# Signings of Group Divisible Designs and Projective Planes

P.B. Gibbons
*Department of Computer Science*
*School of Mathematical and Information Sciences*
*University of Auckland*
*Private Bag 92019, Auckland*
*New Zealand*

R. Mathon
*Department of Computer Science*
*University of Toronto*
*Toronto, Ontario*
*Canada M5S 1A4*

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**Abstract**

We investigate signings of the all-ones matrix $J_v$ for $2 \leq v \leq 16$ over various admissible groups. For $v < 16$ all signings have been enumerated. For the case $v = 16$ only some classes have been completely enumerated. Of particular significance are signings over the group $Z_2$, since these can also be considered as $GDD(16 \times 2, 16, 8)$'s. Signing these over $Z_2$, and expanding, produces $GDD(16 \times 4, 16, 4)$'s. We continue in this way until we obtain $GDD(16 \times 16, 16, 1)$'s, which can be extended to projective planes of order 16.

We discuss the properties of classes of $GDD$'s obtained during this search, as well as identifying the projective planes obtained at the final stage.

## 1. Definitions and Background

A *balanced incomplete block design*, $BIBD(v, b, r, k, \lambda)$, is an arrangement of $v$ *elements* into $b$ *blocks* such that (i) each element appears in exactly $r$ blocks; (ii) each block contains exactly $k$ elements; and (iii) each pair of distinct elements appear together in exactly $\lambda$ blocks. Well known necessary conditions for a $BIBD(v, b, r, k, \lambda)$ to exist are $vr = bk$ and $\lambda(v-1) = r(k-1)$. Because of this dependence we shall use the abbreviated notation $BIBD(v, k, \lambda)$ to denote a $BIBD(v, b, r, k, \lambda)$. A $BIBD$ is nontrivial if $3 \leq k < v$. A *symmetric BIBD* ($SBIBD$) is a $BIBD$ with $v = b$. A *projective plane* of order $v$ is an $SBIBD(v^2 + v + 1, v + 1, 1)$.

Two $BIBD(v, k, \lambda)$'s, with element sets $V_1$ and $V_2$ respectively, are said to be *isomorphic* if there is a bijection $\theta : V_1 \to V_2$ which preserves blocks. An *automorphism* of a $BIBD$ is an isomorphism of the $BIBD$ with itself. The set of all automorphisms, under the usual composition of mappings, forms the *automorphism group* of the $BIBD$.

A *generalized Bhaskar Rao design* (*GBRD*) is defined as follows. Let $W$ be a $v \times b$ matrix with elements from $G \cup \{0\}$, where $G = \{h_1 = e, h_2, ..., h_g\}$ is a finite group of order $g$, with identity element $e$. Then $W$ can be expressed as a sum $W = h_1A_1 + h_2A_2 + ... + h_gA_g$, where $A_1, ..., A_g$ are $v \times b$ (0,1)-matrices such that the Hadamard product $A_i \times A_j = 0$ for any $i \neq j$. (That is, for $i \neq j$, no 1 of $A_i$ occurs in the same position as a 1 of $A_j$). Denote by $W^+$ the transpose of $h_1^{-1}A_1 + ... + h_g^{-1}A_g$ and let $N = A_1 + ... + A_g$. Then $W$ is a *generalized Bhaskar Rao design* denoted by $GBRD(v, b, r, k, \lambda; G)$ if

(i) $WW^+ = reI + \lambda/g(h_1 + ... + h_g)(J - I)$, and

(ii) $NN^T = (r - \lambda)I + \lambda J$.

The second condition merely prescribes that $N$ be the incidence matrix of a $BIBD(v, b, r, k, \lambda)$. Because of the parameter dependencies for $BIBD$'s mentioned above we shall use the shorter notation $GBRD(v, k, \lambda; G)$ for a generalized Bhaskar Rao design.

A $GBRD(v, k, \lambda; G)$ with $v = b$ is a *symmetric GBRD* or *generalized weighing matrix.* If $W$ has no 0 entries then the *GBRD* is also known as a *generalized Hadamard matrix* or *GHM*. A *GHM* over the group $Z_2$ is a *Hadamard matrix* (which provided the original motivation for studying *GHM*'s). Producing a $GBRD(v, k, \lambda; G)$ from a $BIBD(v, k, \lambda)$ is generally referred to as *signing* the *BIBD* over the group $G$.

A *group-divisible design*, $GDD(v \times g, k, \lambda)$ is an incidence structure $(X, B)$ consisting of a set $X$, $|X| = vg$, partitioned into $v$ disjoint $g$-subsets (*groups*), $X = X_1 \cup ... \cup X_v$, and a collection $B$ of $k$-subsets of $X$ (*blocks*) such that:

(i) Each point $x \in X$ is incident with $r$ blocks.

(ii) $| L \cap X_i | \leq 1$ for every block $L \in B$ and $i = 1, ..., v$.

(iii) If $x \in X_i, y \in X_j, i \neq j$, there are exactly $\lambda$ blocks incident with $x$ and $y$.

If $|B| = bg$ then $bk = rv$ and $\lambda g(v - 1) = r(k - 1)$. Note that the groups in the above definition correspond to subsets rather than algebraic groups.

An incidence structure $(X, B)$ (which can be a *BIBD* or a *GDD*) is said to be *resolvable* if there exists a partition $R$ of the set of blocks $B$ into subsets $R_1, ..., R_u$, called *parallel classes*, such that each $R_i$ is a partition of $X$. It is well known that a the dual of a symmetric *GDD* (where $v = b$ and $r = k$) is again a *GDD*, which means that the blocks of the original *GDD* partition into $v$ sets of $g$ disjoint blocks each (corresponding to groups in the dual) constituting a resolution. Hence a symmetric *GDD* and its dual are both resolvable. In this paper the property of resolvability is used in the final step of obtaining a projective plane of order 16 from a $GDD(16 \times 16, 16, 1)$.

A $GDD(v \times g, k, \lambda)$ with $g = 1$ is a $BIBD(v, k, \lambda)$. *GDD* isomorphism is defined in the same way as *BIBD* isomorphism.

From a $GBRD(v, k, \lambda; G)$, $|G| = g$, we can form a $GDD(v \times g, k, \lambda/g)$ as follows. For any $h \in G$ let $P_h$ denote the corresponding $g \times g$ permutation matrix, $P_{h_1} + ... + P_{h_g} =$

$J$. If $W$ is the $v \times g$ matrix of a *GBRD* let $N$ be the $vg \times bg$ (0,1)-matrix obtained from $W$ by replacing any group element $h$ by $P_h$ and any 0 entry by a $g \times g$ all-zero matrix. Then $N$ is the incidence matrix of a *GDD*$(v \times g, k, \lambda/g)$.

Two *GBRD*$(v, k, \lambda; G)$'s $W$ and $W'$ are *isomorphic* if there exist two $G$-permutation matrices $P$ and $Q$, and an automorphism $\sigma$ of the group $G$ such that $W' = P\sigma(W)Q$. The isomorphism itself will be denoted by the triple $(P, \sigma, Q)$. An isomorphism of $W$ with itself is called an automorphism of $W$. The set of automorphisms of a *GBRD* form a group under the operations of matrix multiplication and mapping composition, i.e. if $(P_1, \sigma_1, Q_1)$ and $(P_2, \sigma_2, Q_2)$ are two automorphisms, then so is $(P_1P_2, \sigma_1\sigma_2, Q_1Q_2)$.

Bhaskar Rao designs have been studied by a number of authors. They are important because of their relationship to other structures, such as *BIBD*'s, codes, *GDD*'s, multi-dimensional Howell cubes, generalized Room squares, equidistant permutation arrays, and doubly resolvable two-fold triple systems. For example, Bhaskar Rao [1], Street and Rodger[29], and Seberry[26] have examined such designs in connection with construction of partially balanced block designs. Gibbons and Mathon [15] have described mathematical and computational techniques for enumerating *GBRD*'s and provided examples of their connection with many of the structures mentioned above. Generalized Hadamard designs have been studied by Butson [3] [4] and by Shrikhande [28] in connection with combinatorial designs, by Delsarte and Goethals [8] in connection with codes, and by Drake [11] in connection with $\lambda$-geometries. Generalized weighing matrices were first introduced by Yates [32] in determining the accuracy of measurements. Since then they have been studied extensively [12] [13] [20] [30] [31].

In this paper we consider the properties of generalized Hadamard matrices of order $v$ in the range $2 \leq v \leq 16$. Such *GHM*'s have been completely enumerated for $v < 16$. The properties of these signings are described in Section 3. The case $v = 16$ is considered in Section 4, where we construct signings of $J_{16}$ over various groups. Of particular significance are signings over the group $Z_2$, since these can also be considered as $GDD(16 \times 2, 16, 8)$'s. Signing these over $Z_2$, and expanding, produces $GDD(16 \times 4, 16, 4)$'s. We continue in this way until we obtain $GDD(16 \times 16, 16, 1)$'s, which can be extended to projective planes of order 16. The properties of such projective planes are discussed.

## 2. Methods and algorithms

In this section we describe the computational method used to enumerate signings of a $GDD(v \times g, k, \lambda)$ over a group $G$. This is an adaptation of the algorithm used by the authors in [15] to enumerate Bhaskar Rao designs. We shall assume that $N$ is the incidence matrix of the given $GDD(v \times g, k, \lambda)$, and $W$ is the matrix of the signed *GDD* which we are trying to construct.

The major simplification from the algorithm described in [15] is achieved by noting that the input *GDD*'s we are dealing with do not have repeated blocks. This means that all $N$-cells (and hence $W$-cells) described in [15] are of size 1, and can therefore be eliminated in a revamped, shorter and faster program. The description of this simplified 2-level backtrack algorithm is as follows.

We begin by defining an ordering $h_1 < h_2 < ... < h_g$ on the elements of $G$, and then proceed to sign the matrix $N$ row by row, replacing the "1" entries of $N$ by elements from the group $G$ subject to the constraint $\sum_{l=1}^{b} w_{il}(w_{jl})^{-1} = \lambda/g(h_1 + ... + h_g)$ (for $i \neq j$). Individual rows are considered in strictly increasing lexicographical order, with the result that any completed matrices will be output in increasing order.

This simple algorithm would be impractical for most problems unless some form of isomorph rejection procedure was implemented during the search. To do this we make use of $A = Aut(G)$, the automorphism group of the signing group $G$, and $H = Aut(N)$, the automorphism group of the incidence matrix $N$ of the given *GDD*. $H$ is precalculated using a group-constructing backtrack algorithm discussed in [14, 17]. In an attempt to produce the smallest isomorphic copy of the current partial configuration we introduce a *minimisation* operation $m$. Suppose $W^t$ represents the first $t$ rows of $W$. For any row (column) of $W^t$, define the row (column) header as the first non-zero entry of $W^t$ in that row (column). Then $m(W^t)$ is formed from $W^t$ in the following way. Scan $W^t$ row by row, from left to right. For each row (column) header $w_{ij} = x$, reduce $w_{ij}$ to $e$ by pre- (post-) multiplying row $i$ (column $j$) by $x^{-1}$.

