Minimally k-factor-critical graphs

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Abstract

A graph G of order n is k-factor-critical, where k is an integer of the same parity as n with $0 \leq k \leq n$, if $G - X$ has a perfect matching for any set X of k vertices of G. A $k-$factor-critical graph G is called minimal if for any edge $e \in E(G)$, $G - e$ is not $k-$factor-critical. In this paper we study some properties of minimally $k-$factor-critical graphs, in particular a bound on the minimum degree, and characterize $(n-4)-$ and minimally $(n-4)-$factor-critical graphs.

1. Introduction

The graphs $G = (V(G), E(G))$ we consider here are undirected, simple and finite of order $|V(G)| = n$. A graph is *even* if its order is even and *odd* if its order is odd. The *neighborhood* of a vertex x is $N(x) = \{y; y \in V(G) \text{ and } xy \in E(G)\}$, its *closed neighborhood* is $N[x] = N(x) \cup \{x\}$, and its *degree* is the integer $d_G(x) = |N(x)|$. The *minimum degree* of G is $\delta(G) = \min\{d_G(x); x \in V(G)\}$. When no confusion may arise, we write V and $d(x)$ instead of $V(G)$ and $d_G(x)$. For any set $A \subseteq V$, $G[A]$ denotes the subgraph induced by A in G, $G - A$ stands for $G[V - A]$. Similarly, if $e = uv$ is an edge of G, $G - e$ or $G - uv$ stands for $(V(G), E(G) - \{e\})$. A *claw* of G is an induced subgraph isomorphic to the star $K_{1,3}$. If $G - A$ is not connected, that is if A is a cutset of G, we denote by $c_o(G - A)$ the number of odd components of $G - A$. A *matching* F of G is a set of independent edges and a *perfect matching* is a matching covering all the vertices of G. Clearly if G has a perfect matching F, its order n is even and F consists of $\frac{n}{2}$ edges. We adopt the convention that a graph of order 0 has a perfect matching. A graph G of even order n is $q-$*extendable* [9], where

q is an integer with $1 \leq q \leq \frac{n}{2}$, if G is connected, has a perfect matching and every set of q independent edges is contained in a perfect matching. A graph G of order n is k*–factor-critical* [5], where k is an integer of same parity as n with $0 \leq k \leq n$, if $G - X$ has a perfect matching for any set X of k vertices of G. Graphs which are 0–factor-critical, 1–factor-critical, 2–factor-critical are respectively graphs with a perfect matching, factor-critical graphs as defined in [6], bicritical graphs as defined in [7]. For k and thus n even, a k–factor-critical graph is clearly $\frac{k}{2}$–extendable. A k–factor-critical (q–extendable resp.) graph G is called *minimal* if for every edge $e \in E(G)$, $G - e$ is not k–factor-critical (q–extendable resp.).

Minimally bicritical graphs have been extensively studied (see [8]). In [1], [2] and [3], Anunchuen and Caccetta gave general properties of minimal q–extendable graphs and characterized q–extendable and minimally q–extendable graphs of even order n for $q = \dfrac{n}{2} - 1$ and $q = \dfrac{n}{2} - 2$.

Our purpose is to study some properties of minimally k–factor-critical graphs and to characterize $(n-4)$– and minimally $(n-4)$–factor-critical graphs.

2. Basic properties of minimally k–factor-critical graphs

Let us first recall some properties of k–factor-critical graphs.

Lemma 1 [2] If G is k–factor-critical for some $1 \leq k < n$ with $n + k$ even, then G is k-connected, $(k+1)$-edge-connected (and thus $\delta \geq k+1$ which is still true when $k = 0$), and $(k-2)$–factor-critical if $k \geq 2$.

Definition : A graph G has Property Q_k if $c_o(G - B) \leq |B| - k$ for every $B \subseteq V$ with $|B| \geq k$.

Lemma 2 [2] A graph G is k–factor-critical if and only if it has Property Q_k.

The following Lemma 3 and Theorem 2.1 are simple adaptations of similar results for $k = 1$ or 2 (cf [8]).

Lemma 3 Let G be a k–factor-critical graph. Then G is minimal if and only if for each $e = uv \in E(G)$, there exists $S_e \subseteq V - \{u, v\}$ with $|S_e| = k$ such that every perfect matching of $G - S_e$ contains e.

