Some Ramsey Numbers for Complete Bipartite Graphs

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Abstract

Upper bounds are determined for the Ramsey number $r(K_{2,2}, K_{m,n})$, $2 \leq m \leq n$. These bounds are attained for infinitely many n in case of $m \leq 3$ and are fairly close to the exact value for every m if n is sufficiently large.

1 Introduction

For complete bipartite graphs G and H only few exact values of the Ramsey number $r(G, H)$ are known. Harary [4] determined the numbers $r(K_{1,t}, K_{1,n})$. Parsons [7,8,9] and Stevens [10] investigated the numbers $r(K_{1,t}, K_{m,n})$. Parsons determined an upper bound for the case $m = 2$, which is attained if n is small relative to t and certain regular graphs exist. Stevens completely solved the case when n is sufficiently large depending on t and m. In [3] it was shown that $r(K_{2,n}, K_{2,n}) \leq 4n - 2$ with equality for infinitely many n. Moreover, Chung and Graham [1] derived a general upper bound for $r(K_{m,n}, K_{m,n})$. But up to now, besides $r(K_{3,3}, K_{3,3}) = 18$ determined in [3], no exact values of $r(K_{s,t}, K_{m,n})$ are known when $s, t, m, n \geq 3$.

Here we focus on the numbers $r(K_{2,2}, K_{m,n}) = r(C_4, K_{m,n})$. The case $m = 1$ was already studied by Parsons. He showed that $r(C_4, K_{1,n}) \leq n + \lceil\sqrt{n}\rceil + 1$ with equality for infinitely many n. Here we will derive corresponding upper bounds for the case $m \geq 2$. These bounds are attained for infinitely many n in case of $m \leq 3$ and are fairly close to the exact value for fixed m and sufficiently large n.

As usual, the vertex set of a graph G is denoted by V and the edge set by E. $N_G(v)$ denotes the set of neighbors of a vertex $v \in G$ in G and $d_G(v)$ the degree of v in G. The minimum degree of the vertices in G is denoted by δ_G and the maximum degree by Δ_G. In a 2-coloring of the complete graph K_n we always use green and red as colors. The green subgraph is denoted by $G(g)$ and the red subgraph by $G(r)$. We write $N_g(v)$, $d_g(v)$, δ_g and Δ_g instead of $N_{G(g)}(v)$, $d_{G(g)}(v)$, $\delta_{G(g)}$ and $\Delta_{G(g)}$ and use the corresponding notations for $G(r)$. If A and B are two sets of vertices from K_n, $g(A, B)$ denotes the number of green edges from A to B. If A consists of a single vertex u, we write $g(u, B)$. A 2-coloring of K_n is said to be a (G, H)-coloring if there is neither a green subgraph G nor a red subgraph H.

2 Properties of $(C_4, K_{m,n})$-colorings

The following lemmas will be used later to establish upper bounds for $r(C_4, K_{m,n})$.

Lemma 1. Let χ be a $(C_4, K_{m,n})$-coloring of K_p and $v \in V$. Then the following assertions hold.
(i)

$$\sum_{u \in N_g(v)} g(u, N_r(v)) \leq p - d_g(v) - 1. \tag{1}$$

(ii) If $d_g(v) \geq m$ and if S is an m-element subset of $N_g(v)$ then

$$\sum_{u \in S} g(u, N_r(v)) \geq p - n - d_g(v) + |\bigcap_{u \in S} N_r(u) \cap N_g(v)|. \tag{2}$$

(iii) If $d_g(v) \geq m-1$ and if S is an $(m-1)$-element subset of $N_g(v)$ then

$$\sum_{u \in S} g(u, N_r(v)) \geq p - n - d_g(v). \tag{3}$$

Proof. Since no green C_4 occurs in χ, each vertex in $N_r(v)$ can be joined by at most one green edge to $N_g(v)$ and this yields (1). Moreover, there is no red $K_{m,n}$ in χ. Thus, in $N_g(v)$ there are no m vertices with n common red neighbors in $N_r(v) \cup N_g(v)$ and no $m-1$ with n common red neighbors in $N_r(v)$. This implies (2) and (3). ■

