A counterexample to a conjecture of Jackson and Wormald

Aung Kyaw

Department of Mathematics
University of Yangon
Yangon
Myanmar

Abstract

A counterexample is presented to the following conjecture of Jackson and Wormald: If $j \geq 1$, $k \geq 2$ and a graph is connected, locally j-connected and $K_{1,(j+1)(k-1)+2}$-free then it has a k-tree.

Preliminaries

All graphs considered here are finite and without loops or multiple edges. As usual, we let $V(G)$ and $E(G)$ denote respectively the vertex set and the edge set of the graph G. The cardinality of the set S is denoted by $|S|$. A $K_{1,k}$-free graph is a graph containing no copy of $K_{1,k}$ as an induced subgraph. Also, a graph is locally j-connected if every subgraph induced by the set of neighbours of a vertex v is j-connected. A k-tree of a graph is a spanning tree with maximum degree at most k.

The join of two disjoint graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained by joining each vertex of G_1 to each vertex of G_2. The union of m disjoint copies of the same graph G is denoted by mG.

In [1], Bill Jackson and Nicholas C. Wormald made the following conjecture: If $j \geq 1$, $k \geq 2$ and a graph is connected, locally j-connected and $K_{1,(j+1)(k-1)+2}$-free then it has a k-tree.

A counterexample

For any integers $\delta \geq 2$ and $k \geq 2$, first we construct the graph $G_1 + G_2$, where $G_1 = K_\delta$ and $G_2 = \{\delta(k-1)+1\}K_\delta$. Then join a $\overline{K}_{\delta(k-1)}$ to each copy of K_δ in G_2; the graph is depicted in Figure 1.

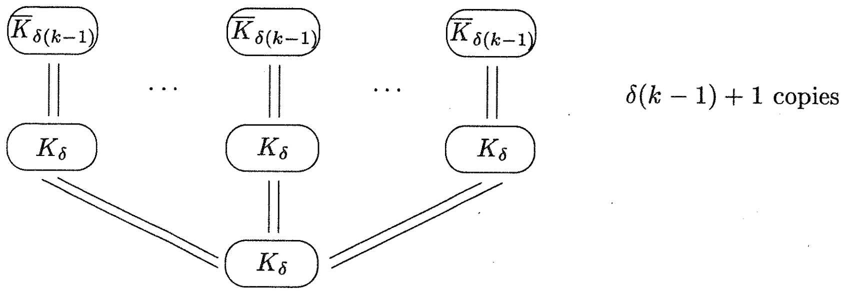

Figure 1

It is easily seen that the graph G shown in Figure 1 is connected, locally $(\delta-1)$-connected and $K_{1,\,\delta(k-1)+2}$-free.

Next, we show it does not have a k-tree. Suppose, for a contradiction, that G does have a k-tree, T. Denote $A = E(T) \cap E(G_1)$ and $B = E(T) \cap E(G_2)$. Since every edge in $E(T) - A$ is incident with at least one vertex of G_2,

$$\begin{aligned} |E(T)| - |A| &\leq k\delta\{\delta(k-1)+1\} - |B| \\ |V(T)| - 1 - |A| &\leq k\delta\{\delta(k-1)+1\} - |B| \\ \delta + k\delta\{\delta(k-1)+1\} - 1 - |A| &\leq k\delta\{\delta(k-1)+1\} - |B| \\ \delta - 1 &\leq |A| - |B|. \end{aligned}$$

But $|A| \leq \delta - 1$, so $|A| = \delta - 1$ and $|B| = 0$. So the degree-sum of the vertices of G_1 in T is at least $\delta(k-1)+1+2|A| = k\delta + \delta - 1$ which contradicts the fact that the degree-sum of the vertices of G_1 in $|T|$ is at most $k\delta$. $\square$

But I feel that Jackson and Wormald's conjecture can be changed as follows:

Conjecture: If $j \geq 1$, $k \geq 2$ and a graph is connected, locally j-connected and $K_{1,\,(j+1)(k-1)+1}$-free then it has a k-tree.

If true, this conjecture is sharp, in view of the graph shown in Figure 1.

Reference

[1] Bill Jackson and Nicholas C. Wormald, *k-walks of graphs*, Australasian Journal of Combinatorics **2** (1990), 135–146.

(Received 8/6/98)