $v \in \{5, 9, 12, 17, 20, 21, 24\}$, then it is obvious from Theorem 3.1 that any $B(\{4,8\},1;v)$ must contain at least one block

A of size 8 and some point x not on A. From Lemma 9.1, it follows that $v \geq 25$.

Lemma 9.3 Let$(X,\mathcal{B})$ be a PBD $B(\{4,8\},1;v)$. Let x be a point of X and let r_x denote the number of blocks of $\mathcal{B}$ containing x. Let $r_4(x)$ and $r_8(x)$ be the number of 4-blocks and 8-blocks in $\mathcal{B}$, respectively, which contain x. Then the following hold:

(i) $3\, r_4(x) + 7\, r_8(x) = v-1$,

(ii) $\lceil (v-1)/7 \rceil \leq r_x \leq \lfloor (v-1)/3 \rfloor$ where $\lceil x \rceil$ denotes the greatest integer less than or equal to x and $\lfloor x \rfloor$ is the least integer greater than or equal to x.

Proof: The proof of (i) follows directly from the fact that every other element must occur is exactly one block with x, and the inequality of (ii) is an immediate consequence of (i).

Lemma 9.4 There does not exist a PBD $B(\{4,8\},1;33)$.

Proof: Suppose $(X, \mathcal{B})$ is a PBD $B(\{4,8\},1;33)$. Then by applying Lemma 9.3, we determine that each point of X must be contained in precisely six 4-blocks and two 8-blocks. So there are 33/4 8-blocks, which is clearly impossible.

Lemma 9.5 There does not exist a PBD $B(\{4,8\},1;41)$.

Proof: Suppose $(X, \mathcal{B})$ is a PBD $B(\{4,8\},1;41)$. Then we can apply Lemma 9.3 to determine that each point x of X is one of two types:

Type I - x is contained in 11 4-blocks and 1 8-block.

Type II - x is contained in 4 4-blocks and 4 8-blocks.

Now, not all points of X can be of type I, since this would imply that there are 41/8 8 blocks, which is impossible. By deleting a point of type II, we obtain a $\{4,8\}$ - GDD of group-type $(3)^4(7)^4$. It is easy to see that no point of this GDD is contained in 4 8-blocks, because of the 3-groups. Consequently, the 12 points in the 3-groups are all of type I and the PBD contains exactly 7 8-blocks. But there can be no such configuration on 41 points with 7 8-blocks, where each point is contained in 1 or 4 8-blocks.

Lemma 9.6 There does not exist a PBD $B(\{4,8\},1;44)$.

Proof: Suppose $(X, \mathcal{B})$ is a PBD $B(\{4,8\},1;44)$. From Lemma 9.3, it is easy to determine that each point z of X is one of two types:

Type I - z is contained in 12 4-blocks and 1 8-block.

Type II - z is contained in 5 4-blocks and 4 8-blocks.

Let b_4 and b_8 denote the number of 4-blocks and 8-blocks respectively in $\mathcal{B}$. Let x and y denote the number of type I and type II points. Then the following equations hold:

(1) $x + y = 44$

(2) $3b_4 + 14b_8 = 473$

(3) $12x + 5y = 4b_4$

(4) $x + 4y = 8b_8$

The first equation comes from counting points; the second comes from counting pairs of points. The third (respectively, fourth) equation comes from counting pairs (z, A) where z is a point and A is 4-block (respectively, 8-block) containing z. Now, equations (1) - (4) are not independent. There is a solution in terms of the integer parameter s given by the following:

$x = 56 - 8s$

$y = 8s - 12$

$b_4 = 153 - 14s$

$b_8 = 3s + 1$

Since x and y are non-negative integers, we must have $2 \le s \le 7$. Let t denote the maximum over all 8-blocks A, of the number of type II points in A. If A is an 8-block with t type II points, then the number of 8-blocks intersecting A, including A itself, is $3t + 1$; so $3t + 1 \le b_8$ and $t \le s$. By counting pairs (z, A) where z is a type II point and A is any 8-block containing z, we readily obtain that $4y \le tb_8$ and $32s - 48 \le t(3s + 1)$. Since $t \le s$, we get $32s - 48 \le s(3s + 1)$ or $3s^2 - 31s + 48 \ge 0$, which is impossible for $s \in \{2, 3, 4, 5, 6,7\}$.

Lemma 9.7 There does not exist a PBD $B(\{4,8\},1;45)$.

Proof: Suppose $(X, \mathcal{B})$ is a PBD $B(\{4,8\},1;45)$. By applying Lemma 9.3, it is readily determined that each point x of X is one of two types:

Type I - x is contained in 10 4-blocks and 2 8-blocks.

Type II - x is contained in 3 4-blocks and 5 8-blocks.

Now, proceeding as we did in Lemma 9.5, we arrive at the conclusion that the PBD on 45 points must contain exactly 11 8-blocks, where each point is in either 2 or 5 of them. However, no such configuration exists.

For convenience, let

$E = \{5, 9, 12, 17, 20, 21, 24, 33, 41, 44, 45\}$,

$S = \{48, 53, 60, 65, 68, 69, 72, 77, 81, 89, 93, 96, 101, 105, 108, 117, 129, 153, 156, 161, 164, 165, 168, 173, 177\}$.

Combining the results of Theorems 3.1, 5.10, 6.9, 7.7 and 8.6 with Lemmas 9.2, 9.4 - 9.7, we obtain our main theorem:

Theorem 9.8 The necessary condition $v \equiv 0$ or $1 \pmod 4$ for $v \in B(4,8)$ is sufficient for all $v \geq 4$ with the exception of $v \in E$ and the possible exception of $v \in S$.

As already mentioned, the spectrum of $B(4,8)$ has useful applications to the construction of other types of combinatorial structures (see, for example, [1, 2, 9, 13, 14, 16]). In particular, we wish to briefly mention an application to the spectrum of a variety of two-variable quasigroup identities, which will supplement the results of [2]. It is fairly well-known that there are idempotent models of the Schröder identity $(xy)(yx) = x$ for orders 4 and 8 (see [13]). Also, there are models for orders 4 and 8 of quasigroups satisfying the identity $(xy)y = x(xy)$, which implies the idempotent law (see [2]). From Theorem 9.8 we can then conclude the following (see [2, Theorem 2.7]).

Theorem 9.9 There are idempotent Schröder quasigroups for all orders $n \equiv 0$ or $1 \pmod 4$ all of whose two-generated subquasigroups are of order 4 or 8, with the possible exception of those orders listed in $E \cup S$.

Theorem 9.10 The spectrum of the quasigroup identity $(xy)y = x(xy)$ contains all integers $n \geq 1$ where $n \equiv 0$ or $1 \pmod 4$, with the possible exception of those values listed in $E \cup S$.

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