Estimates on Strict Hall Exponents*

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Abstract

Let B_n be the set of all n by n Boolean matrices, and let $H_n^* = \{A \in B_n : A^k \text{ is a Hall matrix for every sufficiently large integer } k\}$. We provide upper estimates on the strict Hall exponents of microsymmetric matrices in H_n^*; furthermore, we obtain the maximum value of the strict Hall exponents of symmetric matrices in H_n^*.

1 Introduction

Let B_n be the set of all n by n matrices over the Boolean algebra $\{0, 1\}$. A matrix A in B_n is said to be a *Hall matrix* provided that there is a permutation matrix Q such that $Q \leq A$ (entrywise order with $0 \leq 1$).

In 1973, Schwarz [1] introduced the concept of Hall exponent: for $A \in B_n$, if there is a positive integer k such that A^k is a Hall matrix, then the least such positive integer is called the *Hall exponent* of A, denoted by $h(A)$. When they made a further study of Hall exponents in 1990, Brualdi and Liu [2] found that there exist $A \in B_n$ and integer $m > h(A)$ such that A^m is not a Hall matrix. Therefore they introduced the concept of the strict Hall exponent.

For $A \in B_n$, if there is a positive integer k such that A^i is a Hall matrix for every integer $i \geq k$, then the least such positive integer is called the *strict Hall exponent* of A, denoted by $h^*(A)$.

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It should be noted that $h(A)$ or $h^*(A)$ does not exist for some $A \in B_n$. Let

$$H_n^* = \{A \in B_n : A^k \text{ is a Hall matrix for every sufficiently large integer } k\}.$$

Then $h^*(A)$ exists if and only if $A \in H_n^*$, and $h(A)$ exists if $A \in H_n^*$.

A matrix $A = (a_{ij}) \in B_n$ is said to be microsymmetric if there is a pair i, j with $i \neq j$ such that $a_{ij} = a_{ji} = 1$ (for such i and j, we call a_{ij} and a_{ji} a pair of symmetric ones of A); $A = (a_{ij})$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i, j. We denote the set of all microsymmetric matrices in H_n^* by MH_n^*, and the set of all symmetric matrices in H_n^* by SH_n^*. Clearly, $SH_n^* \subset MS_n^*$.

A matrix $A \in B_n$ is primitive provided that for some positive integer m, $A^m = J_n$, the all 1's matrix in B_n. The set of primitive matrices in B_n is denoted by P_n.

Recently, we proved that $h^*(A) \leq \lfloor \frac{n^2}{4} \rfloor$ for $A \in P_n$ and $n \geq 2$. (This was conjectured in [2]). This upper estimate seems, however, far from satisfactory for some special classes of matrices in $P_n \subset H_n^*$.

In the present paper, we provide upper estimates on the strict Hall exponents of matrices in MH_n^*; furthermore we obtain the maximum value of the strict Hall exponents of matrices in SH_n^* .

2 Preliminaries

Recall that the matrix A is reducible provided that there is a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix};$$

otherwise A is irreducible.

The digraph of $A = (a_{ij}) \in B_n$, $D(A)$, is defined by $D(A) = (V,E)$ where $V = VD(A) = \{1, 2, \cdots, n\}$ and the arc $(i,j) \in E = ED(A)$ if and only if $a_{ij} = 1$ for all i, j. Thus loops are permitted in $D(A)$, but multiple arcs are not allowed.

It is well known that $A \in B_n$ is irreducible if and only if $D(A)$ is strongly connected, and $A \in B_n$ is primitive if and only if $D(A)$ is strongly connected and the greatest common divisor of the lengths of all cycles of $D(A)$ is 1.

For an irreducible $A \in B_n$, let R be a set of some distinct lengths of cycles of $D(A)$. For i, $j \in VD(A)$, $d_R(i,j)$ denotes the length of the shortest walk from i to j meeting at least one cycle of each length in R, and $d(i,j)$ denotes the distance from i to j , i.e., the length of the shortest path from i to j.

For $X \subseteq VD(A)$, let $R_t(X)$ be the set of vertices of $D(A)$ which can be reached by a walk of length t from a vertex in X. In particular, $R_0(X) = X$. It follows from Hall's theorem ([4]) that A^t is a Hall matrix if and only if $|R_t(X)| \geq |X|$ for every nontrivial subset X of $VD(A)$.

We have

Lemma 2.1 ([5]) *Suppose A is an irreducible matrix in B_n and $X \subseteq VD(A)$. Then for every positive integer t,*

$$|\bigcup_{i=0}^{t} R_i(X)| \geq \min\{|X|+t, n\}.$$

Let a, b be coprime positive integers. The Frobenius number $\phi(a,b)$ is defined to be the least integer ϕ such that every integer $m \geq \phi$ can be expressed in the form $xa+yb$ where x, y are nonnegative integers. It is well known that $\phi(a,b) = (a-1)(b-1)$.

