On Odd Primitive Graphs*

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Abstract

In this paper, we prove that every odd primitive graph must contain two vertex disjoint odd cycles. We also characterize a family of odd primitive graphs whose exponent achieves the upper bound.

We follow the notation and terminology of Bondy and Murty [1], unless otherwise stated. A digraph D is said to be *primitive* if there exists a positive integer k such that for each ordered pair of vertices u, v there is a directed walk of length k from vertex u to vertex v in D. The smallest such integer k is called the *exponent* of D, denoted by $\gamma(D)$. A primitive digraph is said to be *odd primitive* if its exponent is odd. It is well known that a digraph is primitive if and only if it is strongly connected and the greatest common divisor of the lengths of all its directed cycles is one.

In this paper, we consider only symmetric digraphs without multiple edges, which we will call *graphs*. Let G be a graph. The *odd girth* of G is the length of a shortest odd cycle in G and is denoted by $g_o(G)$. For two vertices u, v of G, we let $\gamma(u, v)$ denote the smallest positive integer k such that there is a walk of length t from u to v in G for all $t \geq k$. Obviously, if G is primitive, then $\gamma(G) = \max_{u,v \in V(G)} \gamma(u, v)$. The basic properties of a primitive graph and its exponent given in the following propositions are well known.

Proposition 1 [3]. *A graph is primitive if and only if it is connected and contains an odd cycle.*

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Proposition 2 [2]. *Let G be a primitive graph. If there are two walks from vertex u to vertex v with odd length k_1 and even length k_2 respectively, then*

$$\gamma(u,v) \leq max\{k_1, k_2\} - 1.$$

The following theorem on primitive graphs is due to J.Z. Wang and D.J. Wang [4].

Theorem 3 [4]. *The set of exponents of all primitive graphs with order n and all odd girth g_o is*

$$\{g_o - 1, g_o, \ldots, 2n - g_o - 1\} - S$$

where S is the set of zero and all odd integers s with $n - g_o + 1 \leq s \leq 2n - g_o - 1$.

Our objective in this paper is to study the structural properties of odd primitive graphs. For convenience, we give some further definitions and notation. The cartesian product $X \times Y$ of two graphs is the graph which has vertex-set $V(X) \times V(Y)$ and two vertices $(x_1, y_1), (x_2, y_2)$ of $X \times Y$ adjacent whenever either $x_1 = x_2$ and $y_1 y_2 \in E(Y)$ or $y_1 = y_2$ and $x_1 x_2 \in E(X)$. A $v - v'-$lollipop is the graph obtained by taking the union of a cycle C and a $v - v'$ path P such that $C \cap P = \{v'\}$. Clearly, $P \cup C \cup P$ is a walk from v to v. For a walk W, we will denote its length by $l(W)$. Our main results are the following theorems.

Theorem 4. *Let G be an odd primitive graph. Then G contains two vertex disjoint odd cycles.*

Proof. Assume that G is an odd primitive graph. Then there exists a pair of vertices u, v such that $\gamma(G) = \gamma(u,v)$ and thus there are no walks of length $\gamma(G) - 1$ from u to v in G. Let W_u and W_v be walks of the shortest odd lengths from vertex u to vertex u and from vertex v to vertex v, respectively. Then the lengths of W_u and W_v are both not greater that $\gamma(G)$. Since any walk of odd length from a vertex of G to itself must contain an odd cycle, we can assume that $W_u = P_u \cup C_u \cup P_u$ is an $u - u'-$lollipop and $W_v = P_v \cup C_v \cup P_v$ is a $v - v'-$lollipop. Since both $l(W_u)$ and $l(W_v)$ are odd, C_u and C_v are two odd cycles in G. Next, we show that $C_u \cap C_v = \emptyset$. Otherwise, let $x \in C_u \cap C_v$. If C_u and C_v are the same cycle, then u' and v' divide C_u into two parts, C_u' and C_u'' say (possibly, $u' = v'$) and so $P_u \cup C_u' \cup P_v$ and $P_u \cup C_u'' \cup P_v$ are two walks from vertex u to v whose lengths have different parity. We apply Proposition 2 to obtain that

$$\begin{aligned}
\gamma(u,v) &\leq \max\{l(P_u \cup C_u' \cup P_v), l(P_u \cup C_u'' \cup P_v)\} - 1 \\
&\leq l(P_u \cup C_u \cup P_v) - 1 \\
&\leq \frac{l(W_u) + l(W_v)}{2} - 1 \\
&\leq \gamma(G) - 1,
\end{aligned}$$