Now suppose our algorithm has constructed $W^t$. Then this partial configuration may be rejected, thereby forcing a backtrack, if there is an isomorphic partial configration which is lexicographically smaller than $W^t$. This will be true if there exists a $\sigma \in A, \phi \in H$ such that $m(\phi(\sigma(W^t))) < W^t$.

In the previous expression, note that $\phi$ is restricted to act on the first $t$ rows of $W$. Not all elements of $H$ will map $\{1, 2, ..., t\} \rightarrow \{1, 2, ..., t\}$, or indeed to an image containing $\{1, 2, ..., t'\}$ for some $t' < t$. To be more precise, in checking $H$ for applicable mappings we can only use mappings $\phi$ with the property that $\phi(\{1, 2, ..., t\})$ contains a set $\{1, 2, ..., t'\}$ for some $t' \leq t$, that is, if there exists a $t' \leq t$ such that $\phi^{-1}(i) \leq t \; \forall i \in \{1, 2, ..., t'\}$. Then the first $t'$ rows of the image of $W^t$ under $\phi$ can be checked against the first $t'$ rows of $W^t$. This criteria is used after the construction of each row of $W$ to select automorphisms of $N$ that can be used in the isomorphism check.

Often the group $H$ is very large and cannot be completely stored. In this case we need to store a subgroup which is effective for isomorph checking. To determine such a subgroup we carry out a preliminary analysis of $H$. Suppose we are carrying out isomorph checking of partial configurations up to row $h$ of $W$. Then, we compute a table $T[0..h][0..h]$ where $T_{ij}$ $(i > j)$ is undefined, and $T_{ij}$ $(i \leq j)$ is the cardinality of the set $S_{ij}$ of elements $\phi$ of $H$ which have the property that $\phi(\{1, 2, ..., j\})$ contains a set $\{1, 2, ..., i\}$. Necessarily an element in $S_{ij}$ will also be in $S_{il}$ for any $l > j$, and an element in $S_{ij}$ will be in $S_{lj}$ for any $l < i$. So the elements we should store for use in our isomorph checking will be the elements in the set

$$\bigcup_{1 \leq i,j \leq h} S_{ij} = \bigcup_{1 \leq j \leq h} S_{1j} = S_{1h}$$

For example, when generating the signings of the expanded HM16.3 (the third signing of $J_{16}$ over $Z_2$), which has a group $H$ of order 49,152, we generated a table $T$, part of which is reproduced in Table 1.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1536 | 3072 | 4608 | 6144 | 7680 | 9216 | 10752 | 12288 |
| 2 | 0 | 1536 | 1536 | 3072 | 3072 | 4608 | 4608 | 6144 |
| 3 | 0 | 0 | 128 | 512 | 768 | 1536 | 1920 | 3072 |
| 4 | 0 | 0 | 0 | 512 | 512 | 1536 | 1536 | 3072 |
| 5 | 0 | 0 | 0 | 0 | 64 | 384 | 576 | 1536 |
| 6 | 0 | 0 | 0 | 0 | 0 | 384 | 384 | 1536 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 192 | 1536 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1536 |

**Table 1: Group analysis table**

If we are intending to perform isomorph rejection up to row 6, then we should include the permutation set $S_{1,6}$ comprising a total of 9216 permutations. In practice, however, we omit from $S_{1,6}$ any mapping $\phi$ such that $\phi(\{1, 2, ..., 6\})$ does not contain the set $\{1, 2\}$. Since the non-zero entries of the first row of the constructed matrix $W$ will be the group identity $e$, this row is in minimal form so that we can never get an isomorph rejection on row 1. A further refinement would be to check, at row $j$, only those mappings $\phi$ such that $\phi(\{1, 2, ..., j\})$ does contains the set $\{1, 2\}$. However, we did not implement this.

Another variation on the standard backtrack method was the use of a "lookahead" heuristic. After the completion of each row $t$ we check that each row $t' > t$ can also be completed separately with respect to the first $t$ rows already constructed. Obviously if this not the case , then we need to backtrack on row $t$.

The above heuristics proved to be essential in the extensive searches covered by this paper. For example, the generation of all signings over $Z_2$ of GDD32.3, the group divisible design resulting from the expansion of HM16.3, took a total of two weeks computation time on a Sun Sparcstation 2. During the search there were more than 30,000 isomorph rejections, and more than one million lookahead rejections. Clearly the search would have been infeasible without these checks.

In the following sections we display the results of these searches. The generating programs were written in the programming language C and run on a variety of UNIX platforms. To determine the isomorphism classes and properties of designs generated we used Brendan McKay's **nauty** software [23]. The designs are considered as point-block bipartite graphs. **nauty** can then be used to find the point and block orbits, as well as the generators of the automorphism group. To calculate the $p$-ranks of the incidence matrices we have written a short program based on sparse Gaussian elimination in modular arithmetic. This requires $O(n^3)$ operations, where $n$ is the order of the matrix. General programs for calculating $p$-ranks are available in most symbolic algebra systems.

## 3. Generalized Hadamard matrices of order $v < 16$

In this section we consider possible signings of the all-ones matrix $J_v$ over various groups to produce *GHM*'s of orders $2 \leq v < 16$. De Launey [7] has investigated the existence of signings over the elementary abelian groups $Z_q$ where $q$ is a prime power. Here we enumerate signings over all feasible groups. The results are summarized in Table 2.

| $v$ | $Exists$ | $\neg Exist$ |
|---|---|---|
| 2 | $Z_2(1)$ | |
| 3 | $Z_3(1)$ | |
| 4 | $Z_2(1), Z_2 \times Z_2(1)$ | $Z_4$ |
| 5 | $Z_5(1)$ | |
| 6 | $Z_3(1)$ | $Z_2, Z_6, S_3$ |
| 7 | $Z_7(1)$ | |
| 8 | $Z_2(1), Z_2 \times Z_2(1), Z_2^3(1)$ | $Z_4, Z_8, Z_2 \times Z_4, D_4, Q$ |
| 9 | $Z_3(2), Z_3 \times Z_3(2)$ | $Z_9$ |
| 10 | $Z_5(1)$ | $Z_2, Z_{10}, D_5$ |
| 11 | $Z_{11}(1)$ | |
| 12 | $Z_2(1), Z_3(1), Z_2 \times Z_2(1)$ | $Z_4, Z_6, S_3, G_{12.1-5}$ |
| 13 | $Z_{13}(1)$ | |
| 14 | $Z_7(1)$ | $Z_2, Z_{14}, D_7$ |
| 15 | | $Z_3, Z_5, Z_{15}$ |

**Table 2: Generalized Hadamard matrices of order $v < 16$**

Table 2 indicates whether *GHM*'s exist for various admissible signing groups. In the "Exists" column a bracketed number after a group name indicates the number of non-isomorphic signings over that group. The notation $G_{12.1-5}$ stands for the 5 non-isomorphic groups of order 12. As can be seen from the table, there is no *GHM* of order $v = 15$. For all other orders, except $v = 9$, the *GHM*'s are unique for the stated signing groups. For $v = 9$ there are 2 signings over $Z_3$, and 2 signings over $Z_3 \times Z_3$. The properties of these signings are listed in Table 3. The designs themselves are listed in Appendix A1.

| $No$ | $G$ | $Rk3$ | $\mid G \mid$ | $APpt$ | $APbl$ | $Sd$ | $Ex$ | $Com$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $Z_3$ | 10 | 23328 | $1 * 27$ | $1 * 27$ | $YES$ | $YES$ | |
| 2 | $Z_3$ | 10 | 2916 | $1 * 27$ | $1 * 27$ | $YES$ | $NO$ | |
| 3 | $Z_3 \times Z_3$ | 36 | 93312 | $1 * 81$ | $1 * 81$ | $YES$ | | $Des$ |
| 4 | $Z_3 \times Z_3$ | 40 | 15552 | $1 * 81$ | $1 * 9, 1 * 72$ | $NO$ | | $Hall$ |

**Table 3: Generalized Hadamard matrices of order 9**

In the above Table 3, the column headings are defined as follows:

*Rk*3: 3-rank of the point-block incidence matrix (rank over the field GF(3))
$\mid G \mid$: order of the automorphism group
*APpt*: cell sizes of automorphism partitioning of points ($n$*$m$ means $n$ cells of size $m$)
*APbl*: cell sizes of automorphism partitioning of blocks
*Sd*: self-dual (YES or NO)
*Ex*: extendable (YES or NO)
*Com*: Comments. *Des*, *Hall* stand for the Desargesian and Hall planes respectively

## 4. Generalized Hadamard matrices of order 16

### 4.1. Repeated signings over $Z_2$

When the all-ones matrix $J_{16}$ is signed over the group $Z_2$ it produces 5 non-isomorphic *HM*'s of order 16. These are listed on pages 419-421 in [31]. Their characteristics are summarised in Table 4.

| $GHM$ | $Dual$ | $GDD(16 \times 2, 16, 8)$ | $\mid G \mid$ |
|---|---|---|---|
| $HM16.1$ | $HM16.1$ | $GDD32.1$ | $10,231,920$ |
| $HM16.2$ | $HM16.2$ | $GDD32.2$ | $294,912$ |
| $HM16.3$ | $HM16.3$ | $GDD32.3$ | $49,152$ |
| $HM16.4$ | $HM16.5$ | $GDD32.4$ | $86,016$ |
| $HM16.5$ | $HM16.4$ | $GDD32.5$ | $86,016$ |

**Table 4: Generalized Hadamard matrices of order 16**

In Table 4 above, $|G|$ is the order of the automorphism group of the expanded $GDD(16 \times 2, 16, 8)$.

We now consider signings of the 5 $GDD(16 \times 2, 16, 8)$'s over $Z_2$. The results are summarised in Table 5:

| $GDD(16 \times 2, 16, 8)$ | $GDD32.1$ | $GDD32.2$ | $GDD32.3$ | $GDD32.4$ | $GDD32.5$ |
|---|---|---|---|---|---|
| $Signings$ | $\geq 514$ | $\geq 300$ | 50 | 0 | 0 |