Proof **1.** Let G be a minimally k–factor-critical graph, then for each $e = uv \in E(G)$, $G - e$ is not k–factor-critical. Therefore, there exists $S_e \subseteq V$ with $|S_e| = k$ such that $G - e - S_e$ has no perfect matching. But $G - S_e$ has a perfect matching since G is k–factor-critical. Hence neither u nor v belong to S_e and any perfect matching of $G - S_e$ contains e.

2. Conversely, suppose that for each $e = uv \in E(G)$, there exists $S_e \subseteq V - \{u, v\}$ with $|S_e| = k$ such that any perfect matching of $G - S_e$ contains e. So, $G - e - S_e$ has no perfect matching and thus $G - e$ is not k–factor-critical. Therefore, G is minimally k–factor-critical. ∎

Theorem 2.1 Let G be a k–factor-critical graph. Then G is minimal if and only if for each $e = uv \in E(G)$, there exists $B_e \subseteq V - \{u, v\}$ with $|B_e| \geq k$ such that $c_o(G - B_e - e) = |B_e| - k + 2$ and u and v belong respectively to two different odd components of $G - B_e - e$.

Proof 1. If G is a minimally k–factor-critical graph, then for each $e = uv \in E(G)$, $G - e$ is not k–factor-critical. By Lemma 2, there exists $B_e \subseteq V$ with $|B_e| \geq k$ such that $c_o(G - e - B_e) > |B_e| - k$ and by parity, $c_o(G - B_e - e) \geq |B_e| - k + 2$. Since G is k–factor-critical, by Lemma 2, $c_o(G - B_e) \leq |B_e| - k$ and thus u and v do not belong to B_e. But $c_o(G - B_e - e) \leq c_o(G - B_e) + 2 \leq |B_e| - k + 2$. Therefore, $c_o(G - B_e - e) = |B_e| - k + 2$, $c_o(G - B_e) = |B_e| - k$, and e is an edge connecting two odd components of $G - B_e - e$. So u and v belong respectively to two different odd components of $G - B_e - e$.

2. Conversely if for each $e \in E(G)$ there exists $B_e \subseteq V$ with $|B_e| \geq k$ and such that $c_o(G - B_e - e) = |B_e| - k + 2$, then B_e contradicts Property Q_k for the graph $G - e$ and $G - e$ is not k–factor-critical. ■

For $n \geq k + 4$, the classes of minimally k–factor-critical graphs and of $(k + 2)$–factor-critical graphs are both contained in the class of k–factor-critical graphs (cf Lemma 1). The next result shows that these two classes are disjoint.

Theorem 2.2 Let G be a minimally k–factor-critical graph of order $n \geq k + 4$. Then G is not $(k + 2)$–factor-critical.

Proof Let $e = uv$ be an edge of a minimally k–factor-critical graph G of order $n \geq k + 4$, and B_e a subset of V as in Theorem 2.1.

Case 1 $|B_e| \geq k + 2$. Let $B = B_e$, then $|B| \geq k + 2$ and $c_o(G - B) = |B_e| - k > |B| - (k + 2)$.

Case 2 $|B_e| = k + 1$. Let $B = B_e \cup \{u\}$, then $|B| \geq k + 2$ and $c_o(G - B) \geq c_o(G - B_e) + 1 = |B_e| - k + 1 = |B| - k > |B| - (k + 2)$.

Case 3 $|B_e| = k$. If $G - B_e$ has more than one even component, let w belong to an even component which does not contain the edge e and $B = B_e \cup \{w, u\}$. Then $|B| = k+2$ and $c_o(G - B) \geq c_o(G - B_e) + 2 = |B_e| - k + 2 = |B| - k > |B| - (k+2)$. If $G - B_e$ has just one even component, then $G - B_e - e$ has exactly two components, say C_u which contains u and C_v which contains v, and both are odd. Since $n > k + 2$, we may assume $|C_u| > 1$. By parity, $|C_u| \geq 3$. Let $w \in C_u - \{u\}$ and $B = B_e \cup \{w, u\}$. Then $|B| = k+2$ and $c_o(G - B) \geq c_o(G - B_e) + 2 = |B_e| - k + 2 = |B| - k > |B| - (k+2)$.