contradicting the assumption. Hence $C_u \neq C_v$. Then x and u' divide C_u into two parts C_u' and C_u'', x and v' divide C_v into two parts C_v' and C_v''. It is seen easily that either $P_u \cup C'_u \cup C'_v \cup P_v$ and $P_u \cup C''_u \cup C''_v \cup P_v$ are both even or $P_u \cup C'_u \cup C''_v \cup P_v$ and $P_u \cup C''_u \cup C'_v \cup P_v$ are both even. Since

$$l(P_u \cup C'_u \cup C'_v \cup P_v) + l(P_u \cup C''_u \cup C''_v \cup P_v) \leq l(W_u) + l(W_v) \leq 2\gamma(G),$$

we have

$$\min\{l(P_u \cup C'_u \cup C'_v \cup P_v), l(P_u \cup C''_u \cup C''_v \cup P_v)\} \leq \gamma(G),$$

and similarly,

$$\min\{l(P_u \cup C'_u \cup C''_v \cup P_v), l(P_u \cup C'_u \cup C''_v \cup P_v)\} \leq \gamma(G).$$

Therefore, one of the four walks from u to v has even length at most $\gamma(G) - 1$ and hence there is a walk of length $\gamma(G) - 1$ from u to v in G, a contradiction. Thus $C_u \cap C_v = \emptyset$. This completes the proof of the theorem. □

Corollary 5. *Let G be an odd primitive graph of order n. Then $g_o(G) \leq \left[\frac{n}{2}\right]$.*

Corollary 6. *Let G be a primitive graph. If G contains an unique odd cycle, then $\gamma(G)$ is even.*

We shall next give a characterization of the primitive graphs of order n and odd girth $\frac{n}{2}$ whose exponent is $\frac{n}{2}$. It follows from Theorem 3 that $\frac{n}{2}$ is the maximum value possible for the girth of such a graph.

Theorem 7. *Let G be an odd primitive graph of order n and $g_o(G) = \frac{n}{2}$. Then $\gamma(G) = \frac{n}{2}$ if and only if*

$$V(G) = V(C_{\frac{n}{2}} + e + C'_{\frac{n}{2}}) = V(C_{\frac{n}{2}} \times P_1) \ \textit{and}\ E(C_{\frac{n}{2}} + e + C'_{\frac{n}{2}}) \subseteq E(G) \subseteq E(C_{\frac{n}{2}} \times P_1).$$

Proof. Assume that G is an odd primitive graph of order n and $g_o(G) = \frac{n}{2}$ and $\gamma(G) = \frac{n}{2}$. So $n \equiv 2 \pmod 4$. Since G is connected, it follows from Theorem 4 that G contains a spanning subgraph isomorphic to $C_{\frac{n}{2}} + e + C'_{\frac{n}{2}}$,the graph obtained from two vertex disjoint cycles $C_{\frac{n}{2}}$ and $C'_{\frac{n}{2}}$ by adding an edge e with one end in $C_{\frac{n}{2}}$ and one end in $C'_{\frac{n}{2}}$. Since $g_o(G) = \frac{n}{2}$, each cycle of length $\frac{n}{2}$ contains no chord edge. Let $C_1 = C_{\frac{n}{2}}$ and $C_2 = C'_{\frac{n}{2}}$. We denote by $[C_1, C_2]$ the set of edges with one end in C_1 and the other in C_2. For two vertices u, v of $C_i, 1 \leq i \leq 2$, we denote by $d_{C_i}(u, v)$ the distance between u and v in C_i.

Claim 1. $\gamma(C_{\frac{n}{2}} + e + C'_{\frac{n}{2}}) = \frac{n}{2}$.

Let $e = c_1c_2$ with $c_i \in C_i, i = 1, 2$. By the definition, it is easy to check that $\gamma(c_1, c_2) = \frac{n}{2}$. Hence in order to show this claim, it suffices to prove that

$$\gamma(c_1, c_2) = \max\{\gamma(u, v) | u, v \in V(C_{\frac{n}{2}} + e + C'_{\frac{n}{2}})\}.$$

Let $u, v \in V(G)$. If u and v are either both in C_1 or both in C_2, then we see easily that there are two paths from u to v whose lengths have different parity and are both not exceeding $\frac{n}{2}$. Thus we apply Proposition 2 to obtain that $\gamma(u, v) < \frac{n}{2} = \gamma(c_1, c_2)$. Assume that $u \in C_1$ and $v \in C_2$. Then u and c_1 divide C_1 into two parts C'_1 and C''_1,

v and c_2 divide C_2 into two parts C_2' and C_2''. Without loss of generality, suppose that

$$l(C_1') = \min\{l(C_1'), l(C_1''), l(C_2'), l(C_2'')\}.$$

Notice that $C_1' \cup C_2' + e$ and $C_1' \cup C_2'' + e$ are two walks from u to v whose lengths have different parity. Hence it follows from Proposition 2 that

$$\gamma(u,v) \leq \max\{l(C_1' \cup C_2' + e), l(C_1' \cup C_2'' + e)\} - 1 \leq l(C_2' \cup C_2'' + e) - 1 \leq \frac{n}{2} \leq \gamma(c_1, c_2).$$

This establishes the claim.