**Table 5: Signings of $GDD(16 \times 2, 16, 8)'s$**

Properties of the 50 $GDD(16\times4, 16, 4)$'s resulting from the signings and expansions of $GDD32.3$ are summarized in Table 6:

| *No* | *Rk*2 | \| *G* \| | *APpt* | *APbl* | *Sd* | *Ex* | *Rd* |
|---|---|---|---|---|---|---|---|
| 1 | 24 | 4 | 16 * 4 | 16 * 2, 8 * 4 | *NO* | *NO* | *NO* |
| 1 | 24 | 4 | 16 * 4 | 16 * 2, 8 * 4 | *NO* | *NO* | *NO* |
| 2 | 25 | 8 | 8 * 8 | 8 * 4, 4 * 8 | *NO* | *NO* | *NO* |
| 3 | 25 | 16 | 4 * 16 | 2 * 4, 3 * 8, 2 * 16 | *NO* | *NO* | *NO* |
| 4 | 24 | 4 | 16 * 4 | 16 * 2, 8 * 4 | *NO* | *NO* | *NO* |
| 5 | 25 | 16 | 4 * 16 | 2 * 4, 3 * 8, 2 * 16 | *NO* | *NO* | *NO* |
| 6 | 24 | 4 | 16 * 4 | 16 * 2, 8 * 4 | *NO* | *NO* | *NO* |
| 7 | 25 | 8 | 8 * 8 | 8 * 4, 4 * 8 | *NO* | *NO* | *NO* |
| 8 | 25 | 8 | 8 * 8 | 4 * 2, 6 * 4, 4 * 8 | *NO* | *NO* | *NO* |
| 9 | 24 | 4 | 16 * 4 | 16 * 2, 8 * 4 | *NO* | *NO* | *NO* |
| 10 | 23 | 4 | 16 * 4 | 16 * 2, 8 * 4 | *NO* | *NO* | *NO* |
| 11 | 23 | 4 | 16 * 4 | 16 * 2, 8 * 4 | *NO* | *NO* | *NO* |
| 12 | 22 | 4 | 16 * 4 | 16 * 4 | *YES* | *NO* | *NO* |
| 13 | 23 | 4 | 16 * 4 | 16 * 4 | *NO* | *NO* | *NO* |
| 14 | 23 | 4 | 16 * 4 | 16 * 4 | *NO* | *NO* | *NO* |
| 15 | 24 | 8 | 8 * 8 | 8 * 8 | *YES* | *NO* | *NO* |
| 16 | 24 | 8 | 8 * 8 | 8 * 8 | *NO* | *NO* | *NO* |
| 17 | 22 | 4 | 16 * 4 | 16 * 4 | *YES* | *NO* | *NO* |
| 18 | 24 | 16 | 4 * 16 | 4 * 16 | *YES* | *NO* | *NO* |
| 19 | 24 | 16 | 4 * 16 | 4 * 16 | *YES* | *NO* | *NO* |
| 20 | 22 | 32 | 2 * 8, 1 * 16, 1 * 32 | 2 * 8, 1 * 16, 1 * 32 | *NO* | *NO* | *NO* |
| 21 | 23 | 16 | 4 * 16 | 4 * 4, 4 * 8, 1 * 16 | *NO* | *NO* | *NO* |
| 22 | 22 | 16 | 4 * 8, 2 * 16 | 4 * 8, 2 * 16 | *NO* | *NO* | *NO* |
| 23 | 23 | 16 | 4 * 16 | 4 * 4, 4 * 8, 1 * 16 | *NO* | *NO* | *NO* |
| 24 | 22 | 32 | 4 * 16 | 1 * 8, 2 * 16, 1 * 32 | *NO* | *NO* | *NO* |
| 25 | 23 | 8 | 8 * 8 | 4 * 2, 6 * 4, 4 * 8 | *NO* | *NO* | *NO* |
| 26 | 22 | 64 | 1 * 64 | 1 * 64 | *NO* | *NO* | *YES* |
| 27 | 22 | 64 | 1 * 64 | 2 * 4, 1 * 8, 1 * 16, 1 * 32 | *NO* | *NO* | *YES* |
| 28 | 23 | 32 | 2 * 16, 1 * 32 | 2 * 4, 3 * 8, 1 * 32 | *NO* | *NO* | *NO* |
| 29 | 23 | 32 | 2 * 16, 1 * 32 | 2 * 4, 3 * 8, 1 * 32 | *NO* | *NO* | *NO* |
| 30 | 22 | 16 | 4 * 8, 2 * 16 | 4 * 4, 2 * 8, 2 * 16 | *NO* | *NO* | *NO* |
| 31 | 22 | 32 | 2 * 32 | 2 * 8, 3 * 16 | *NO* | *NO* | *YES* |
| 32 | 24 | 16 | 4 * 8, 2 * 16 | 4 * 8, 2 * 16 | *YES* | *NO* | *NO* |
| 33 | 22 | 64 | 1 * 64 | 2 * 4, 1 * 8, 1 * 16, 1 * 32 | *NO* | *NO* | *NO* |
| 34 | 22 | 64 | 1 * 64 | 2 * 4, 1 * 8, 1 * 16, 1 * 32 | *NO* | *NO* | *NO* |
| 35 | 20 | 32 | 4 * 16 | 8 * 8 | *NO* | *NO* | *YES* |
| 36 | 22 | 64 | 1 * 64 | 2 * 32 | *NO* | *NO* | *YES* |
| 37 | 22 | 64 | 1 * 64 | 2 * 32 | *NO* | *NO* | *YES* |
| 38 | 22 | 64 | 1 * 64 | 4 * 16 | *NO* | *NO* | *YES* |
| 39 | 22 | 128 | 1 * 64 | 4 * 16 | *NO* | *NO* | *YES* |
| 40 | 24 | 32 | 2 * 32 | 2 * 4, 1 * 8, 1 * 16, 1 * 32 | *NO* | *NO* | *NO* |
| 41 | 22 | 64 | 1 * 64 | 2 * 8, 1 * 16, 1 * 32 | *NO* | *NO* | *YES* |
| 42 | 22 | 32 | 2 * 32 | 4 * 16 | *NO* | *NO* | *YES* |
| 43 | 20 | 64 | 2 * 32 | 4 * 8, 2 * 16 | *NO* | *NO* | *YES* |
| 44 | 20 | 512 | 1 * 64 | 2 * 32 | *NO* | *NO* | *YES* |
| 45 | 20 | 512 | 1 * 64 | 1 * 64 | *YES* | *NO* | *YES* |
| 46 | 20 | 1024 | 1 * 64 | 1 * 64 | *YES* | *NO* | *YES* |
| 47 | 20 | 32 | 4 * 16 | 8 * 8 | *NO* | *NO* | *YES* |
| 48 | 21 | 64 | 2 * 32 | 2 * 32 | *YES* | *NO* | *YES* |
| 49 | 23 | 8 | 8 * 8 | 8 * 8 | *NO* | *NO* | *NO* |
| 50 | 25 | 128 | 1 * 64 | 2 * 32 | *NO* | *NO* | *NO* |

**Table 6: Signings of HM16.3 over $Z_2$**

In the preceding Table 6, new column headings used are defined as follows:

*Rk*2: 2-rank of the point-block incidence matrix (rank over field GF(2))

*Rd*: reducible (YES/NO), i.e. can GDD be obtained by signing $J_{16}$ over $Z_2^2$?

As can be seen none of the 50 *GDD*$(16 \times 4, 16, 4)$'s can be signed over $Z_2$. In order to continue the process we therefore turn our attention to the signings of *GDD*32.1, corresponding to the *HM* of order 16 with the largest automorphism group. This class contains at least 514 *GDD*$(16 \times 4, 16, 4)$'s and has not yet been enumerated completely.

Many of these designs are not able to be signed over $Z_2$. However, the 130th non-isomorphic design in this enumeration, which we shall denote by *GDD*64.130, can be signed over $Z_2$ to produce a total of 8 $GDD(16 \times 8, 16, 2)$'s (otherwise known as *semibiplanes* [21]). The properties of *GDD*64.130 and its 8 "offspring" are summarized in Table 7. *GDD*64.130 itself is listed in Appendix A3.

| *GDD* | *Rk*2 | \| *G* \| | *APpt* | *APbl* | *Sd* | *Parent* |
|---|---|---|---|---|---|---|
| *GDD*64.130 | 18 | 13824 | $1*64$ | $1*4, 1*12, 1*48$ | *NO* | *GDD*32.1 |
| *GDD*128.1 | 44 | 256 | $1*128$ | $4*8, 6*16$ | *NO* | *GDD*64.130 |
| *GDD*128.2 | 44 | 384 | $1*128$ | $4*8, 4*24$ | *NO* | *GDD*64.130 |
| *GDD*128.3 | 44 | 1152 | $1*128$ | $1*8, 2*24, 1*72$ | *NO* | *GDD*64.130 |
| *GDD*128.4 | 44 | 4608 | $1*128$ | $1*8, 3*24, 1*96$ | *NO* | *GDD*64.130 |
| *GDD*128.5 | 44 | 768 | $1*128$ | $2*8, 1*16, 2*24, 1*48$ | *NO* | *GDD*64.130 |
| *GDD*128.6 | 44 | 3072 | $1*128$ | $1*8, 1*24, 1*96$ | *NO* | *GDD*64.130 |
| *GDD*128.7 | 44 | 768 | $1*128$ | $1*8, 3*24, 1*48$ | *NO* | *GDD*64.130 |
| *GDD*128.8 | 44 | 3072 | $1*128$ | $2*8, 1*16, 1*96$ | *NO* | *GDD*64.130 |

**Table 7: GDD64.130 and its signings over $Z_2$**

Signing these 8 $GDD(16 \times 8, 16, 2)$'s over $Z_2$ produces a set of 6 $GDD(16 \times 16, 16, 1)$'s as shown in Table 8.

| *GDD* | *Rk*2 | \|*G*\| | *APpt* | *APbl* | *Sd* | *Parent* |
|---|---|---|---|---|---|---|
| *GDD*256.1 | 105 | 4608 | $1*256$ | $2*32, 2*48, 1*96$ | *NO* | *GDD*128.1, 2 |
| *GDD*256.2 | 105 | 27648 | $1*256$ | $1*16, 3*48, 1*192$ | *NO* | *GDD*128.3, 4 |
| *GDD*256.3 | 99 | 18432 | $1*256$ | $1*16, 3*48, 1*192$ | *NO* | *GDD*128.5, 6 |
| *GDD*256.4 | 101 | 92160 | $1*256$ | $1*16, 1*240$ | *NO* | *GDD*128.6 |
| *GDD*256.5 | 105 | 18432 | $1*256$ | $2*16, 1*32, 1*192$ | *NO* | *GDD*128.7, 8 |
| *GDD*256.6 | 105 | 18432 | $1*256$ | $2*161*32, 1*192$ | *NO* | *GDD*128.8 |

**Table 8: Signings of GDD's 128.1-8 over $Z_2$**

These 6 $GDD(16 \times 16, 16, 1)$'s can each be extended to produce a projective plane of order 16. The extension is completed in the following way. Take a $GDD(16 \times 16, 16, 1)$ $(X, B)$ with groups $X_1, ..., X_{16}$ and resolve $B$ into parallel classes $R_1, ..., R_{16}$. Then we can form the projective plane $P$ $(X', B')$ where:

$X' = X \cup \{\infty_1, \infty_2, ..., \infty_{17}\}$

$B' = \{\{b \cup \{\infty_i\} \mid b \in R_i, i \in \{1, ..., 16\}\} \cup \{X_i \cup \{\infty_{17}\} \mid i \in \{1, ..., 16\}\} \cup \{\infty_1, ..., \infty_{17}\}\}$

It can be easily verified that $(X', B')$ is a $BIBD(16^2+16+1, 16+1, 1)$ or projective plane of order 16.