Lemma 2. Let χ be a $(C_4, K_{m,n})$-coloring of K_p, $m \geq 2$ and $p \geq \max\{n+m+1, n+m^2-m-1\}$. Then

$$\left\lceil \frac{p-n-1}{m} \right\rceil + 1 \leq \Delta_g \leq m + \left\lfloor (m+n-1)/(\lceil \frac{p-n}{m} \rceil - 1) \right\rfloor. \tag{4}$$

Furthermore,

$$d_g(v) \leq m - 1 + \left\lfloor (n-1)/\lceil \frac{p-n-d_g(v)}{m-1} \rceil \right\rfloor \tag{5}$$

for every $v \in V$ with $m - 1 \leq d_g(v) \leq p - n - 1$.

Proof. No red $K_{m,n}$ implies green edges in χ. Take m vertices at least two of them adjacent in green. They have at least $p - m - m\Delta_g + 2$ and at most $n-1$ common red neighbors. This implies $\Delta_g \geq m-1$. Consider now a vertex v with $d_g(v) = \Delta_g$. Let $N_g(v) = \{u_1, ..., u_{\Delta_g}\}$ and $g_i = g(u_i, N_r(v))$. We may assume that $g_1 \leq g_2 \leq ... \leq g_{\Delta_g}$. Using inequality (3) and $g_i \leq \Delta_g - 1$ we obtain that $(m-1)(\Delta_g - 1) \geq \sum_{i=1}^{m-1} g_i \geq p - n - \Delta_g$. This yields the first inequality in (4). To prove the second inequality in (4) note that $\Delta_g \geq m$. Moreover, $|\bigcap_{i=1}^{m} N_r(u_i) \cap N_g(v)| \geq \Delta_g - 2m$ since each u_i can have at most one green neighbor in $N_g(v)$. Now inequality (2) implies that $\sum_{i=1}^{m} g_i \geq p - n - 2m$ which yields $g_m \geq \lceil (\sum_{i=1}^{m} g_i)/m \rceil \geq \lceil (p-n)/m \rceil - 2$. Using that $g_m \leq g_{m+1} \leq ... \leq g_{\Delta_g}$ and inequality (1) we obtain that $p-n-2m+(\Delta_g - m)(\lceil (p-n)/m \rceil - 2) \leq \sum_{i=1}^{\Delta_g} g_i \leq p - \Delta_g - 1$ and this yields the second inequality of (4).

To prove inequality (5) consider a vertex v with $d = d_g(v) \geq m-1$ and $d \leq p-n-1$. Let $N_g(v) = \{u_1, ..., u_d\}$ and $g_i = g(u_i, N_r(v))$. Again we may assume that $g_1 \leq ... \leq g_d$. Then inequalities (3) and (1) imply that $p-n-d+(d-(m-1))\lceil (p-n-d)/(m-1) \rceil \leq p-d-1$ yielding inequality (5). ∎

3 Erdös-Rényi and Moore graphs

Here we consider two classes of graphs which will be useful to establish lower bounds for $r(C_4, K_{m,n})$.

For a prime power q the **Erdös-Renyi graph** $ER(q)$, first constructed by Erdös and Rényi in [2], is defined to be the graph whose vertices are the points of the projective plane $PG(2,q)$ where two vertices (x,y,z) and (x',y',z') are adjacent iff $xx'+yy'+zz'=0$. The Erdös-Rényi graph was studied in detail by Parsons in [9]. Here we will use the following properties of $ER(q)$.

(α) $ER(q)$ has q^2+q+1 vertices, $q+1$ of degree q and q^2 of degree $q+1$.
(β) $ER(q)$ does not contain a subgraph C_4.
(γ) In $ER(q)$ there are no two adjacent vertices of degree q.
(δ) In $ER(q)$ no vertex of degree q belongs to a subgraph K_3.