Claim 2. For any two edges $e_1 = u_1v_1, e_2 = u_2v_2 \in [C_1, C_2]$, we have $d_{C_1}(u_1, u_2) = d_{C_2}(v_1, v_2)$.

Suppose on the contrary, that $d_{C_1}(u_1,u_2) \neq d_{C_2}(v_1,v_2)$. Then we shall prove the contradiction that $\gamma(G) < \frac{n}{2}$. Let u, v be any two vertices of G. If u and v are either both in C_1 or both in C_2, then we see easily that there are two paths from u to v whose lengths have different parity and both are not exceeding $\frac{n}{2}$. Thus we apply Proposition 2 to obtain that $\gamma(u,v) < \frac{n}{2}$. Next, assume that $u \in C_1$ and $v \in C_2$. Since $d_{C_1}(u_1,u_2) \neq d_{C_2}(v_1,v_2)$, either we have $d_{C_1}(u,u_1) \neq d_{C_2}(v,v_1)$ or $d_{C_1}(u,u_2) \neq d_{C_2}(v,v_2)$. Thus we may assume without loss of generality that $d_{C_1}(u,u_1) < d_{C_2}(v,v_1)$. Then it is easy to see that there exist two walks from vertex u to vertex v with lengths $d_{C_1}(u,u_1)+d_{C_2}(v,v_1)+1$ and $d_{C_1}(u,u_1)+l(C_2)-d_{C_2}(v,v_1)+1$, respectively. Since $l(C_2)$ is odd, $d_{C_1}(u,u_1) + d_{C_2}(v,v_1) + 1$ and $d_{C_1}(u,u_1) + l(C_2) - d_{C_2}(v,v_1) + 1$ have different parity and hence, by Proposition 2 we obtain

$$\begin{aligned}
\gamma(u,v) &\leq \max\{d_{C_1}(u,u_1) + d_{C_2}(v,v_1) + 1, d_{C_1}(u,u_1) + l(C_2) - d_{C_2}(v,v_1) + 1\} - 1 \\
&\leq d_{C_1}(u,u_1) + l(C_2) - d_{C_2}(v,v_1) \\
&< l(C_2) = \frac{n}{2}.
\end{aligned}$$

Thus it follows that $\gamma(G) \leq \max_{u,v \in V(G)}\{\gamma(u,v)\} < \frac{n}{2}$. Therefore, we have the desired contradiction. This proves Claim 2.

Since G contains a spanning subgraph isomorphic to $C_{\frac{n}{2}} + e + C'_{\frac{n}{2}}$, by Claims 1 and 2, it follows that

$$V(G) = V(C_{\frac{n}{2}} + e + C'_{\frac{n}{2}}) = V(C_{\frac{n}{2}} \times P_1) \text{ and } E(C_{\frac{n}{2}} + e + C'_{\frac{n}{2}}) \subseteq E(G) \subseteq E(C_{\frac{n}{2}} \times P_1).$$

Conversely, suppose that G satisfies the hypothesis of the theorem. Then we see easily that

$$\gamma(C_{\frac{n}{2}} \times P_1) \leq \gamma(G) \leq \gamma(C_{\frac{n}{2}} + e + C'_{\frac{n}{2}}).$$

It is easy to show that $\gamma(C_{\frac{n}{2}} \times P_1) = \frac{n}{2}$, and then by Claim 1, it follows that $\gamma(G) = \frac{n}{2}$. The proof of the theorem is complete. □

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, MacMillan Co., New York, 1976.

[2] Liu Bolian, B. McKay, N. Wormald and Zhang Kemin, The exponent set of symmetric primitive matrices with zero trace, *Linear Alg. Appl.*, 133(1990),121-131.

[3] Shao Jiayu, The exponent set of symmetric primitive matrices, *Sci Sinica A*, 9(1986), 931-939.

[4] Wang Jianzhong and Wang Dianjun, A characterization for the exponent set of symmetric primitive matrices, *Advances in Math. (China)*, 22(1993), 516-523.

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