The four projective planes obtained in this way are all translation planes, and are described in Table 9. For listings of these planes see [5, 10].

| GDD | Resulting projective plane |
|---|---|
| GDD256.1 | Derived semifield plane |
| GDD256.2 | Derived semifield plane |
| GDD256.3 | Johnson-Walker plane |
| GDD256.4 | Dempwolff plane |
| GDD256.5 | Lorimer-Rahilly plane |
| GDD256.6 | Derived semifield plane |

**Table 9: Projective planes obtained by extending GDD256.1-6**

### 4.2. Signings over groups other than $Z_2$

Returning to the question of signings of $J_{16}$ we now consider admissible groups other than $Z_2$. First consider admissible groups of order 4, namely $Z_2 \times Z_2$ and $Z_4$. A class of such signings are summarized in Table 10. The designs themselves are listed in Appendix A2.

| No | Rk2 | $\mid G \mid$ | APbl | Sd | Ex | Comments |
|---|---|---|---|---|---|---|
| 1 | 16 | 1105920 | 1*64 | YES | YES | cover |
| 2 | 16 | 4608 | 1*16,1*48 | YES | YES | |
| 3 | 20 | 864 | 1*4,1*12,1*48 | YES | NO | cover |
| 4 | 20 | 288 | 1*4,1*12,1*48 | YES | NO | |
| 5 | 16 | 6144 | 1*64 | YES | YES | cover |
| 6 | 16 | 1536 | 1*16,1*48 | YES | YES | |
| 7 | 20 | 96 | 1*4,1*12,1*48 | YES | NO | |
| 8 | 20 | 1536 | 1*64 | YES | YES | cover |
| 9 | 20 | 128 | 1*64 | YES | YES | |
| 10 | 20 | 288 | 1*64 | YES | NO | |
| 11 | 16 | 73728 | 1*64 | YES | YES | |
| 12 | 20 | 512 | 1*64 | YES | YES | |
| 13 | 20 | 128 | 2*16,1*32 | YES | NO | |
| 14 | 20 | 72 | 1*4,2*12,1*36 | YES | NO | |
| 15 | 20 | 1024 | 1*64 | YES | NO | |
| 16 | 21 | 512 | 1*64 | YES | NO | |
| 17 | 20 | 512 | 1*64 | YES | NO | =C45 |
| 18 | 21 | 64 | 2*32 | YES | NO | =C48 |
| 19 | 20 | 512 | 1*64 | YES | NO | |
| 20 | 20 | 256 | 1*64 | YES | NO | |
| 21 | 21 | 512 | 1*64 | YES | NO | |
| 22 | 20 | 512 | 1*64 | YES | NO | |
| 23 | 19 | 1152 | 1*16,1*48 | YES | NO | |
| 24 | 20 | 1536 | 1*64 | YES | NO | |
| 25 | 20 | 512 | 1*64 | YES | NO | |
| 26 | 20 | 768 | 1*64 | YES | NO | |
| 27 | 20 | 64 | 4*16 | YES | NO | |
| 28 | 20 | 1024 | 1*64 | YES | NO | =C46 |
| 29 | 20 | 128 | 2*16,1*32 | YES | NO | |
| 30 | 23 | 192 | 1*16,1*48 | YES | NO | |
| 31 | 22 | 384 | 1*16,1*48 | YES | NO | |
| 32 | 22 | 256 | 1*64 | YES | NO | |
| 33 | 22 | 512 | 1*64 | YES | NO | |
| 34 | 23 | 64 | 4*16 | YES | NO | |
| 35 | 22 | 384 | 1*64 | YES | NO | |
| 36 | 22 | 512 | 1*64 | YES | NO | |
| 37 | 22 | 128 | 2*16,1*32 | YES | NO | |
| 38 | 22 | 384 | 1*64 | YES | NO | |
| 39 | 22 | 512 | 1*64 | YES | NO | |
| 40 | 22 | 7680 | 2*32 | YES | NO | |

**Table 10: Symmetric signings of $J_{16}$ over $Z_2^2$ (1-29) and $Z_4$ (30-40)**

Note: In the above Table 10, C45, C46 and C48 refer to signings 45, 46 and 48 respectively of HM16.3 over $Z_2$.

Table 10 contains all symmetric (i.e. self-dual) extensions of $J_{16}$ over $Z_2 \times Z_2$ and $Z_4$. They were obtained using a symmetric version of the program described in Section 2, with only elementary isomorph rejection (lexicographical ordering of rows) and no lookahead since all rows of $J_{16}$ are the same. Reducing the signings modulo $Z_2$ (see [15]) we have determined that the first 29 signings (over $Z_2 \times Z_2$) are extensions of HM16.1, and the remaining 11 signings (over $Z_4$) are extensions of HM16.2. Note that these are all self-dual extensions of the Hadamard matrices HM16.1, HM16.2 and HM16.3 over $Z_2$. Since these are all self-dual extensions, they must also contain the self-dual extensions of HM16.3 (viz. numbers 45, 46 and 47). Note that a symmetric extension can be obtained from more than one matrix of order 32, but the reduction of say No.17 to C45 is obviously not via a subgroup. We have also seen this phenomenon in the extensions of $GDD64.130$ to $GDD(16 \times 8, 16, 2)$'s and $GDD(16 \times 16, 16, 1)$'s in Tables 7 and 8.

If an extension has a zero diagonal (corresponding to the unit matrix) then after blowing up by using the permutation matrices from the signing group and changing the all one diagonal to all zero we obtain the adjacency matrix of a graph (note that it is symmetric) which corresponds to an antipodal cover of $K_{16}$ with parameters (16,4,4) (see [18]). There are exactly 4 nonisomorphic covers with these parameters, and they are indicated on the right of the table.

From Table 10 it can be seen that none of the 11 signings of $J_{16}$ over $Z_4$ are extendible (i.e. further signable over $Z_2$). However, some of the 29 signings over $Z_2 \times Z_2$ are. We present the results of these further signings in Table 11 below.

| No | Rk2 | $\mid G \mid$ | APpt | APbl | Sd | Ex | Parent |
|---|---|---|---|---|---|---|---|
| 1 | 44 | 6144 | 1*128 | 1*32,1*96 | NO | YES | 1 |
| 2 | 53 | 96 | 2*4,2*12,2*48 | 2*4,2*12,2*48 | YES | NO | 1 |
| 3 | 54 | 96 | 1*32,1*96 | 1*2,1*6,1*8,1*16,1*96 | NO | NO | 1 |
| 4 | 56 | 12 | 2*2,1*4,2*6,9*12 | 2*2,1*4,2*6,9*12 | NO | NO | 1 |
| 5 | 55 | 12 | 4*2,12*6,4*12 | 4*2,12*6,4*12 | YES | NO | 1 |
| 6 | 56 | 48 | 4*8,4*24 | 4*8,4*24 | YES | NO | 1 |
| 7 | 52 | 384 | 2*64 | 2*16,1*96 | NO | NO | 1 |
| 8 | 56 | 192 | 1*16,1*48,1*64 | 1*16,1*48,1*64 | YES | NO | 1 |
| 9 | 44 | 2304 | 1*128 | 1*8,1*48,1*72 | NO | YES | 1 |
| 10 | 55 | 64 | 4*16,1*64 | 4*8,1*32,1*64 | NO | NO | 1 |
| 11 | 56 | 96 | 1*8,1*24,3*32 | 1*8,1*24,3*32 | YES | NO | 1 |
| 12 | 53 | 96 | 2*16,2*48 | 2*16,2*48 | YES | NO | 1 |
| 13 | 56 | 20 | 4*2,12*10 | 4*2,12*10 | YES | NO | 1 |
| 14 | 54 | 32 | 6*16,1*32 | 6*16,1*32 | YES | NO | 1 |
| 15 | 54 | 96 | 1*32,2*48 | 2*16,2*48 | NO | NO | 1 |
| 16 | 54 | 192 | 1*8,1*24,1*96 | 1*8,1*24,1*96 | YES | NO | 1 |
| 17 | 56 | 16 | 8*8,4*16 | 8*8,4*16 | YES | NO | 1 |
| 18 | 48 | 3072 | 1*128 | 1*128 | YES | NO | 1 |
| 19 | 50 | 192 | 1*32,2*48 | 1*32,2*48 | YES | NO | 1 |
| 20 | 54 | 32 | 6*16,1*32 | 6*16,1*32 | YES | NO | 1 |
| 21 | 52 | 768 | 1*128 | 1*128 | YES | NO | 1 |
| 22 | 54 | 480 | 1*48,1*80 | 1*48,1*80 | YES | NO | 1 |
| 23 | 44 | 1536 | 1*128 | 1*8,1*24,2*48 | NO | YES | 1 |
| 24 | 52 | 384 | 1*32,1*96 | 1*32,1*96 | YES | NO | 1 |
| 25 | 54 | 384 | 1*128 | 1*128 | YES | NO | 1 |
| 26 | 54 | 768 | 1*128 | 1*128 | YES | NO | 1 |
| 27 | 44 | 2560 | 1*128 | 1*8,1*40,1*80 | NO | YES | 1 |
| 28 | 54 | 1280 | 1*128 | 1*128 | YES | NO | 1 |
| 29 | 52 | 768 | 1*128 | 1*32,1*96 | NO | NO | 1 |
| 30 | 48 | 6144 | 1*128 | 1*128 | YES | NO | 1 |
| 31 | 44 | 36864 | 1*128 | 1*128 | YES | YES | 1 |
| 32 | 42 | 49152 | 1*128 | 1*128 | YES | YES | 1 |
| 33 | 40 | 122880 | 1*128 | 1*128 | YES | YES | 1 |
| 34 | 52 | 384 | 1*32,1*96 | 1*32,1*96 | YES | NO | 2 |
| 35 | 50 | 192 | 1*32,2*48 | 1*32,2*48 | YES | NO | 2 |
| 36 | 52 | 128 | 2*8,1*16,3*32 | 2*8,1*16,3*32 | YES | NO | 2 |
| 37 | 46 | 512 | 2*32,1*64 | 2*32,1*64 | YES | NO | 2 |
| 38 | 48 | 768 | 1*32,1*96 | 1*32,1*96 | YES | NO | 2 |
| 39 | 48 | 256 | 4*32 | 4*32 | YES | NO | 2 |
| 40 | 48 | 1536 | 1*32,1*96 | 1*32,1*96 | YES | YES | 2 |