Lemma 3. Let q be a prime power, $G = ER(q)$, $\overline{G}$ the complement of G and let $S = \{v_1, ..., v_m\} \subset V = V(G)$. Then

$$|\bigcap_{v \in S} N_{\overline{G}}(v)| \leq q^2 - (m-1)q + \binom{m-1}{2}. \tag{6}$$

Proof. Let $T = V \backslash S$ and, for $v \in S$, let $T_v = N_G(v) \cap T$ and $S_v = N_G(v) \cap S$. Then $|\bigcap_{v\in S} N_{\overline{G}}(v)| = |T| - |\bigcup_{v \in S} T_v|$ and, by property (α), $|T| = q^2+q+1-m$. Thus, inequality (6) is proved if we can show that

$$|\bigcup_{v \in S} T_v| \geq qm - \binom{m}{2}. \tag{7}$$

Let $M = \{(i,j); 1 \leq i < j \leq m\}$. Trivially,

$$|\bigcup_{v \in S} T_v| \geq \sum_{v \in S} |T_v| - \sum_{(i,j) \in M} |T_{v_i} \cap T_{v_j}|. \tag{8}$$

Let $M_1 = \{(i,j) \in M; \{v_i, v_j\} \in E(G), \ \min\{d_G(v_i), d_G(v_j)\} = q\}$ and $M_2 = \{(i,j) \in M; |S_{v_i} \cap S_{v_j}| = 1\}$. By properties ($\beta$) and ($\delta$),

$$\sum_{(i,j) \in M} |T_{v_i} \cap T_{v_j}| \leq \binom{m}{2} - |M_1| - |M_2|. \tag{9}$$

Let $S' = \{v \in S; d_G(v) = q\}$ and $S'' = S \backslash S'$. By property (γ), $|M_1| = \sum_{v\in S'} |S_v|$. Furthermore, by property (β), $|M_2| = \sum_{v \in S} \binom{|S_v|}{2}$. Thus, inequalities (8) and (9) imply

$$|\bigcup_{v\in S} T_v| \geq \sum_{v\in S'}(|S_v| + |T_v| + \binom{|S_v|}{2}) + \sum_{v\in S''}(|T_v| + \binom{|S_v|}{2}) - \binom{m}{2}. \quad (10)$$

Note that $|S_v| + |T_v| = d_G(v)$ and, by property (α), the vertices in S'' have degree $q+1$. Thus, every summand of the two sums in (10) is at least q. This proves (7) and the proof of Lemma 3 is complete. ■

For integers $\delta \geq 3$ and $g \geq 3$ a (δ, g)- **Moore graph** is defined to be a graph regular of degree δ with girth g and $p(\delta, g)$ vertices where

$$p(\delta, g) = \begin{cases} 1 + \frac{\delta}{\delta-2}\{(\delta-1)^{(g-1)/2} - 1\} & \text{if } g \text{ is odd} \\ \frac{2}{\delta-2}\{(\delta-1)^{g/2} - 1\} & \text{if } g \text{ is even.} \end{cases} \quad (11)$$

It is well known that every graph with minimum degree δ and girth g has at least $p(\delta, g)$ vertices.

In the following section we will use a result of Hoffman and Singleton [6] concerning $(\delta, 5)$-Moore graphs. They showed that there are no such graphs with $\delta \geq 3$ and $\delta \notin \{3, 7, 57\}$ whereas $(3,5)$- and $(7,5)$-Moore graphs do exist (the Petersen graph and the so-called Hoffman-Singleton graph). Up to now it is unknown whether a $(57,5)$-Moore graph exists.

4 Ramsey numbers $r(C_4, K_{m,n})$

We will determine bounds and values for $r(C_4, K_{m,n})$ which depend in case of $2 \leq m \leq 4$ on the difference s between n and $(\lceil\sqrt{n}\rceil - 1)^2$, the greatest square less than n $(1 \leq s \leq 2\lceil\sqrt{n}\rceil - 1)$.

Theorem 1. Let $n \geq 2$, $q = \lceil\sqrt{n}\rceil$, $s = n - (q-1)^2$ and $M = \{2, 5, 37, 3137\}$. Then

$$r(C_4, K_{2,n}) \leq \begin{cases} n + 2\lceil\sqrt{n}\rceil - 1; & s = 1 \text{ and } n \notin M, \\ n + 2\lceil\sqrt{n}\rceil; & n \in M \text{ or } 2 \leq s \leq q-1, \\ n + 2\lceil\sqrt{n}\rceil + 1; & q \leq s \leq 2q - 1. \end{cases} \quad (12)$$

