HAMILTON DECOMPOSITIONS OF LINE GRAPHS OF PERFECTLY 1-FACTORISABLE GRAPHS OF EVEN DEGREE

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Abstract

The proof of the following theorem is the main result of this paper: If G is a $2k$-regular graph that has a perfect 1-factorisation, then the line graph, $L(G)$, of G is Hamilton decomposable. Consideration is given to Hamilton decompositions of $L(K_{2k} - F)$.

1 Introduction

All graphs considered in this paper are finite and have no loops or multiple edges. By $V(G)$ and $E(G)$ we denote the vertex set and edge set, respectively, of the graph G. By K_n we denote the complete graph on n vertices.

A *cycle* is a 2-regular connected graph. A *Hamilton cycle* in a graph G is a 2-regular connected spanning subgraph of G.

A *1-factorisation* of a graph G is a partition of $E(G)$ into 1-factors. A *perfect 1-factorisation* of G is a 1-factorisation of G in which the union of any pair of 1-factors is a Hamilton cycle of G. A graph is said to be *perfectly 1-factorisable* if it has at least one perfect 1-factorisation.

The *line graph*, denoted by $L(G)$, of a graph G is the graph with vertex set $E(G)$, where two vertices of $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges in G are incident with a common vertex in G.

A *Hamilton decomposition* of a regular graph G consists of a set of Hamilton cycles (plus a 1-factor if $\Delta(G)$ is odd) of G such that these cycles (and the 1-factor when $\Delta(G)$ is odd) partition the edges of G. If G has a Hamilton decomposition, it is said to be *Hamilton decomposable*.

Definitions omitted in this paper can be found in [5].

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While decompositions of line graphs into 1-factors have been well studied [1, 10, 11], Hamilton decompositions remain an area of continuing research, much of which is motivated by a conjecture made by Bermond [4]:

Conjecture 1 *If G is Hamilton decomposable, then $L(G)$ is Hamilton decomposable.*

Bermond's conjecture has been shown to hold when G is a Hamilton decomposable graph satisfying any of the following criteria [8, 11, 13, 15, 16, 17]:

1. $\Delta(G) \leq 5$,
2. $\Delta(G) \equiv 0 \pmod 4$,
3. $\Delta(G)$ is odd and G is bipartite, or
4. $G = K_{2k+1}$ for $k \geq 0$.

2 Main Result

We prove the following theorem, which serves to further support Bermond's conjecture:

Theorem 1 *If G is a $2k$-regular graph that has a perfect 1-factorisation, then $L(G)$ is Hamilton decomposable.*

Proof. G is $2k$-regular, so $L(G)$ is $(4k-2)$-regular. To show that $L(G)$ is Hamilton decomposable, $(2k-1)$ edge-disjoint Hamilton cycles of $L(G)$ will be constructed. We accomplish this task by noting that each Euler tour in G corresponds to a Hamilton cycle in $L(G)$, and so we need only find $(2k-1)$ Euler tours in G such that each pair of incident edges in G occurs consecutively in exactly one of these Euler tours (ie. such that the Euler tours partition the 2-paths of G).

We begin by fixing a proper edge-colouring of G such that the edges of each colour class correspond to the edges of a 1-factor in a perfect 1-factorisation of G. We use the colours $0, \ldots, (2k-2)$ and ∞. Additionally we select some vertex v of G at which we will begin and end each Euler tour.

Each of the Euler tours that we construct will be obtained by starting at v and then travelling along the k Hamilton cycles of a Hamilton decomposition of G. Each of these Hamilton cycles will be obtained from the union of two of the 1-factors in the perfect 1-factorisation of G. The set of 1-factor pairs thus used for each Euler tour will correspond to a 1-factor in K_{2k} where $V(K_{2k}) = \{0, \ldots, 2k-2\} \cup \{\infty\}$.

Consider now the following 1-factor, F, in K_{2k}:

$$\{\infty, 0\}, \{2k-2, 1\}, \{2k-3, 2\}, \{2k-4, 3\}, \ldots, \{k+1, k-2\}, \{k, k-1\}$$

We treat each pair of colours as an ordered pair, with the first coordinate being the colour of the edge that we use when departing v, and the second coordinate being the colour of the edge used when returning to v.

Let σ denote the permutation $(0, \ldots, 2k-2)(\infty)$. Then the 1-factors F, $\sigma(F)$, $\ldots$, $\sigma^{2k-2}(F)$ partition the 2-sets of $V(K_{2k})$, and so each pair of edges in G that meet at a vertex other than v will be used consecutively in exactly one of the $(2k-1)$ resultant Euler tours.

Edge pairs that meet at v are described by the 1-factors F', $\sigma(F')$, $\ldots$, $\sigma^{2k-2}(F')$ where F' denotes the 1-factor:

$$\{0, 2k-2\}, \{1, 2k-3\}, \{2, 2k-4\}, \{3, 2k-5\}, \ldots, \{k-2, k\}, \{k-1, \infty\}$$

Again, the 2-sets of $V(K_{2k})$ are partitioned.

Hence each 2-path in G will occur in exactly one of the $(2k-1)$ Euler tours. The $(2k-1)$ Euler tours thus correspond to $(2k-1)$ edge-disjoint Hamilton cycles in $L(G)$. □

3 Discussion

Kotzig [14] has posed the following conjecture:

Conjecture 2 *K_{2k} has a perfect 1-factorisation for all $k \geq 2$.*

Kotzig's conjecture has been shown to hold when k is prime, or when $(2k-1)$ is prime, or when $2k$ is one of 16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, or 6860. (See references [2, 6, 9, 12, 14, 18, 19].)

Corollary 1 *$L(K_{2k} - F)$ is Hamilton decomposable, where F is a 1-factor of K_{2k}, provided that any of the following conditions are satisfied:*

1. *k is prime,*
2. *$(2k-1)$ is prime, or*
3. *$2k$ is one of 16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, or 6860.*

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