In the three cases above, we have $|B| \geq k + 2$ and $c_o(G - B) > |B| - (k + 2)$. By Lemma 2, G is not $(k + 2)$–factor-critical. ■

3. Minimally k–factor-critical graphs and degrees

Theorem 3.1 Let G be a minimally k–factor-critical graph of order n. Then for each $e = uv \in E(G)$, there exists $S_e \subseteq V - \{u, v\}$ with $|S_e| = k$ such that $d_G(u) + d_G(v) \leq n + |N(u) \cap N(v) \cap S_e|$. In particular, $d_G(u) + d_G(v) \leq n + k$.

Proof Since G is a minimally k–factor-critical graph, for each $e = uv \in E(G)$, there exists by Lemma 3 a set $S_e \subseteq V - \{u, v\}$ with $|S_e| = k$ such that any perfect matching of $G - S_e$ contains e.

If $N(u) \cap N(v) \subseteq S_e$ then $|N(u) \cap N(v) \cap S_e| = |N(u) \cap N(v)|$ and thus $d_G(u) + d_G(v) = |N(u) \cup N(v)| + |N(u) \cap N(v)| \leq n + |N(u) \cap N(v) \cap S_e|$. Otherwise, let F be a perfect matching of $G - S_e$. For each $w \in N(u) \cap N(v) - S_e$, there exists $w' \in V - S_e - \{u, v\}$ such that $ww' \in E(F)$. If $w' \in N(u) \cup N(v)$, say $w' \in N(v)$, then $F' = (F - \{uv, ww'\}) \cup \{uw, vw'\}$ is a perfect matching of $G - S_e$ which does not contain e, in contradiction to the definition of S_e. Hence $w' \notin N(u) \cup N(v)$. Since F is a matching, we have $|N(u) \cup N(v)| \leq n - |(N(u) \cap N(v)) \setminus S_e| = n - |N(u) \cap N(v)| + |N(u) \cap N(v) \cap S_e|$. Therefore, $d_G(u) + d_G(v) \leq n + |N(u) \cap N(v) \cap S_e|$. ■

Corollary 3.2 Let G be a $k-$factor-critical graph of order n and maximum degree $\Delta(G) = n - 1$. Then G is minimal if and only if G contains one vertex of degree $n - 1$ and $n - 1$ vertices of degree $k + 1$.

Proof Let G be a $k-$factor-critical graph of order n, and $u \in V$ such that $d_G(u) = n - 1$. Then for any $v \in V \setminus \{u\}$, we have $e = uv \in E(G)$.

If G is minimal, then for any $v \in V \setminus \{u\}$, by Theorem 2.3, $d_G(u) + d_G(v) \leq n + k$. So $d_G(v) \leq n + k - (n-1) = k+1$. By Lemma 1, $\delta(G) \geq k+1$ and thus $d_G(v) = k+1$.

Conversely, if G has $n - 1$ vertices of degree $k + 1$, then for any $e \in E(G)$, we have $\delta(G - e) < k + 1$ and thus $G - e$ is not $k-$factor-critical. ■

Theorem 3.3 In a minimally $k-$factor-critical graph G of order $n \geq k+4$, $\delta(G) \leq \frac{n+k}{2} - 1$. If moreover $n \geq k + 6$, then $\delta(G) \leq \frac{n+k}{2} - 2$.