Proof. Suppose first that we have a $(C_4, K_{2,n})$- coloring of K_p where $p = n + 2q$ and $1 \leq s \leq q - 1$. The two inequalities in (4) imply $\Delta_g = q + 1$ for $1 \leq s \leq q - 3$, $q + 1 \leq \Delta_g \leq q + 2$ for $q - 2 \leq s \leq q - 1, q \geq 3$, and $q + 1 \leq \Delta_g \leq q + 3$ for $q = 2$, i. e. $n = 2$. Inequality (5) yields that $d_g(v) \neq q + 1$ for all $v \in V$. Moreover, $\Delta_g = q + 3$ for $n = 2$ would immediately lead to a green C_4 or to a red $K_{2,2}$. Thus, only $\Delta_g = q + 2$ and $s = q - 2$ or $s = q - 1$ remains. Consider a vertex v with $d_g(v) = q + 2$. Let $N_g(v) = \{u_1, \dots, u_{q+2}\}$ and $g_i = g(u_i, N_r(v))$. Inequality (3) implies that $g_i \geq q - 2$. We may assume that $g_1 \leq \dots \leq g_{q+2}$. Then $g_{q+2} \geq q - 1$ in case of $s = q - 2$ would yield a contradiction to (1) and the same holds for $g_{q+1} \geq q-1$ or $g_{q+2} \geq q$ in case of $s = q-1$. Thus, $g_1 = \ldots = g_{q+2} = q-2$ if $s = q-2$ and $g_1 = \ldots = g_{q+1} = q-2$, $q-2 \leq g_{q+2} \leq q-1$ if $s = q-1$. Note that there must be vertices u_i and u_j in $N_g(v)$ with q common red neighbors in $N_g(v)$. But then (2) implies $g_i + g_j \geq 2q-2$, a contradiction, and the second case of inequality (12) is proved.

To prove the first case consider now $n \geq 10$ with $s = 1$. Suppose that we have a $(C_4, K_{2,n})$-coloring of K_p with $p = n + 2q - 1$ (i. e., $p = q^2 + 1$). Because of $q \geq 4$, inequality (4) implies that $q \leq \Delta_g \leq q + 1$.

First assume that $\Delta_g = q+1$. Let v be a vertex with $d_g(v) = q+1$, $N_g(v) = \{u_1, \ldots, u_{q+1}\}$ and $g_i = g(u_i, N_r(v))$. By (3), $g_i \geq q-2$. We may assume that $g_1 \leq \ldots \leq g_{q+1}$. Then (1) yields $g_1 = \ldots = g_q = q-2$ and $q-2 \leq g_{q+1} \leq q-1$. Moreover, there must be two vertices in $\{u_1, \ldots, u_q\}$ with $q-1$ common red neighbors in $N_g(v)$, and we obtain a contradiction to (2).

It remains that $\Delta_g = q$. Assume that $\delta_g \leq q-1$ and let w be a vertex with $d_g(w) = \delta_g$. If $\delta_g \geq 1$, inequality (3) yields $g(u, N_r(w)) \geq q$ for every $u \in N_g(w)$, contradicting $\Delta_g = q$. If $\delta_g = 0$, w and any other vertex have more than n common red neighbors and a red $K_{2,n}$ would occur. Thus, the green subgraph of K_p must be regular of degree q. Moreover, its girth g must be at least 5 since no green C_4 occurs and since a green C_3 would immediately lead to a red $K_{2,n}$. A girth $g \geq 6$ is impossible since then at least $p(q,g)$ vertices would occur in K_p (compare section 3) and $p(q,g) > q^2 + 1 = p$ if $g \geq 6$. Since $q^2 + 1 = p(q,5)$, it remains that the green subgraph is a $(q,5)$-Moore graph. But this yields a contradiction for $q \geq 4, q \neq 7, q \neq 57$ (i. e., $n \neq 37, n \neq 3137$) since such Moore graphs do not exist, and the first case in (12) is proved.