**Table 11: Semibiplanes from symmetric GHM(16) over $Z_2 \times Z_2$**

| No | Rk2 | $\|G\|$ | APpt | APbl | Sd | Ex | Parent |
|---|---|---|---|---|---|---|---|
| 41 | 46 | 6144 | 1*128 | 1*128 | YES | YES | 5 |
| 42 | 52 | 128 | 2*8,1*16,3*32 | 2*8,1*16,3*32 | YES | NO | 5 |
| 43 | 48 | 1024 | 1*128 | 1*128 | YES | NO | 5 |
| 44 | 46 | 1024 | 1*128 | 1*128 | YES | NO | 5 |
| 45 | 48 | 512 | 1*128 | 1*128 | NO | YES | 5 |
| 46 | 52 | 128 | 2*8,1*16,3*32 | 2*8,1*16,3*32 | YES | NO | 5 |
| 47 | 48 | 512 | 2*32 1*64 | 2*32,1*64 | YES | NO | 5 |
| 48 | 48 | 2048 | 1*128 | 1*128 | YES | NO | 5 |
| 49 | 48 | 512 | 2*32,1*64 | 2*32,1*64 | YES | NO | 5 |
| 50 | 56 | 128 | 2*64 | 2*64 | YES | NO | 5 |
| 51 | 54 | 256 | 1*128 | 1*128 | YES | NO | 5 |
| 52 | 56 | 64 | 4*16,2*32 | 4*16,2*32 | YES | NO | 5 |
| 53 | 47 | 256 | 2*16,3*32 | 2*16,3*32 | YES | NO | 6 |
| 54 | 46 | 512 | 2*16,1*32,1*64 | 2*16,1*32,1*64 | YES | NO | 6 |
| 55 | 48 | 1536 | 1*32,1*96 | 1*32,1*96 | YES | NO | 6 |
| 56 | 46 | 256 | 4*32 | 4*32 | YES | NO | 6 |
| 57 | 48 | 256 | 4*32 | 4*32 | YES | NO | 6 |
| 58 | 46 | 256 | 4*32 | 4*32 | YES | NO | 6 |
| 59 | 48 | 768 | 2*16,1*96 | 2*16,1*96 | YES | NO | 6 |
| 60 | 48 | 5376 | 1*128 | 1*128 | YES | NO | 8 |
| 61 | 48 | 768 | 1*128 | 1*128 | YES | NO | 8 |
| 62 | 46 | 1024 | 1*128 | 1*128 | YES | NO | 8 |
| 63 | 52 | 64 | 2*64 | 2*64 | YES | NO | 9 |
| 64 | 50 | 128 | 2*64 | 2*64 | YES | NO | 9 |
| 65 | 52 | 64 | 2*64 | 2*64 | YES | NO | 9 |
| 66 | 52 | 64 | 2*64 | 2*64 | YES | NO | 9 |
| 67 | 40 | 2288 | 1*128 | 1*128 | YES | YES | 11 |
| 68 | 43 | 8192 | 1*128 | 1*128 | YES | YES | 11 |
| 69 | 44 | 6144 | 1*128 | 1*128 | NO | NO | 11 |
| 70 | 41 | 344064 | 1*128 | 1*128 | YES | YES | 11 |
| 71 | 42 | 49152 | 1*128 | 1*128 | YES | YES | 11 |
| 72 | 48 | 6144 | 1*128 | 1*128 | YES | NO | 11 |
| 73 | 44 | 1536 | 1*128 | 2*16,1*96 | NO | YES | 11 |
| 74 | 44 | 36864 | 1*128 | 1*128 | YES | YES | 11 |
| 75 | 48 | 768 | 1*128 | 1*128 | YES | NO | 12 |
| 76 | 48 | 256 | 1*128 | 1*128 | YES | NO | 12 |
| 77 | 48 | 1024 | 1*128 | 1*128 | YES | NO | 12 |

**Table 11(cont.): Semibiplanes from symmetric GHM(16) over $Z_2 \times Z_2$**

Signings of the semibiplanes produced in Table 11 are listed in Table 12 below. Projective planes resulting from the extensions of these semibiplanes are indicated in the right hand column. For listings of these planes see [5]. The *Parent* column in Table 12 contains the identification numbers of the parent semibiplanes of Table 11.

| No | Rk2 | $\mid G \mid$ | APpt | APbl | Sd | Parent | Plane |
|---|---|---|---|---|---|---|---|
| 1 | 97 | 184320 | 1*256 | 1*64,1*192 | NO | 1 | HALL |
| 3 | 97 | 76800 | 1*256 | 1*16,1*80,1*160 | NO | 27 | HALL |
| 2 | 97 | 27648 | 1*256 | 1*16,1*96,1*144 | NO | 9,23 | SF4 |
| 4 | 97 | 73728 | 1*256 | 1*256 | YES | 31,67,68,74 | SF2 |
| 5 | 97 | 442368 | 1*256 | 1*256 | YES | 31,32,71,74 | SF4 |
| 6 | 81 | 3686400 | 1*256 | 1*256 | YES | 33 | DES |
| 7 | 109 | 9216 | 1*64 1*192 | 1*64,1*192 | YES | 40 | DH1 |
| 8 | 108 | 12288 | 1*256 | 1*256 | NO | 41,45 | MAT |
| 10 | 105 | 86016 | 1*256 | 1*32,1*224 | NO | 70,73 | LRdual |

**Table 12: Signings of semibiplanes**

| | | | |
|---|---|---|---|
| DES | Desargues plane | HALL | Hall plane |
| SF4 | semifield plane with kernel 4 | SF2 | semifield plane with kernel 2 |
| DH1 | first derived Hall plane | LR | Lorimer-Rahilly plane |
| MAT | new plane | | |

Finally we present the results of signing $J_{16}$ over the group $Z_2^3$ in Table 13. The 4 signings themselves are listed in Appendix A4.

| No | Rk2 | $\mid G \mid$ | APbl | Sd | Ex | Comments |
|---|---|---|---|---|---|---|
| 1 | 40 | 12288 | 1*128 | YES | YES | |
| 2 | 44 | 36864 | 1*128 | YES | YES | |
| 3 | 48 | 5376 | 1*128 | YES | NO | cover |
| 4 | 48 | 768 | 1*128 | YES | NO | |

**Table 13: Symmetric signings of $J_{16}$ over $Z_2^3$**

The *GDD*'s of orders 64, 128 and 256 in Tables 11, 12 and 13 are all *GHM*'s of order 16 over the elementary abelian groups of orders 4, 8 and 16, respectively.

## 5. Concluding Remarks

In this paper we have described an effective backtracking algorithm for signing symmetric group divisible designs. The algorithm makes use of lookahead and isomorph rejection techniques. The application of this algorithm has allowed us to present a complete enumeration of generalized Hadamard matrices of order $v < 16$ for all possible underlying groups.

We have also achieved a complete enumeration of several classes of generalized Hadamard matrices and symmetric group divisible designs of order 16. The search proceeds bottom up from the all-ones matrix $J_{16}$ by successive signings over $Z_2$ all the way up to symmetric $GDD(16 \times 16, 16, 1)$'s, which can always be extended to projective planes of order 16. This approach allows one to generate all projective planes of order 16 that have an elation of order 2, which includes all the known planes and possibly some new ones.

During the search many other interesting configurations were generated. For example we generated semibiplanes which are symmetric $GDD(128, 16, 2)$'s, corresponding to signings of $J_{16}$ over groups of order 8 (see [25], particularly for a polynomial algorithm for signing a semibiplane to a projective plane, or showing that it cannot be done). We also have generated covers of complete graphs, which correspond to symmetric $GDD$'s with symmetric incidence matrices with a constant diagonal (see [18]). Examples of new semibiplanes and covers are presented.

With the continual advent of more powerful machines it is hoped to continue this work to eventually provide a complete enumeration of signings of $J_{16}$. We would begin by a complete enumeration of the signings of the $GDD(16 \times 2, 16, 8)$'s 32.1 and 32.2 over $Z_2$. Of course there is a great amount of further computational work required to continue the signings and classify the outputs. However we are helped by the fact that many of the signings of $GDD$'s 32.1 and 32.2 cannot be signed over $Z_2$. The goal of this work would include a classification of all projective planes of order 16 possessing an involutary elation.

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## A1. Generalized Hadamard matrices of order 9

### Group tables

$Z_3$

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 2 | 0 | 1 |
| 1 | 2 | 0 |

$Z_3 \times Z_3$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 0 | 1 | 5 | 3 | 4 | 8 | 6 | 7 |
| 1 | 2 | 0 | 4 | 5 | 3 | 7 | 8 | 6 |
| 6 | 7 | 8 | 0 | 1 | 2 | 3 | 4 | 5 |
| 8 | 6 | 7 | 2 | 0 | 1 | 5 | 3 | 4 |
| 7 | 8 | 6 | 1 | 2 | 0 | 4 | 5 | 3 |
| 3 | 4 | 5 | 6 | 7 | 8 | 0 | 1 | 2 |
| 5 | 3 | 4 | 8 | 6 | 7 | 2 | 0 | 1 |
| 4 | 5 | 3 | 7 | 8 | 6 | 1 | 2 | 0 |

### The 4 designs

1 and 2 are signings over $Z_3$. 3 and 4 are signings over $Z_3 \times Z_3$.

| **1** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 |
| 0 | 0 | 0 | 2 | 2 | 2 | 1 | 1 | 1 |
| 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 |
| 0 | 1 | 2 | 1 | 2 | 0 | 2 | 0 | 1 |
| 0 | 1 | 2 | 2 | 0 | 1 | 1 | 2 | 0 |
| 0 | 2 | 1 | 0 | 2 | 1 | 0 | 2 | 1 |
| 0 | 2 | 1 | 1 | 0 | 2 | 2 | 1 | 0 |
| 0 | 2 | 1 | 2 | 1 | 0 | 1 | 0 | 2 |

| **2** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 |
| 0 | 0 | 0 | 2 | 2 | 2 | 1 | 1 | 1 |
| 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 |
| 0 | 1 | 2 | 1 | 2 | 0 | 2 | 0 | 1 |
| 0 | 1 | 2 | 2 | 0 | 1 | 1 | 2 | 0 |
| 0 | 2 | 1 | 0 | 2 | 1 | 1 | 0 | 2 |
| 0 | 2 | 1 | 1 | 0 | 2 | 0 | 2 | 1 |
| 0 | 2 | 1 | 2 | 1 | 0 | 2 | 1 | 0 |

| **3** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | 2 | 1 | 6 | 8 | 7 | 3 | 5 | 4 |
| 0 | 3 | 6 | 2 | 5 | 8 | 1 | 4 | 7 |
| 0 | 4 | 8 | 5 | 6 | 1 | 7 | 2 | 3 |
| 0 | 5 | 7 | 8 | 1 | 3 | 4 | 6 | 2 |
| 0 | 6 | 3 | 1 | 7 | 4 | 2 | 8 | 5 |
| 0 | 7 | 5 | 4 | 2 | 6 | 8 | 3 | 1 |
| 0 | 8 | 4 | 7 | 3 | 2 | 5 | 1 | 6 |

| **4** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | 2 | 1 | 6 | 8 | 7 | 3 | 5 | 4 |
| 0 | 3 | 6 | 2 | 7 | 4 | 1 | 8 | 5 |
| 0 | 4 | 8 | 5 | 2 | 6 | 7 | 3 | 1 |
| 0 | 5 | 7 | 8 | 3 | 2 | 4 | 1 | 6 |
| 0 | 6 | 3 | 1 | 5 | 8 | 2 | 4 | 7 |
| 0 | 7 | 5 | 4 | 6 | 1 | 8 | 2 | 3 |
| 0 | 8 | 4 | 7 | 1 | 3 | 5 | 6 | 2 |

## A2. Symmetric signings of $J_{16}$ over $Z_2 \times Z_2$ and over $Z_4$

### Group tables

$Z_2 \times Z_2$

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 1 | 0 | 3 | 2 |
| 2 | 3 | 0 | 1 |
| 3 | 2 | 1 | 0 |

$Z_4$

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 3 | 0 | 1 | 2 |
| 2 | 3 | 0 | 1 |
| 1 | 2 | 3 | 0 |