Proof Let G be $k-$factor-critical of order $n \geq k + 4$. By [4], if $\delta(G) \geq \frac{n+k}{2}$ then G is k-hamiltonian, i.e. $G - X$ contains a hamiltonian cycle for every set of at most k vertices of G. Let e be any edge of G and X any set of k vertices of G. Since $G - X$ contains an even hamiltonian cycle, $G - X - e$ contains a hamiltonian path of even order, and thus a perfect matching. Therefore $G - e$ is k-factor-critical. Hence if G is minimally k-factor-critical then $\delta(G) \leq \frac{n+k}{2} - 1$, which is the first part of the theorem. To show the second part, we give another and direct proof of the first part, without using the result of [4], in order to point out all the possible cases of equality $\delta(G) = \frac{n+k}{2} - 1$. Since G is a minimally $k-$factor-critical graph, for each $e = uv \in E(G)$ there exists by Theorem 2.1 a set $B_e \subseteq V - \{u, v\}$ with $|B_e| \geq k$ such that $c_o(G - e - B_e) = |B_e| - k + 2$. Let $C_1, C_2, \cdots, C_p, C_u$ and C_v be the odd components of $G - e - B_e$, where $p = |B_e| - k$, C_u is the component which contains u and C_v the component which contains v. We may assume $|C_1| \leq |C_2| \leq \cdots \leq |C_p|$ and $|C_u| \leq |C_v|$. We note that $\delta(G) \leq |B_e| + |C_1| - 1$ and that $\delta(G) \leq |B_e| + |C_u| - 1$ if $|C_u| > 1$ (i.e. by parity $|C_u| \geq 3$), $\delta(G) \leq |B_e| + 1$ if $|C_u| = 1$.

Case 1. $|B_e| \geq k + 2$ i.e. $p \geq 2$.

Since $|C_p| \geq \cdots \geq |C_1|$ and $|C_v| \geq |C_u| \geq 1$, we have $n \geq |B_e| + (|B_e| - k)|C_1| + 2$, that is $n \geq |B_e| + (|B_e| - k) + 2(|C_1| - 1) + (|B_e| - k - 2)(|C_1| - 1) + 2$, with $|B_e| - k - 2 \geq 0$

and $|C_1|-1 \geq 0$. Hence $2(|B_e|+|C_1|) \leq n+k$ and $\delta(G) \leq |B_e|+|C_1|-1 \leq \frac{n+k}{2}-1$. The equality $\delta(G) = \frac{n+k}{2} - 1$ implies here that $n = |B_e| + (|B_e| - k)|C_1| + 2$, $(|B_e|-k-2)(|C_1|-1) = 0$, $|C_1| = |C_2| = \cdots = |C_p|$, $|C_u| = |C_v| = 1$, and $G-B_e-e$ contains no even component. If $|C_1| > 1$, that is $|C_1| \geq 3$, then $|B_e| = k+2$ and $n \geq |B_e|+8 = k+10$. On the other hand, $\frac{n+k}{2}-1 = \delta(G) \leq |B_e|+1 = k+3$ and thus $n \leq k+8$, which yields a contradiction. Hence $|C_1| = |C_2| = \cdots = |C_p| = 1$, $n = 2|B_e| - k + 2 \geq k+6$ and $|B_e| = \frac{n+k}{2} - 1 = \delta(G)$. So for $1 \leq i \leq p$, the only vertex z_i of C_i is adjacent to every vertex of B_e, and each vertex u, v is adjacent to at least $|B_e|-1$ vertices of B_e. Therefore in this first case, the equality $\delta(G) = \frac{n+k}{2}-1$ implies $n \geq k+6$, $|N(u) \setminus N[v]| \leq 1$ and $|N(v) \setminus N[u]| \leq 1$.

Case 2. $|B_e| = k+1$ i.e. $p = 1$.

Subcase 2.1 $|C_1| \leq |C_u|$.

If $|C_1| \geq \frac{n-k}{4}$ then $|B_e|+|C_1| \leq n-|C_u|-|C_v| \leq n-2|C_1| \leq n-\frac{n-k}{2} = \frac{n+k}{2}$. Hence $\delta(G) \leq |B_e| + |C_1| - 1 \leq \frac{n+k}{2} - 1$. The equality $\delta(G) = \frac{n+k}{2} - 1$ requires $|C_u| = |C_v| = |C_1| = \frac{n-k}{4}$, and $|B_e|+|C_1| = n-|C_u|-|C_v|$ and thus $G-B_e-e$ has no even component. Therefore $n = |B_e|+3|C_1| = k+1+\frac{3(n-k)}{4}$, that is $n = k+4$.

If $|C_1| < \frac{n-k}{4}$ i.e. $|C_1| \leq \frac{n-k-2}{4}$, then $\delta(G) \leq |B_e|+|C_1|-1 \leq \frac{n+3k-2}{4} < \frac{n+k}{2} - 1$, with a strict inequality.

Subcase 2.2 $|C_u| < |C_1|$ and thus by parity, $|C_1| \geq |C_u|+2$.