Note that B_n forms a finite multiplicative semigroup of order 2^{n^2}. Let $A \in B_n$. The sequence of powers A^1, A^2, $\cdots$ clearly forms a subsemigroup $\langle A \rangle$ of B_n, and there is a least positive integer $k = k(A)$ such that $A^k = A^{k+t}$ for some $t > 0$, and there is a least positive integer $p = p(A)$ such that $A^k = A^{k+p}$. We call the integer $k = k(A)$ the index of A, and the integer $p = p(A)$ the period of A. It should be noted that this definition of the index of a Boolean matrix is a little different from that in [6] where $k(A)$ was permitted to be zero; however, they are the same for an irreducible Boolean matrix whose associated digraph is not a cycle of length n. It is well known that $p(A)$ equals the greatest common divisor of the distinct lengths of all the cycles of $D(A)$ if A is irreducible. And it is easy to see that $h^*(A) \leq k(A)$ for $A \in H_n^*$.

Let $A \in B_n$ with $p(A) = p$. For all i and j, $k_A(i,j)$ is defined to be the least positive integer k such that $(A^{l+p})_{ij} = (A^l)_{ij}$ for every integer $l \geq k$, and $m_A(i,j)$ is defined to be the least positive integer m such that $(A^{a+mp})_{ij} = 1$ for every integer $a \geq 0$. It is easy to verify that

$$k(A) = \max_{1 \leq i,j \leq n} k_A(i,j),$$

and

$$k_A(i,j) = \max\{m_A(i,j) - p + 1, 1\}.$$

3 Main Results

Theorem 3.1 *Let $A \in P_n \cap MH_n^*$, $n \geq 2$. Then $h^*(A) \leq 2n-3$.*

Proof. Since $A \in P_n \cap MH_n^*$, $D(A)$ must contain a cycle C_2 with length 2 and a cycle C_r with length r where r is odd. Let $X \subseteq VD(A)$ with $|X| = k$, $1 \leq k < n$. We will prove that $|R_t(X)| \geq k$ for $t \geq 2n-3$. Note that this is obvious for $k=1$. We assume $k > 1$.

There exist $x' \in X$, $y' \in VC_2$ such that

$$d(x',y') = \min_{x \in X, y \in VC_2} d(x,y).$$

Therefore $d(x',y') \leq n-k-1$.

There also exists $z' \in VC_r$ such that

$$d(y', z') = \min_{z \in VC_r} d(y', z),$$

and $d(y', z') \le n - r$.

Setting $R = \{2, r\}$, we have

$$d_R(x', z') \le d(x', y') + d(y', z') \le n - k - 1 + n - r = 2n - k - r - 1.$$

By the definition of the Frobenius number, for every integer $m \ge 2n - k - r - 1 + \phi(2, r) = 2n - k - r - 1 + (r - 1) = 2n - k - 2$, there is a walk from x' to z' with length m. Hence for $t \ge (2n - k - 2) + k - 1 = 2n - 3$, we have

$$\bigcup_{a=0}^{k-1} R_a(\{z'\}) \subseteq R_t(\{x'\}).$$

By Lemma 2.1,

$$\begin{aligned} |R_t(X)| &\ge |R_t(\{x'\})| \\ &\ge |\textstyle\bigcup_{a=0}^{k-1} R_a(\{z'\})| \\ &\ge 1 + (k - 1) = k, \quad k > 1. \end{aligned}$$

Thus we have proved that $h^*(A) \le 2n - 3$.

Note that A has at least two symmetric ones for $A \in MH_n^*$. We can generalize Theorem 3.1 to Theorem 3.2.

Theorem 3.2 *Suppose $A \in P_n \cap MH_n^*$, and there are exactly s rows in A containing symmetric ones, $2 \le s \le n$. Then $h^*(A) \le 2n - s - 1$.*

Furthermore we have

Theorem 3.3 *Suppose $A \in MH_n^*$, A is irreducible, and there are exactly s rows in A containing symmetric ones, $2 \le s \le n$. Then $h^*(A) \le 2n - s - 1$.*

Proof. By Theorem 3.2, we need only to prove $h^*(A) \le 2n - s - 1$ for irreducible but not primitive $A \in MH_n^*$. In this case $p(A) = 2$. For any vertices $i, j \in VD(A)$, there is a walk starting from vertex i to some vertex u of a cycle of length 2 of $D(A)$ with length $\le n - s$; and vertex j can be reached by a walk starting from u with length $\le n - 1$. Hence for some positive integer $m \le n - s + n - 1 = 2n - s - 1$ and any integer $a \ge 0$, there is a walk from i to j with length $m + 2a$. Thus $m_A(i, j) \le m \le 2n - s - 1$, and $k_A(i, j) \le m_A(i, j) - 2 - 1 \le 2n - s - 2 < 2n - s - 1$. Now it follows that

$$h^*(A) \le k(A) = \max_{1 \le i, j \le n} k_A(i, j) < 2n - s - 1,$$

as desired.