### The 40 signings. 1-29 are over $Z_2 \times Z_2$, 30-40 over $Z_4$

| 1 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 0 2 2 2 2 3 3 3 3 1 1 1 1 |
| 0 0 0 0 3 3 3 3 1 1 1 1 2 2 2 2 |
| 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| 0 1 2 3 1 0 3 2 2 3 0 1 3 2 1 0 |
| 0 1 2 3 2 3 0 1 3 2 1 0 1 0 3 2 |
| 0 1 2 3 3 2 1 0 1 0 3 2 2 3 0 1 |
| 0 2 3 1 0 2 3 1 0 2 3 1 0 2 3 1 |
| 0 2 3 1 1 3 2 0 2 0 1 3 3 1 0 2 |
| 0 2 3 1 2 0 1 3 3 1 0 2 1 3 2 0 |
| 0 2 3 1 3 1 0 2 1 3 2 0 2 0 1 3 |
| 0 3 1 2 0 3 1 2 0 3 1 2 0 3 1 2 |
| 0 3 1 2 1 2 0 3 2 1 3 0 3 0 2 1 |
| 0 3 1 2 2 1 3 0 3 0 2 1 1 2 0 3 |
| 0 3 1 2 3 0 2 1 1 2 0 3 2 1 3 0 |

| 2 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 0 2 2 2 2 3 3 3 3 1 1 1 1 |
| 0 0 0 0 3 3 3 3 1 1 1 1 2 2 2 2 |
| 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| 0 1 2 3 1 0 3 2 2 3 0 1 3 2 1 0 |
| 0 1 2 3 2 3 0 1 3 2 1 0 1 0 3 2 |
| 0 1 2 3 3 2 1 0 1 0 3 2 2 3 0 1 |
| 0 2 3 1 0 2 3 1 0 2 3 1 0 2 3 1 |
| 0 2 3 1 1 3 2 0 2 0 1 3 3 1 0 2 |
| 0 2 3 1 2 0 1 3 3 1 0 2 1 3 2 0 |
| 0 2 3 1 3 1 0 2 1 3 2 0 2 0 1 3 |
| 0 3 1 2 0 3 1 2 0 3 1 2 1 2 0 3 |
| 0 3 1 2 1 2 0 3 2 1 3 0 2 1 3 0 |
| 0 3 1 2 2 1 3 0 3 0 2 1 0 3 1 2 |
| 0 3 1 2 3 0 2 1 1 2 0 3 3 0 2 1 |

| 3 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 0 2 2 2 2 3 3 3 3 1 1 1 1 |
| 0 0 0 0 3 3 3 3 1 1 1 1 2 2 2 2 |
| 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| 0 1 2 3 1 0 3 2 2 3 0 1 3 2 1 0 |
| 0 1 2 3 2 3 0 1 3 2 1 0 1 0 3 2 |
| 0 1 2 3 3 2 1 0 1 0 3 2 2 3 0 1 |
| 0 2 3 1 0 2 3 1 0 2 3 1 0 3 1 2 |
| 0 2 3 1 1 3 2 0 2 0 1 3 3 0 2 1 |
| 0 2 3 1 2 0 1 3 3 1 0 2 1 2 0 3 |
| 0 2 3 1 3 1 0 2 1 3 2 0 2 1 3 0 |
| 0 3 1 2 0 3 1 2 0 3 1 2 0 2 3 1 |
| 0 3 1 2 1 2 0 3 3 0 2 1 2 0 1 3 |
| 0 3 1 2 2 1 3 0 1 2 0 3 3 1 0 2 |
| 0 3 1 2 3 0 2 1 2 1 3 0 1 3 2 0 |

| 4 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 0 2 2 2 2 3 3 3 3 1 1 1 1 |
| 0 0 0 0 3 3 3 3 1 1 1 1 2 2 2 2 |
| 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| 0 1 2 3 1 0 3 2 2 3 0 1 3 2 1 0 |
| 0 1 2 3 2 3 0 1 3 2 1 0 1 0 3 2 |
| 0 1 2 3 3 2 1 0 1 0 3 2 2 3 0 1 |
| 0 2 3 1 0 2 3 1 0 2 3 1 0 3 1 2 |
| 0 2 3 1 1 3 2 0 2 0 1 3 3 0 2 1 |
| 0 2 3 1 2 0 1 3 3 1 0 2 1 2 0 3 |
| 0 2 3 1 3 1 0 2 1 3 2 0 2 1 3 0 |
| 0 3 1 2 0 3 1 2 0 3 1 2 1 3 2 0 |
| 0 3 1 2 1 2 0 3 3 0 2 1 3 1 0 2 |
| 0 3 1 2 2 1 3 0 1 2 0 3 2 0 1 3 |
| 0 3 1 2 3 0 2 1 2 1 3 0 0 2 3 1 |

| 5 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 0 2 2 2 2 3 3 3 3 1 1 1 1 |
| 0 0 0 0 3 3 3 3 1 1 1 1 2 2 2 2 |
| 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| 0 1 2 3 1 0 3 2 2 3 0 1 3 2 1 0 |
| 0 1 2 3 2 3 0 1 3 2 1 0 1 0 3 2 |
| 0 1 2 3 3 2 1 0 1 0 3 2 2 3 0 1 |
| 0 2 3 1 0 2 3 1 0 2 3 1 1 3 2 0 |
| 0 2 3 1 1 3 2 0 2 0 1 3 2 0 1 3 |
| 0 2 3 1 2 0 1 3 3 1 0 2 0 2 3 1 |
| 0 2 3 1 3 1 0 2 1 3 2 0 3 1 0 2 |
| 0 3 1 2 0 3 1 2 1 2 0 3 0 3 1 2 |
| 0 3 1 2 1 2 0 3 3 0 2 1 3 0 2 1 |
| 0 3 1 2 2 1 3 0 2 1 3 0 1 2 0 3 |
| 0 3 1 2 3 0 2 1 0 3 1 2 2 1 3 0 |

| 6 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 0 2 2 2 2 3 3 3 3 1 1 1 1 |
| 0 0 0 0 3 3 3 3 1 1 1 1 2 2 2 2 |
| 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| 0 1 2 3 1 0 3 2 2 3 0 1 3 2 1 0 |
| 0 1 2 3 2 3 0 1 3 2 1 0 1 0 3 2 |
| 0 1 2 3 3 2 1 0 1 0 3 2 2 3 0 1 |
| 0 2 3 1 0 2 3 1 0 2 3 1 1 3 2 0 |
| 0 2 3 1 1 3 2 0 2 0 1 3 2 0 1 3 |
| 0 2 3 1 2 0 1 3 3 1 0 2 0 2 3 1 |
| 0 2 3 1 3 1 0 2 1 3 2 0 3 1 0 2 |
| 0 3 1 2 0 3 1 2 1 2 0 3 1 2 0 3 |
| 0 3 1 2 1 2 0 3 3 0 2 1 2 1 3 0 |
| 0 3 1 2 2 1 3 0 2 1 3 0 0 3 1 2 |
| 0 3 1 2 3 0 2 1 0 3 1 2 3 0 2 1 |

| 7 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0000222233331111 |
| 0000333311112222 |
| 0123012301230123 |
| 0123103223013210 |
| 0123230132101032 |
| 0123321010322301 |
| 0231023103120231 |
| 0231132030212013 |
| 0231201312033102 |
| 0231310221301320 |
| 0312031202311203 |
| 0312120320132130 |
| 0312213031020312 |
| 0312302113203021 |

| 8 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0000222233331111 |
| 0000333311112222 |
| 0123012301230123 |
| 0123103223013210 |
| 0123230132101032 |
| 0123321010322301 |
| 0231023103120312 |
| 0231132030212130 |
| 0231201312033021 |
| 0231310221301203 |
| 0312031202310231 |
| 0312120331022013 |
| 0312213013203102 |
| 0312302120131320 |

| 9 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0000222233331111 |
| 0000333311112222 |
| 0123012301230123 |
| 0123103223013210 |
| 0123230132101032 |
| 0123321010322301 |
| 0231023103120312 |
| 0231132030212130 |
| 0231201312033021 |
| 0231310221301203 |
| 0312031202311320 |
| 0312120331023102 |
| 0312213013202013 |
| 0312302120130231 |

| 10 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0000222233331111 |
| 0000333311112222 |
| 0123012301230123 |
| 0123103223013210 |
| 0123230132101032 |
| 0123321010322301 |
| 0231023112030231 |
| 0231132021302013 |
| 0231201303123102 |
| 0231310230211320 |
| 0312031202311203 |
| 0312120320132130 |
| 0312213031020312 |
| 0312302113203021 |

| 11 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0000222233331111 |
| 0000333311112222 |
| 0123012301230123 |
| 0123103223013210 |
| 0123230132101032 |
| 0123321010322301 |
| 0231023113201320 |
| 0231132031022013 |
| 0231201320130231 |
| 0231310202313102 |
| 0312031212031203 |
| 0312120330212130 |
| 0312213021300312 |
| 0312302103123021 |

| 12 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0000222233331111 |
| 0000333311112222 |
| 0123012301230123 |
| 0123103232102301 |
| 0123230110323210 |
| 0123321023011032 |
| 0231031212030231 |
| 0231120321302013 |
| 0231213003123102 |
| 0231302130211320 |
| 0312023102311203 |
| 0312132020132130 |
| 0312201331020312 |
| 0312310213203021 |

| 13 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0000222233331111 |
| 0001233301112223 |
| 0122112330130230 |
| 0123103202313102 |
| 0123230113201320 |
| 0123321031022013 |
| 0230301311230122 |
| 0231023110323210 |
| 0231132023011032 |
| 0231310232102301 |
| 0312031203120312 |
| 0312213012033021 |
| 0312302121301203 |
| 0313020320212131 |

| 14 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0000222233331111 |
| 0001233301112223 |
| 0122112330130230 |
| 0123103202313102 |
| 0123230113201320 |
| 0123321031022013 |
| 0230301311231202 |
| 0231023110322130 |
| 0231132023010312 |
| 0231310232103021 |
| 0312031212030123 |
| 0312213021301032 |
| 0312302103122301 |
| 0313020320213211 |

| 15 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222301331123 |
| 0010233332112201 |
| 0122023011012333 |
| 0123201310323210 |
| 0123310232100123 |
| 0133032022231011 |
| 0203113213031202 |
| 0212102231303031 |
| 0231031203122130 |
| 0231120330210312 |
| 0312230113201320 |
| 0312321020133102 |
| 0320312103312012 |
| 0331303121020221 |

| 16 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222301331123 |
| 0010233332112201 |
| 0122123133230100 |
| 0123210310322310 |
| 0123301232101023 |
| 0133132100013222 |
| 0203313021130212 |
| 0212302012313130 |
| 0231231013201302 |
| 0231320131022031 |
| 0312021303123012 |
| 0312130221300321 |
| 0320012213031231 |
| 0331003220212113 |

| 17 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222301331123 |
| 0011233223012310 |
| 0122023011102333 |
| 0123201310233210 |
| 0123310232010123 |
| 0132032133221100 |
| 0202113313310202 |
| 0213102331022031 |
| 0230120230131312 |
| 0231031212303021 |
| 0312230102130231 |
| 0313321120302102 |
| 0321312003123012 |
| 0330303021211221 |

| 18 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222301331123 |
| 0011233223012310 |
| 0122023012131330 |
| 0123201313200213 |
| 0123310231023120 |
| 0132032130212103 |
| 0202113333110022 |
| 0213231032321001 |
| 0230120213302131 |
| 0231302112033012 |
| 0312103201233201 |
| 0313321100102322 |
| 0321312020310231 |
| 0330030321121212 |