2.2.1 If $|C_u| = 1$ then $n \geq |B_e| + |C_u| + |C_v| + |C_1| \geq k+6$ and thus $\delta(G) \leq |B_e| + 1 = k+2 \leq \frac{n+k}{2} - 1$. The equality $\delta(G) = \frac{n+k}{2} - 1$ requires $n = k+6$, $|C_u| = |C_v| = 1$, $|C_1| = 3$, $G - C_1 - e$ has no even component, u and v are adjacent to every vertex of B_e, and each of the three vertices of C_1 is adjacent to at least k of the $k+1$ vertices of B_e. In particular, $N(u) \setminus N[v] = N(v) \setminus N[u] = \emptyset$.

2.2.2 Suppose now $|C_u| \geq 3$ (thus $n \geq k+12$).

If $|C_u| \geq \frac{n-k-2}{4}$, then $|B_e| + |C_u| \leq n - |C_1| - |C_v| \leq n - \frac{n-k-2}{2} - 2 = \frac{n+k}{2} - 1$ and $\delta(G) < \frac{n+k}{2} - 1$, strictly.

If $|C_u| \leq \frac{n-k-4}{4}$, then $|B_e| + |C_u| \leq k+1+\frac{n-k}{4} - 1 = \frac{n+3k}{4}$ and $\delta(G) \leq \frac{n+3k}{4} - 1 < \frac{n+k}{2} - 1$, strictly.

Case 3. $|B_e| = k$ i.e. $p = 0$ and thus $|C_u| \leq \frac{n-k}{2}$.

Subcase 3.1 If $|C_u| > 1$ then $\delta(G) \leq |B_e| + |C_u| - 1 \leq \frac{n+k}{2} - 1$. The equality $\delta(G) = \frac{n+k}{2} - 1$ requires $|C_u| = |C_v| = \frac{n-k}{2}$ and thus $\frac{n-k}{2}$ is odd ≥ 3 and $n-k \geq 6$, $G - B_e - e$ has no even component, C_u and C_v are cliques, every vertex of $C_u \setminus \{u\}$ and of $C_v \setminus \{v\}$ is adjacent to all the vertices of B_e, u (v resp.) is adjacent to all the vertices of B_e except perhaps to one of them.

Subcase 3.2 If $|C_u| = 1$ then $\delta(G) \leq |B_e| + 1 = k+1 \leq \frac{n+k}{2} - 1$ since $n \geq k+4$. The equality $\delta(G) = \frac{n+k}{2} - 1$ requires $n = k+4$, $|C_u| = |C_v| = 1$ and $G - B_e - e$ contains one even component of order 2, or $|C_u| = 1$, $|C_v| = 3$ and $G - B_e - e$ contains no even component.

To summarize the study, when $n \geq k+6$ the only possible cases of equality $\delta(G) = \frac{n+k}{2} - 1$ occur in 1, 2.2.1 and 3.1. Hence if a minimally k-factor-critical graph G with $n \geq k+6$ satisfies $\delta(G) = \frac{n+k}{2} - 1$, each edge $e = uv$ is of one of the three encountered types, Type 1 described in Case 1, Type 2 described in Case 2.2.1, Type 3 described in Case 3.1. Recall that if $e = uv$ is of Type 1 then $|N(u) \setminus N[v]| \leq 1$ and $|N(v) \setminus N[u]| \leq 1$; if e is of Type 2 then $N(u) \setminus N[v] = N(v) \setminus N[u] = \emptyset$; if e is of Type 3 then there exist two disjoint triangles $K_3(u)$ and $K_3(v)$ such that uv is the only edge of G between $K_3(u)$ and $K_3(v)$.

Let G be a minimally k-factor-critical graph of order $n \geq k+6$ and $\delta(G) = \frac{n+k}{2} - 1$. If G contains an edge $e = uv$ of Type 1, let $x \in B_e \cap N(u)$. Using the notation of Case 1, we have $\{z_1, z_2\} \subseteq N(x) \setminus N[u]$, so the edge ux is not of Type 1 or 2 and thus must be of Type 3. But as in Type 1 each z_i, $1 \leq i \leq p$, is adjacent to every vertex of B_e, and u is adjacent to every vertex of B_e except perhaps to at most one, we cannot find two disjoint triangles $K_3(x)$ and $K_3(u)$ joined by the only edge ux, a contradiction. Hence no edge of G is of Type 1.