To prove the remaining third case, suppose that for $q \leq s \leq 2q-1$ we have a $(C_4, K_{2,n})$-coloring of K_p with $p = n + 2q + 1$. Then inequality (4) implies that $\Delta_g = q+1$ for $q \leq s \leq 2q-3$ and $q+1 \leq \Delta_g \leq q+2$ for $2q-2 \leq s \leq 2q-1$. By inequality (5), $d_g(v) \neq q+1$ for all $v \in V$. Moreover, $d_g(v) = q+2$ is only possible if $s = 2q-1$. Thus, only $s = 2q-1$ and $\Delta_g = q+2$ remains. Consider a vertex v with $d_g(v) = q+2$. Let $N_g(v) = \{u_1, \ldots, u_{q+2}\}$ and $g_i = g(u_i, N_r(v))$. By (3), $g_i \geq q-1$. Then (1) implies that $g_1 = \ldots = g_{q+2} = q-1$. But this yields a contradiction to (2), since there must be two vertices in $N_g(v)$ with q common red neighbors in $N_g(v)$, and the proof of Theorem 1 is complete. ■

Corollary. For $n = 3137$, equality in (12) is attained (i.e. $r(C_4, K_{2,n}) = 3251$) iff there is a $(57,5)$-Moore graph.

Proof. The proof of Theorem 1 shows that a $(57,5)$-Moore graph must exist if equality is attained. Furthermore, the existence of such a graph leads to equality, since a 2-coloring of a K_{3250} where the green subgraph is isomorphic to a $(57,5)$-Moore graph contains no green C_4 and no red $K_{2,n}$. ■

The next theorem shows that the bounds derived in Theorem 1 are attained in certain cases.

Theorem 2. Let $n \geq 2$, $q = \lceil \sqrt{n} \rceil$, $s = n - (q-1)^2$ and $M' = \{2, 5, 37\}$. If q is a prime power then

$$r(C_4, K_{2,n}) = \begin{cases} n + 2\lceil \sqrt{n} \rceil - 1; & s = 1 \quad \text{and} \quad n \notin M' \\ n + 2\lceil \sqrt{n} \rceil; & n \in M' \quad \text{or} \quad s = q - 1 \geq 2 \\ n + 2\lceil \sqrt{n} \rceil + 1; & s = q \end{cases} \tag{13}$$

and

$$n + 2\lceil \sqrt{n} \rceil - 1 \leq r(C_4, K_{2,n}) \leq n + 2\lceil \sqrt{n} \rceil; \quad 1 \leq s \leq q - 2. \tag{14}$$

If $q + 1$ is a prime power then

$$r(C_4, K_{2,n}) = n + 2\lceil \sqrt{n} \rceil + 1; \quad s = 2q - 1 \quad (\text{i. e.} \quad n = q^2). \tag{15}$$

Proof. First suppose that q is a prime power. In view of Theorem 1 it suffices to prove " $\geq$ " for (13) and the left inequality of (14).

Consider a 2-coloring of K_p with $p = q^2 + q + 1$ where the green subgraph is isomorphic to the Erdös-Rényi graph $ER(q)$. Then, by property (β) of $ER(q)$, no green C_4 occurs and, by Lemma 3, no red K_{2,q^2-q+1}. This implies " $\geq$ " for the third case of (13), i.e. for $s = q$, since then $n = q^2 - q + 1$ and $p = n + 2q$.

To settle the second case of (13), first consider $s = q - 1$, i. e. $n = q^2 - q$. Delete from K_p a vertex u with $d_g(u) = q$ (which exists by property (α) of $ER(q)$) and a green neighbor v of u. The remaining K_{n+2q-1} contains no green C_4. Assume that a red $K_{2,n}$ occurs. Then there are vertices x and y with n common red neighbors. By Lemma 3, x and y cannot have more than n common red neighbors in K_p. Thus, u and also v must be joined green to one of the vertices x and y. By property (δ) of $ER(q)$, we may assume that u is joined green to x and red to y and that v is joined green to y and red to x. Then the edge $\{x, y\}$ must be red since otherwise a green C_4 would occur. Moreover, by properties (α) and (γ), x has q green neighbours in K_{n+2q-1} and y at least $q - 1$. No green C_4 implies that in K_{n+2q-1} there are at least $2q - 2$ vertices joined green to x or y. But this yields at most $n - 1$ common red neighbors of x and y, a contradiction, and " $\geq$ " follows for $s = q - 1$. Note that $n = 2$ is included for $q = 2$. To establish " $\geq$ " for $n = 5$ and for $n = 37$, consider a 2-coloring of K_{q^2+1} where the green subgraph is isomorphic to the Petersen graph respectively to the Hoffman-Singleton graph, the two special $(\delta, 5)$- Moore graphs described in section 3.