| 19 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222323330111 |
| 0010233310112223 |
| 0122023011233013 |
| 0123201332103102 |
| 0123310210320231 |
| 0133032022011321 |
| 0221131213032300 |
| 0230120231301123 |
| 0231213003121032 |
| 0231302130213210 |
| 0302330121131202 |
| 0312012331022130 |
| 0312103202310312 |
| 0313321103202021 |

| 20 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222323330111 |
| 0010233310112223 |
| 0122102230133013 |
| 0123013213200231 |
| 0123231002313102 |
| 0133220130021321 |
| 0221310311323020 |
| 0230032011231312 |
| 0231123032102130 |
| 0231301223011203 |
| 0302303131210122 |
| 0312021303121032 |
| 0312130221302301 |
| 0313312102032210 |

| 21 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222323330111 |
| 0010233310112223 |
| 0122123133010230 |
| 0123210332103012 |
| 0123301210320321 |
| 0133132100232102 |
| 0221331021133200 |
| 0230320012311132 |
| 0231013213202310 |
| 0231102331021023 |
| 0302030231213121 |
| 0312203121301302 |
| 0312312003122031 |
| 0313021202031213 |

| 22 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222323330111 |
| 0010233310112223 |
| 0122201233013013 |
| 0123023110323102 |
| 0123132032100231 |
| 0133210200231321 |
| 0221313021133020 |
| 0230302012311312 |
| 0231031213202130 |
| 0231120331021203 |
| 0302330131210122 |
| 0312012303121032 |
| 0312103221302301 |
| 0313321102032210 |

| 23 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0001222323330111 |
| 0011233210012233 |
| 0122123130130023 |
| 0123210331023201 |
| 0123301213200132 |
| 0132132003212301 |
| 0221331021313020 |
| 0230013313122120 |
| 0230102231301213 |
| 0231320112031302 |
| 0302030232113112 |
| 0312021301231230 |
| 0313203022101321 |
| 0313312100322012 |

| 24 |
|---|
| 0000000000000000 |
| 0000111122223333 |
| 0011222200331133 |
| 0011333322112200 |
| 0123012301012323 |
| 0123103223231010 |
| 0123231032100132 |
| 0123320110323201 |
| 0202023131132031 |
| 0202132013311302 |
| 0231021313203120 |
| 0231130231020213 |
| 0312210321303012 |
| 0312301203120321 |
| 0330213030211221 |
| 0330302112032112 |

| 25 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 1 2 2 2 2 0 0 3 3 1 1 3 3 |
| 0 0 1 1 3 3 3 3 2 2 1 1 2 2 0 0 |
| 0 1 2 3 0 1 2 3 0 1 0 1 2 3 2 3 |
| 0 1 2 3 1 0 3 2 2 3 2 3 1 0 1 0 |
| 0 1 2 3 2 3 1 0 3 2 1 0 0 1 3 2 |
| 0 1 2 3 3 2 0 1 1 0 3 2 3 2 0 1 |
| 0 2 0 2 0 2 3 1 3 1 1 3 3 1 2 0 |
| 0 2 0 2 1 3 2 0 1 3 3 1 0 2 1 3 |
| 0 2 3 1 0 2 1 3 1 3 2 0 2 0 3 1 |
| 0 2 3 1 1 3 0 2 3 1 0 2 1 3 0 2 |
| 0 3 1 2 2 1 0 3 3 0 2 1 3 0 1 2 |
| 0 3 1 2 3 0 1 2 1 2 0 3 0 3 2 1 |
| 0 3 3 0 2 1 3 0 2 1 3 0 1 2 2 1 |
| 0 3 3 0 3 0 2 1 0 3 1 2 2 1 1 2 |

| 26 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 1 2 2 3 3 0 0 1 1 2 2 3 3 |
| 0 0 1 2 2 3 0 2 1 2 3 3 1 3 0 1 |
| 0 1 2 2 2 0 2 1 3 3 0 1 0 3 1 3 |
| 0 1 2 3 0 1 3 2 0 1 2 3 1 0 2 3 |
| 0 1 3 0 2 3 3 0 2 1 3 2 2 1 1 0 |
| 0 1 3 2 1 2 0 3 2 3 1 0 3 0 2 1 |
| 0 2 0 1 3 0 2 2 2 3 1 3 1 1 3 0 |
| 0 2 0 2 3 1 1 3 3 1 3 1 2 0 0 2 |
| 0 2 1 3 0 2 3 1 1 3 0 2 3 1 0 2 |
| 0 2 1 3 1 3 2 0 3 1 2 0 0 2 3 1 |
| 0 3 2 1 0 1 2 3 1 2 3 0 3 2 1 0 |
| 0 3 2 3 3 0 1 0 1 0 1 2 2 3 2 1 |
| 0 3 3 0 1 2 1 2 3 0 0 3 1 2 1 2 |
| 0 3 3 1 3 3 0 1 0 2 2 1 0 1 2 2 |

| 27 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 1 2 2 3 3 0 1 2 3 0 1 2 3 |
| 0 0 1 2 0 3 2 2 3 1 1 0 2 3 3 1 |
| 0 1 2 0 2 0 2 3 0 3 1 1 3 3 1 2 |
| 0 1 2 3 0 1 3 2 3 3 2 2 1 1 0 0 |
| 0 1 3 2 2 3 0 1 3 2 0 1 1 0 2 3 |
| 0 1 3 2 3 2 1 0 1 0 2 3 2 3 1 0 |
| 0 2 0 3 0 3 3 1 1 2 1 3 0 2 1 2 |
| 0 2 1 1 3 3 2 0 2 2 3 1 3 1 0 0 |
| 0 2 2 1 1 2 0 2 1 3 0 3 3 0 3 1 |
| 0 2 3 0 1 2 1 3 3 1 3 0 1 2 0 2 |
| 0 3 0 2 3 1 1 2 0 3 3 1 0 2 2 1 |
| 0 3 1 3 3 1 0 3 2 1 0 2 2 0 1 2 |
| 0 3 2 3 1 0 2 1 1 0 3 0 2 1 2 3 |
| 0 3 3 1 2 0 3 0 2 0 1 2 1 2 3 1 |

| 28 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 2 0 2 2 3 0 1 1 3 1 2 3 3 |
| 0 0 2 3 1 2 2 0 3 0 3 1 3 1 1 2 |
| 0 1 0 1 3 2 3 2 0 3 2 1 3 2 0 1 |
| 0 1 2 2 2 0 1 2 3 3 1 0 0 3 1 3 |
| 0 1 2 2 3 1 0 3 1 1 3 2 3 0 2 0 |
| 0 1 3 0 2 2 3 1 3 2 0 3 1 1 2 0 |
| 0 2 0 3 0 3 1 3 3 2 1 1 2 0 2 1 |
| 0 2 1 0 3 3 1 2 2 1 3 3 0 1 0 2 |
| 0 2 1 3 2 1 3 0 1 3 2 0 1 0 3 2 |
| 0 2 3 1 1 0 2 3 1 3 0 2 2 1 0 3 |
| 0 3 1 3 3 0 3 1 2 0 1 2 2 2 1 0 |
| 0 3 2 1 2 3 0 1 0 1 0 1 2 3 3 2 |
| 0 3 3 1 0 1 2 2 2 0 3 0 1 3 2 1 |
| 0 3 3 2 1 3 0 0 1 2 2 3 0 2 1 1 |

| 29 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 2 0 2 2 3 0 1 1 3 1 2 3 3 |
| 0 0 2 3 3 1 2 1 1 3 3 0 0 2 1 2 |
| 0 1 0 3 0 3 2 3 2 2 3 1 2 1 0 1 |
| 0 1 2 1 3 0 3 2 2 0 1 1 3 2 3 0 |
| 0 1 2 2 2 3 1 1 3 1 0 3 3 0 0 2 |
| 0 1 3 1 3 2 1 3 0 3 2 2 0 0 2 1 |
| 0 2 0 1 2 2 3 0 1 1 3 0 3 1 2 3 |
| 0 2 1 3 2 0 1 3 1 0 2 3 2 3 1 0 |
| 0 2 1 3 3 1 0 2 3 2 0 1 1 0 2 3 |
| 0 2 3 0 1 1 3 2 0 3 1 3 2 1 0 2 |
| 0 3 1 0 2 3 3 0 3 2 1 2 0 2 1 1 |
| 0 3 2 2 1 2 0 0 1 3 0 1 2 3 3 1 |
| 0 3 3 1 0 3 0 2 2 1 2 0 1 3 1 2 |
| 0 3 3 2 1 0 2 1 3 0 3 2 1 1 2 0 |

| 30 |
|---|
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 0 2 2 3 3 1 1 3 3 1 1 2 2 |
| 0 0 0 1 0 3 2 3 3 3 1 2 2 2 1 1 |
| 0 1 2 0 3 0 2 2 0 1 3 1 2 3 1 3 |
| 0 1 2 3 0 2 3 0 3 2 2 1 1 0 3 1 |
| 0 1 3 2 2 3 0 2 1 0 1 2 1 3 3 0 |
| 0 1 3 3 2 0 2 1 2 3 0 0 3 1 2 1 |
| 0 2 1 3 0 3 1 2 0 2 1 3 0 1 2 3 |
| 0 2 1 3 1 2 0 3 2 0 3 1 3 2 1 0 |
| 0 2 3 1 3 2 1 0 1 3 2 0 0 3 1 2 |
| 0 2 3 2 1 1 2 0 3 1 0 3 1 2 0 3 |
| 0 3 1 2 2 1 1 3 0 3 0 1 2 0 3 2 |
| 0 3 1 2 3 0 3 1 1 2 3 2 0 2 0 1 |
| 0 3 2 1 1 3 3 2 2 1 1 0 3 0 0 2 |
| 0 3 2 1 3 1 0 1 3 0 2 3 2 1 2 0 |

| 31 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 0 2 2 3 3 1 1 3 3 1 1 2 2 |
| 0 0 0 2 1 3 0 2 3 3 1 3 2 2 1 1 |
| 0 1 2 1 0 0 3 2 0 1 3 2 2 3 1 3 |
| 0 1 2 3 0 2 0 3 3 2 2 1 1 0 3 1 |
| 0 1 3 0 3 0 2 2 2 3 0 1 3 1 2 1 |
| 0 1 3 2 2 3 2 0 1 0 1 2 1 3 3 0 |
| 0 2 1 3 0 3 2 1 0 2 1 3 0 1 2 3 |
| 0 2 1 3 1 2 3 0 2 0 3 1 3 2 1 0 |
| 0 2 3 1 3 2 0 1 1 3 2 0 0 3 1 2 |
| 0 2 3 3 2 1 1 2 3 1 0 0 1 2 0 3 |
| 0 3 1 2 2 1 3 1 0 3 0 1 2 0 3 2 |
| 0 3 1 2 3 0 1 3 1 2 3 2 0 2 0 1 |
| 0 3 2 1 1 3 2 3 2 1 1 0 3 0 0 2 |
| 0 3 2 1 3 1 1 0 3 0 2 3 2 1 2 0 |