If G contains an edge $e = uv$ of Type 2 (which implies $n = k+6$), let x be a vertex of B_e adjacent to some vertex z_1 of C_1. Since $z_1 \in N(x) \setminus N[u]$, the edge ux is not of Type 2 and must be of Type 3. The triangle $K_3(u)$ does not contain v since x is adjacent to v, and is of the kind utw with $t, w \in B_e$. Hence $K_3(x)$ contains no vertex of C_1 since each vertex of C_1 is adjacent to at least one of t, w, and $K_3(x)$ is contained in B_e. This gives a contradiction since u is adjacent to every vertex of B_e.

Therefore every edge of G must be of Type 3. Let $e = uv$ be such an edge and x, y two vertices of $C_u \setminus \{u\}$. Since $N(x) \setminus \{y\} = N(y) \setminus \{x\} = (C_u \setminus \{x,y\}) \cup B_e$, the edge xy cannot be of Type 3.

Hence no minimally k-factor-critical graph of order $n \geq k+6$ satisfies $\delta(G) = \frac{n+k}{2} - 1$. This completes the proof of the theorem. ∎

Corollary 3.4 Let G be a minimally k-factor-critical graph of order n. If $k = n-2, n-4$ or $n-6$, then $\delta(G) = k+1$.

Proof: The only $(n-2)$–factor-critical graph of order n is K_n, which proves the corollary for $k = n-2$. For $k = n-4$ or $n-6$, this is a consequence of Theorem 3.3 and the property $\delta(G) \geq k+1$ recalled in Lemma 1. ■

Problem: It is clear from the Ear Decomposition of 1-factor-critical graphs (cf [8]) that every minimally 1-factor-critical graphs has minimum degree 2.

Is it true that every minimally k–factor-critical graph G has minimum degree $\delta(G) = k+1$?

4. Minimally (n – 4)–factor-critical graphs

Theorem 4.1 A graph G of order $n \geq 6$ is $(n-4)$–factor-critical if and only if G is claw-free and $\delta(G) \geq n-3$.

Proof Let G be a $(n-4)$–factor-critical graph. By Lemma 1, $\delta(G) \geq k+1 = n-3$. If there exists a set Y of four vertices inducing a claw, then $G[Y]$ has no perfect matching, contradicting G is $(n-4)$–factor-critical.

Conversely, let Y be any subgraph of G induced by exactly four vertices. Since G is claw-free and $\delta(G) \geq n-3$, $Y \neq K_{1,3}$ and $\delta(Y) \geq 1$ which implies that Y has a perfect matching. ■

Let us remark that the condition for a graph G of order n to be claw-free and have minimum degree $\delta(G) \geq n-3$ is equivalent to the condition to have independence number $\alpha(G) \leq 2$ and $\delta(G) \geq n-3$.

Theorem 4.2 A graph G of order $n \geq 6$ is minimally $(n-4)$–factor-critical if and only if it is claw-free and satisfies one of the following three conditions:

(1) G is $(n-3)$–regular.

(2) G contains one vertex of degree $n-1$ and $n-1$ vertices of degree $n-3$.

(3) G contains $n-2$ vertices of degree $n-3$ and two vertices of degree $n-2$, say u and v, which are such that $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.

Proof Let G be a minimally $(n-4)$–factor-critical graph. By Theorem 4.1 and Corollary 3.4, G is claw-free and $\delta(G) = n-3$.

If $\Delta(G) = n-3$ then G is $(n-3)$-regular.

If $\Delta(G) = n-1$ then by Corollary 3.2, G contains one vertex of degree $n-1$ and $n-1$ vertices of degree $k+1 = n-3$.

If $\Delta(G) = n-2$ then each vertex of G has degree $n-2$ or $n-3$. If n is odd, then $n-2$ is also odd and G has an even number of vertices of degree $n-2$. If n is even, then $n-3$ is odd and G has an even number of vertices of degree $n-3$ and thus also an even number of vertices of degree $n-2$. Therefore, G contains at least two vertices of degree $n-2$. Suppose G has three vertices of degree $n-2$, say u, v and w.