To prove " $\geq$ " for the first case in (13) and the left inequality in (14) delete from K_p a vertex u with $d_g(u) = q$ and $q - s + 1$ of its green neighbors where $1 \leq s \leq q - 2$. The remaining K_{n+2q-2} contains no green C_4. Assume that a red $K_{2,n}$ occurs and consider two vertices x and y with n common red neighbors. Since x and y have at most $q^2 - q = n + q - s - 1$ common red neighbors in K_p, there are at least three among the deleted vertices joined green to x or to y. Thus, x or y must be joined green to two of the deleted vertices, contradicting one of the properties (β) and (δ) of $ER(q)$, and the proof of (13) and (14) is complete.

Now suppose that $q + 1$ is a prime power and $s = 2q - 1$, i. e. $n = q^2$. To prove (15), again it suffices to show " $\geq$ " in view of Theorem 1. Consider a 2-coloring of K_p with $p = (q+1)^2 + (q+1) + 1 = n + 3q + 3$ where the green subgraph is isomorphic to the Erdös-Rényi graph $ER(q+1)$. Delete a vertex u with $d_g(u) = q+2$ and all its green neighbors. By property (β) of $ER(q+1)$, K_p and also the remaining K_{n+2q} contains no green C_4. Assume that a red $K_{2,n}$ occurs in the remaining K_{n+2q} and let x and y be two vertices with n common red neighbors. By Lemma 3, x and y can have at most $q^2 + q = n + q$ common red neighbors in K_p. This implies that at least three of the deleted vertices, i. e., three of the green neighbors of u, must be joined green to x or to y. Thus, x or y has to be joined green to two green neighbors of u. But this yields a green C_4 in K_p, a contradiction, and Theorem 2 is proved. ■

The following lemma shows that equality in (12) also holds for $n = 8$ and that the lower bound in (14) yields the exact value in case of $n = 11$.

Lemma 4.

$$r(C_4, K_{2,8}) = 15, \quad r(C_4, K_{2,11}) = 18. \tag{16}$$

Proof. Figure 1 and inequality (12) imply that $r(C_4, K_{2,8}) = 15$. To prove $r(C_4, K_{2,11}) = 18$, it suffices to show ≤ 18 in view of the left inequality in (14).

Assume that we have a $(C_4, K_{2,11})$-coloring of K_{18}. By (4), $4 \leq \Delta_g \leq 6$. Let v be a vertex with $d_g(v) = \Delta_g$ and $N_g(v) = \{u_1, ..., u_{\Delta_g}\}$. We may assume that all green edges between vertices of $N_g(v)$ belong to the edge-set $\{\{u_i, u_{i+1}\}; 1 \leq i \leq \Delta_g - 1, i \text{ odd}\}$. This implies $|N_r(u_i) \cap N_r(u_{i+1}) \cap N_g(v)| = \Delta_g - 2$, and (2) yields $g(u_i, N_r(v)) + g(u_{i+1}, N_r(v)) \geq 5$. Thus, $\Delta_g = 6$ is impossible since otherwise we would obtain a contradiction to (1).