By Theorem 3.3, we immediately have

Theorem 3.4 *Suppose $A \in MH_n^*$, $n \ge 2$ and A is irreducible. Then $h^*(A) \le 2n - 3$.*

Now we investigate the strict Hall exponents of symmetric matrices in H_n^*. We consider the primitive matrices first.

Theorem 3.5 *For $n \geq 3$, we have*

$$\max\{h^*(A) : A \in P_n \cap SH_n^*\} = \begin{cases} n-2 & n \text{ is odd,} \\ n-1 & n \text{ is even.} \end{cases}$$

Proof. Suppose $A \in P_n \cap SH_n^*$. Then all rows of A contain symmetric ones. By Theorem 3.2, We have $h^*(A) \leq n-1$.

When n is even, we have $R_{n-2}(X) \supseteq X$ since A is symmetric. Hence $|R_{n-2}(X)| \geq |X|$ holds for every $X \subseteq VD(A)$. Combining with the fact that $h^*(A) \leq n-1$, we have $h^*(A) \leq n-2$.

Therefore we have

$$h^*(A) \leq \begin{cases} n-2, & n \text{ is even,} \\ n-1, & n \text{ is odd.} \end{cases}$$

In the following we are going to show that the above upper bound can be achieved for every n.

If n is even, let

$$A_1 = \begin{pmatrix} & & & 1 & & & \\ & 0_{n/2\times n/2} & & \vdots & & 0 & \\ & & & 1 & & & \\ 1 & \cdots & 1 & 0 & 1 & & \\ & & & 1 & 0 & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ & 0 & & & & \ddots & 0 & 1 \\ & & & & & & 1 & 1 \end{pmatrix}_{n\times n}.$$

Clearly $A_1 \in P_n \cap SH_n^*$. It is easy to verify that

$$A_1^{n-3} = \begin{pmatrix} & & & 1 & 0 & & \\ & 0_{n/2\times n/2} & & \vdots & \vdots & & J \\ & & & 1 & 0 & & \\ 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & & & \\ & & & \vdots & & J & \\ & J & & 1 & & & \end{pmatrix}.$$

A_1^{n-3} has a $\frac{n}{2} \times (\frac{n}{2}+1)$ zero submatrix with $\frac{n}{2} + (\frac{n}{2}+1) > n$, so A_1^{n-3} is not a Hall matrix. Thus $h^*(A_1) \geq n-2$. But we have proved that $h^*(A_1) \leq n-2$, so we have $h^*(A_1) = n-2$.

When n is odd, let

$$A_2 = \begin{pmatrix} & 0_{(n+1)/2\times(n+1)/2} & & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} & & & & 0 & \\ 1 & \cdots & 1 & 0 & 1 & & & & \\ & & & 1 & 0 & \ddots & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & 0 & & & & \ddots & 0 & 1 \\ & & & & & & 1 & 1 \end{pmatrix}_{n\times n}.$$

It is easy to see that $A_2 \in P_n \cap SH_n^*$ and

$$A_2^{n-2} = \begin{pmatrix} 0_{(n+1)/2\times(n+1)/2} & J \\ J & J \end{pmatrix}.$$

Hence A_2^{n-2} has a $\frac{n+1}{2} \times \frac{n+1}{2}$ zero submatrix, and it is not a Hall matrix. So $h^*(A_2) \geq n-1$. Note that $h^*(A) \leq n-1$. We have $h^*(A) = n-1$. The proof is now completed.

Theorem 3.6 *For $n \geq 3$, we have*

$$\max\{h^*(A) : \ A \in SH_n^*\} = \begin{cases} n-2, & n \text{ is even}, \\ n-1, & n \text{ is odd}. \end{cases}$$

Proof. Suppose that $A \in SH_n^*$. By Lemma 3.5, we need only to prove

$$h^*(A) \leq \begin{cases} n-2, & n \text{ is even}, \\ n-1, & n \text{ is odd} \end{cases}$$

for $A \in SH_n^* \backslash P_n$. We divide the proof into two cases.

Case 1: A is irreducible. It has been proved in [7] that $k(A) \leq n-2$. Hence $h^*(A) \leq k(A) \leq n-2$.

Case 2: A is reducible. Assume that $A_1, A_2, \cdots, A_t$ $(t \geq 2)$ are the irreducible components of A. Let the order of A_i be n_i for $1 \leq i \leq t$. It is easy to see that $h^*(A_i)$ exists for every i since $A \in SH_n^*$. For $1 \leq i \leq t$, it has been proved that $h^*(A_i) \leq n_i - 1$ if A_i is primitive in Theorem 3.5; and by a similar argument as in Case 1, $h^*(A) \leq n_i - 2$ if A_i is not primitive. Hence

$$h^*(A) = \max_{1\leq i\leq t} h^*(A_i) \leq \max_{1\leq i\leq t} n_i - 1 \leq n - 1 - 1 = n - 2.$$

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