Case 1 Two of them, say u and v are not adjacent.

Then $N(u) = N(v) = V - \{u, v\}$ and $w \in N(u)$. Let $e = uw$. Since $G - e$ is not $(n-4)$–factor-critical and $\delta(G-e) \geq n-3$, $G-e$ has an induced subgraph H isomorphic to $K_{1,3}$ by Theorem 4.1. Since G is claw-free, H must contain u and w

as two pendant vertices. There are only two vertices v and w which are not adjacent to u in $G-e$, so the only other possible pendant vertex is v. But H can not be $K_{1,3}$ since $vw \in E(G-e)$, a contradiction.

Case 2 uv, uw and $vw \in E(G)$. Let $e = uw$. As in Case 1, $G-e$ has an induced subgraph H isomorphic to $K_{1,3}$ and u and w are two pendant vertices of H. If x is the third pendant vertex of H, then $x \in V(G) - \{u, v, w\}$ and $x \notin N(u) \cup N(w)$. Considering similarly the edge uv, there exists $y \in V(G) - \{u, v, w\}$ such that $y \notin N(u) \cup N(v)$. Since $d(u) = n-2$ and $x, y \notin N(u) - \{u\}$, we have $x = y$. Hence, $N(x) \subseteq V \setminus \{u, v, w\}$ and thus $d(x) \leq n-4$, a contradiction.

Therefore, there are exactly two vertices, say u and v, of degree $n-2$. If u and v are not adjacent then $N(u) \setminus \{v\} = N(v) \setminus \{u\} = V \setminus \{u, v\}$. If they are adjacent and if $N(u) \setminus \{v\} \neq N(v) \setminus \{u\}$, then $N(u) = V \setminus \{u'\}$ and $N(v) = V \setminus \{v'\}$ for some vertices $u' \neq v'$. By considering the edge uv', a similar argument as above yields a contradiction.

Conversely, by the hypothesis we have $\delta(G) = n-3$ and G is claw-free. By Theorem 4.1, G is $(n-4)$–factor-critical. Moreover, for any $e = uv \in E(G)$, if $d(u) = n-3$ or $d(v) = n-3$, we have $\delta(G-e) < n-3$ and $G-e$ is not $(n-4)$–factor-critical. Otherwise, we are in the third case with $uv \in E(G)$, $N(u) \setminus \{v\} = N(v) \setminus \{u\}$ and $V \setminus (N(u) \cup N(v)) = \{x\}$ for some vertex x of G. If w is any vertex in $N(u) \cap N(v)$ and since $d(x) = n-3$, then $\{u, v, w, x\}$ induced a subgraph of $G-e$ which isomorphic to $K_{1,3}$. By Theorem 4.1, $G-e$ is not $(n-4)$–factor-critical. Therefore, G is a minimally $(n-4)$–factor-critical graph. ■

In [2] and [3], Anunchuen and Cacetta determined all the $(\frac{n}{2}-2)$– and minimally $(\frac{n}{2}-2)$–extendable graphs of even order $n \geq 6$. Since for n even, every $(n-4)$–factor-critical graph is $(\frac{n}{2}-2)$–extendable, we expected to find in Theorem 4.1 a subclass of non-bipartite $(\frac{n}{2}-2)$–extendable graphs (some p–extendable graphs are bipartite whereas k–factor-critical graphs are never bipartite). Surprisingly, for $n \geq 10$, we found all of them, that is

Corollary 4.3 A non-bipartite graph of even order $n \geq 10$ is $(\frac{n}{2}-2)$–extendable if and only if it is $(n-4)$–factor-critical.

In consequence

Corollary 4.4 A non-bipartite graph of even order $n \geq 10$ is minimally $(\frac{n}{2}-2)$–extendable if and only if it is minimally $(n-4)$–factor-critical.

This last corollary allows us to get from Theorem 4.2 all the non-bipartite minimally $(\frac{n}{2}-2)$–extendable graphs of order $n \geq 10$ which were obtained in [2] after a long proof (the bipartite ones are easily obtained from the bipartite $(\frac{n}{2}-2)$–extendable graphs which are all the bipartite graphs of even order n and minimum degree $\geq \frac{n}{2}-1$).

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