Now suppose that $\Delta_g = 5$. Then (1) implies that $d_g(u_5) \leq 3$, and $\delta_g \leq 3$ follows. Let w be any vertex with $d_g(w) = \delta_g$. If $\delta_g < 3$, w and one of its green neighbors (or any vertex if $\delta_g = 0$) would have at least eleven common red neighbors, a contradiction. Thus, $\delta_g = 3$. Let $N_g(w) = \{1, 2, 3\}$ and $N_r(w) = \{4, ..., 17\}$. $\Delta_g = 5$ and (3) imply $d_g(1) = d_g(2) = d_g(3) = 5$ and only red edges between the vertices 1, 2 and 3. We may assume that $N_g(i) = \{w, 4i, 4i+1, 4i+1, 4i+3\}$ for $1 \leq i \leq 3$. At most ten common red neighbors of 16 and 17 and no green C_4 imply $g(16, N_g(i)) = g(17, N_g(i)) = 1$ for $1 \leq i \leq 3$, and we can assume that $\{4, 8, 12\} \subset N_g(16)$ and $\{7, 11, 15\} \subset N_g(17)$. Consider $y \in \{5, 6, 9, 10, 13, 14\}$. A green C_4 would occur if $d_g(y) > 4$, and eleven common red neighbors of y and w if $d_g(y) < 4$. It remains that $d_g(y) = 4$. This implies red edges $\{5, 6\}$, $\{9, 10\}$ and $\{13, 14\}$, and, without loss of generality, green edges $\{2j, 2j+1\}$ for $2 \leq j \leq 7$. It can be shown that $d_g(z) = 5$ for every $z \in \{4, 7, 8, 11, 12, 15\}$. Since every vertex v with $d_g(v) = 5 = \Delta_g$ must have a green neighbor u with $d_g(u) = 3$, we obtain that $d_g(16) = d_g(17) = 3$, and, as for the vertex w, only red edges between the vertices in $N_g(x)$ for $x = 16$ and $x = 17$. The interdiction of a green C_4 yields red edges {4,13},{8,13}, and {15,6}. Then $d_g(4) = d_g(8) = 5$ implies that, without loss of generality, the edges $\{4, 14\}$ and $\{8, 15\}$ are green, which forces {15,4} to be red. But then all edges from 15 to 4, 6 and 7 are red and $d_g(15) = 5$ implies that $\{15, 5\}$ is green, yielding a green C_4.

The remaining case is $\Delta_g = 4$. Then $\delta_g < 4$ is impossible as otherwise again a vertex w with $d_g(w) = \delta_g$ and one of its green neighbors (or any other vertex if $\delta_g = 0$) would have at least eleven common red neighbors. We obtain that the green subgraph must be a graph of order 18 regular of degree 4. Moreover, no green triangle can occur, and it is not difficult to see that no such graph exists. Thus, the proof of Lemma 4 is complete. ■

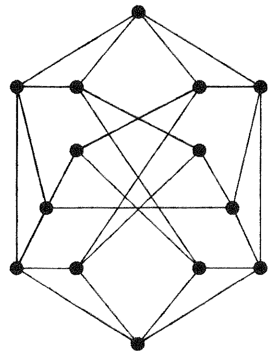

Fig.1. The green edges of a $(C_4, K_{2,8})$-coloring of K_{14}.

The following table summarizes the preceding results for $r(C_4, K_{2,n})$ up to $n = 21$.

n	2	3	4	5	6	7	8	9	10	11	12	13
$r(C_4, K_{2,n})$	6	8	9	11	12	14	15	16	17	18	20	22

n	14	15	16	17	18	19	20	21
$r(C_4, K_{2,n})$	22/23	22/24	25	26	27/28	28/29	30	32

Tab.1. Values and bounds for $r(C_4, K_{2,n})$ up to $n = 21$.

Theorem 3. Given n, let $q = \lceil \sqrt{n} \rceil$ and $s = n - (q-1)^2$.
(i) If $n \geq 3$ then

$$r(C_4, K_{3,n}) \leq \begin{cases} n + 3\lceil \sqrt{n} \rceil; & 1 \leq s \leq q-1, \\ n + 3\lceil \sqrt{n} \rceil + 2; & q \leq s \leq 2q-1. \end{cases} \tag{17}$$

Equality in (17) is attained for $s = 1$ if q is a prime power.
(ii) If $n \geq 4$ then

$$r(C_4, K_{4,n}) \leq n + 4\lceil \sqrt{n} \rceil + 3. \tag{18}$$

Furthermore, if $q + 1$ is a prime power and $s = q + 1$, then

$$r(C_4, K_{4,n}) \geq n + 4\lceil \sqrt{n} \rceil + 2. \tag{19}$$

Proof.(i) First suppose that we have a $(C_4, K_{3,n})$-coloring of K_{n+3q} in case of $1 \le s \le q-1$ which implies $q \ge 3$ because of $n \ge 3$. Then (4) yields $q+1 \le \Delta_g \le q+4$ for $q=3, s=2$ and $q+1 \le \Delta_g \le q+3$ otherwise. But $q+1 \le d_g(v) \le q+3$ is impossible by (5) and the remaining case is $q=3, s=2$ (i.e. $n=6$) and $\Delta_g = q+4 = 7$. Let v be a vertex with $d_g(v) = 7$. Then there must be three vertices u_1, u_2, u_3 in $N_g(v)$ with three common red neighbors u_4, u_5, u_6 in $N_g(v)$. By (1), $g(\{u_1, u_2, u_3\}, N_r(v)) \le 3$ or $g(\{u_4, u_5, u_6\}, N_r(v)) \le 3$. In both cases a red $K_{3,7}$ occurs, a contradiction, and the first case of (17) is proved.

From (4) and (5) it can be deduced that a $(C_4, K_{3,n})$-coloring of K_{n+3q+2} cannot exist for $s \le 2q-1$ and the second case of (17) follows.

Now let $s=1$, i.e. $n = q^2 - 2q + 2$, and let q be a prime power. Consider a 2-coloring of K_{q^2+q+1} where the green subgraph is isomorphic to $ER(q)$. Then no green C_4 occurs and, by (6), any three vertices have at most $q^2 - 2q + 1 = n-1$ common red neighbors. Thus, $r(C_4, K_{3,n}) \ge q^2 + q + 2 = n + 3q$ and equality in (17) follows.

(ii) Suppose now that we have a $(C_4, K_{4,n})$-coloring of K_{n+4q+3}. From (4) it can be deduced that $q+2 \le \Delta_g \le q+5$. By (5), $q+3 \le \Delta_g \le q+5$ is impossible and $\Delta_g = q+2$ is only possible if $s = 2q-1$, i.e., $n = q^2$. The remaining case is $\Delta_g = q+2$ and $n = q^2$. Let v be a vertex with $d_g(v) = q+2$, $N_g(v) = \{u_1, ..., u_{q+2}\}$ and $g_i = g(u_i, N_r(v))$. We may assume that $g_1 \le g_2 \le ... \le g_{q+2}$. If $g_3 \le q$, the vertices v, u_1, u_2 and u_3 have at least n common red neighbors in $N_r(v)$, a contradiction. Taking into account that $\Delta_g = q+2$, the remaining case is $g_3 = ... = g_{q+2} = q+1$. But then, if a green C_4 is avoided, the vertices v, u_1, u_2 and one common red neighbor of $u_3, ..., u_{q+2}$ in $N_r(v)$ have n common red neighbors among the green neighbors of the vertices $u_3, ..., u_{q+2}$ in $N_r(v)$. Thus, there is no $(C_4, K_{4,n})$-coloring of K_{n+4q+3} and inequality (18) is proved.

Now let $s = q+1$, i.e. $n = q^2 - q + 2$, and let $q+1$ be a prime power. Then a 2-coloring of K_{n+4q+1} where the green subgraph is isomorphic to $ER(q+1)$ is a $(C_4, K_{4,n})$-coloring by (6) and inequality (19) follows. ■

In addition to Theorem 3 we can show that equality in (19) holds for $n=4$. It seems to be difficult to decide whether equality holds for all n such that $q+1$ is a prime power and $s = q+1$. The next theorem shows that bounds similar to the preceding ones can be obtained for $r(C_4, K_{m,n})$ for all m if n is sufficiently large (depending on m).

Theorem 4. Let $2 \le m \le n$. Then

$$r(C_4, K_{m,n}) \le n + (m^2+3)/2 + m\sqrt{n + (m^2 + 2 + 1/m^2)/4 - 1/m}. \qquad (20)$$

Moreover, if $q = (m-1)/2 + \sqrt{n - (m^2 - 4m + 7)/4}$ is a prime power (i.e. $n = q^2 - (m-1)q + \binom{m-1}{2} + 1$) then

$$r(C_4, K_{m,n}) \ge n + m + m\sqrt{n - (m^2 - 4m + 7)/4}. \qquad (21)$$

Proof. Inequality (20) is an immediate consequence of (4). Now let q be a prime power. Note that $q^2 + q + 1 = n + m - 1 + m\sqrt{n - (m^2 - 4m + 7)/4}$. Consider a 2-coloring of K_{q^2+q+1} with the green subgraph isomorphic to $ER(q)$. Then no green C_4 occurs and, by Lemma 3, no red $K_{m,n}$. This yields inequality (21) and the proof of Theorem 4 is complete. ■

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