| 32 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 1 0 2 3 3 1 2 3 3 1 1 2 2 |
| 0 0 1 0 2 3 2 3 0 3 1 1 2 3 1 2 |
| 0 1 0 2 2 0 0 3 3 1 2 3 3 2 1 1 |
| 0 1 2 3 0 0 1 2 0 3 3 2 2 1 3 1 |
| 0 1 3 2 0 1 2 2 3 0 1 0 1 3 2 3 |
| 0 1 3 3 3 2 2 0 2 2 0 1 3 1 1 0 |
| 0 2 1 0 3 0 3 2 2 1 3 1 1 2 0 3 |
| 0 2 2 3 1 3 0 2 1 2 1 3 0 0 1 3 |
| 0 2 3 1 2 3 1 0 3 1 0 2 1 0 3 2 |
| 0 2 3 1 3 2 0 1 1 3 2 0 2 3 0 1 |
| 0 3 1 2 3 2 1 3 1 0 1 2 0 2 3 0 |
| 0 3 1 3 2 1 3 1 2 0 0 3 2 0 2 1 |
| 0 3 2 1 1 3 2 1 0 1 3 0 3 2 2 0 |
| 0 3 2 2 1 1 3 0 3 3 2 1 0 1 0 2 |

| 33 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 1 1 2 3 3 0 1 3 3 1 2 2 2 |
| 0 0 1 0 3 3 2 3 3 2 1 1 2 0 1 2 |
| 0 1 1 3 2 0 3 3 2 0 1 2 0 2 3 1 |
| 0 1 2 3 0 2 2 1 1 0 0 3 3 3 1 2 |
| 0 1 3 2 3 2 0 2 0 1 2 1 3 0 3 1 |
| 0 1 3 3 3 1 2 0 2 2 3 0 1 1 2 0 |
| 0 2 0 3 2 1 0 2 0 3 1 3 2 1 1 3 |
| 0 2 1 2 0 0 1 2 3 0 3 1 1 3 2 3 |
| 0 2 3 1 1 0 2 3 1 3 2 0 2 3 0 1 |
| 0 2 3 1 2 3 1 0 3 1 0 2 3 2 1 0 |
| 0 3 1 2 0 3 3 1 2 1 2 3 2 1 0 0 |
| 0 3 2 0 2 3 0 1 1 3 3 2 1 0 2 1 |
| 0 3 2 1 3 1 3 2 1 2 0 1 0 2 0 3 |
| 0 3 2 2 1 2 1 0 3 3 1 0 0 1 3 2 |

| 34 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 1 1 2 3 3 0 1 3 3 1 2 2 2 |
| 0 0 1 2 3 0 0 2 3 3 1 2 1 1 2 3 |
| 0 1 1 3 0 3 3 2 2 1 0 2 3 2 0 1 |
| 0 1 2 0 3 2 3 0 1 3 2 1 0 2 1 3 |
| 0 1 3 0 3 3 2 1 0 2 1 3 2 0 2 1 |
| 0 1 3 2 2 0 1 1 3 0 3 0 3 2 1 2 |
| 0 2 0 3 2 1 0 3 1 3 3 2 2 0 1 1 |
| 0 2 1 3 1 3 2 0 3 0 2 1 1 0 3 2 |
| 0 2 3 1 0 2 1 3 3 2 0 1 2 1 0 3 |
| 0 2 3 2 2 1 3 0 2 1 1 3 0 1 3 0 |
| 0 3 1 1 3 0 2 3 2 1 2 0 2 3 1 0 |
| 0 3 2 1 2 2 0 2 0 0 1 1 3 3 3 1 |
| 0 3 2 2 0 1 2 1 1 3 0 3 1 3 0 2 |
| 0 3 2 3 1 3 1 2 1 2 3 0 0 1 2 0 |

| 35 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 0 1 1 2 3 3 0 1 3 3 1 2 2 2 |
| 0 0 1 2 3 0 1 2 3 0 1 3 2 1 2 3 |
| 0 1 1 3 0 3 3 1 2 0 3 2 2 2 0 1 |
| 0 1 2 0 3 2 0 3 1 3 1 2 0 2 1 3 |
| 0 1 3 1 3 0 2 0 2 2 3 0 3 1 1 2 |
| 0 1 3 2 1 3 0 2 0 3 2 1 3 0 2 1 |
| 0 2 0 3 2 1 2 0 0 3 1 3 2 3 1 1 |
| 0 2 1 0 0 3 2 3 3 2 1 1 1 0 3 2 |
| 0 2 3 1 3 1 3 2 1 1 0 2 2 0 3 0 |
| 0 2 3 3 2 2 0 1 3 1 2 0 1 1 0 3 |
| 0 3 1 2 2 0 3 3 2 1 2 1 0 3 1 0 |
| 0 3 2 1 2 2 1 0 3 0 0 1 3 2 3 1 |
| 0 3 2 2 0 1 1 2 1 3 3 0 1 3 0 2 |
| 0 3 2 3 1 3 2 1 1 2 0 3 0 1 2 0 |

| 36 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 1 0 1 3 3 0 2 3 3 1 2 2 2 |
| 0 0 1 3 3 2 2 2 0 0 1 1 2 1 3 3 |
| 0 1 0 3 2 2 0 3 2 1 0 3 3 1 1 2 |
| 0 1 1 2 2 0 3 0 3 0 2 1 3 3 2 1 |
| 0 1 3 2 0 3 1 2 0 3 1 2 0 3 1 2 |
| 0 1 3 2 3 0 2 1 2 1 3 0 1 2 0 3 |
| 0 2 0 0 2 3 0 2 1 3 3 1 1 2 3 1 |
| 0 2 2 0 1 0 3 1 3 3 1 3 2 1 0 2 |
| 0 2 3 1 0 2 1 3 3 1 2 0 2 0 3 1 |
| 0 2 3 1 3 1 2 0 1 3 0 2 3 1 2 0 |
| 0 3 1 2 3 3 0 1 1 2 2 3 2 0 1 0 |
| 0 3 2 1 1 3 3 2 2 1 0 1 0 0 2 3 |
| 0 3 2 3 1 2 1 0 3 0 3 2 1 2 1 0 |
| 0 3 2 3 2 1 2 3 1 2 1 0 0 3 0 1 |

| 37 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 2 0 0 1 3 2 3 3 3 1 1 2 2 |
| 0 0 2 3 2 3 2 3 1 1 1 2 0 1 0 3 |
| 0 1 0 2 3 1 3 2 1 0 1 3 2 3 0 2 |
| 0 1 0 3 1 2 3 3 0 3 2 1 0 2 2 1 |
| 0 1 1 2 3 3 0 1 3 1 0 2 3 2 2 0 |
| 0 1 3 3 2 3 1 1 0 2 3 0 2 0 1 2 |
| 0 2 2 1 1 0 3 0 3 3 1 2 3 0 1 2 |
| 0 2 3 1 0 3 1 2 3 0 2 1 1 2 0 3 |
| 0 2 3 1 1 2 0 3 1 2 0 3 2 1 3 0 |
| 0 2 3 2 3 1 2 0 2 1 3 1 0 0 3 1 |
| 0 3 1 0 2 0 3 2 3 1 2 0 2 1 3 1 |
| 0 3 1 1 3 2 2 0 0 2 1 0 1 3 2 3 |
| 0 3 2 0 0 2 2 1 1 0 3 3 3 2 1 1 |
| 0 3 2 3 2 1 0 2 2 3 0 1 1 3 1 0 |

| 38 |
| --- |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 2 0 0 2 3 1 2 3 3 1 1 2 3 |
| 0 0 2 1 1 3 2 3 0 3 1 1 2 3 0 2 |
| 0 1 0 1 0 3 3 1 3 0 2 3 2 2 2 1 |
| 0 1 0 3 3 2 2 2 1 0 0 1 3 1 3 2 |
| 0 1 2 2 3 2 1 0 3 1 1 0 0 3 2 3 |
| 0 1 3 3 1 2 0 3 0 2 3 2 0 2 1 1 |
| 0 2 1 0 3 1 3 0 3 2 3 1 2 0 1 2 |
| 0 2 2 3 0 0 1 2 2 3 1 3 3 0 1 1 |
| 0 2 3 1 2 0 1 3 3 1 0 2 2 1 3 0 |
| 0 2 3 1 3 1 0 2 1 3 2 0 1 2 0 3 |
| 0 3 1 2 2 3 0 0 2 3 2 1 0 1 3 1 |
| 0 3 1 3 2 1 3 2 0 0 1 2 1 3 2 0 |
| 0 3 2 0 2 3 2 1 1 1 3 0 3 2 1 0 |
| 0 3 3 2 1 2 3 1 2 1 0 3 1 0 0 2 |

| 39 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 2 0 1 2 3 0 1 3 3 1 2 2 3 |
| 0 0 2 3 1 3 0 2 3 3 1 2 1 1 2 0 |
| 0 1 0 1 3 3 3 1 0 2 1 3 2 0 2 2 |
| 0 1 1 3 3 0 3 2 2 1 0 2 3 2 0 1 |
| 0 1 2 0 3 3 2 0 1 3 2 1 0 2 1 3 |
| 0 1 3 2 1 2 0 1 3 0 3 0 3 2 1 2 |
| 0 2 0 3 0 2 1 3 1 3 3 2 2 0 1 1 |
| 0 2 1 3 2 1 3 0 3 0 2 1 1 0 3 2 |
| 0 2 3 1 1 0 2 3 3 2 0 1 2 1 0 3 |
| 0 2 3 2 3 2 1 0 2 1 1 3 0 1 3 0 |
| 0 3 1 1 2 3 0 3 2 1 2 0 2 3 1 0 |
| 0 3 2 1 0 2 2 2 0 0 1 1 3 3 3 1 |
| 0 3 2 2 2 0 1 1 1 3 0 3 1 3 0 2 |
| 0 3 3 0 2 1 3 2 1 2 3 0 0 1 2 1 |

| 40 |
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| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
| 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 0 0 1 2 0 1 2 3 0 1 3 3 1 2 2 3 |
| 0 0 2 3 2 0 3 1 3 3 1 2 1 1 2 0 |
| 0 1 0 2 2 3 0 1 3 1 0 3 2 3 1 2 |
| 0 1 1 0 3 3 3 3 1 2 1 2 0 2 0 2 |
| 0 1 2 3 0 3 1 2 2 0 3 1 1 0 3 2 |
| 0 1 3 1 1 3 2 0 0 3 2 0 3 1 2 2 |
| 0 2 0 3 3 1 2 0 1 3 3 2 2 0 1 1 |
| 0 2 1 3 1 2 0 3 3 0 2 1 3 2 1 0 |
| 0 2 3 1 0 1 3 2 3 2 0 1 0 3 2 1 |
| 0 2 3 2 3 2 1 0 2 1 1 3 0 1 3 0 |
| 0 3 1 1 2 0 1 3 2 3 0 0 2 2 3 1 |
| 0 3 2 1 3 2 0 1 0 2 3 1 2 1 0 3 |
| 0 3 2 2 1 0 3 2 1 1 2 3 3 0 0 1 |
| 0 3 3 0 2 2 2 2 1 0 1 0 1 3 1 3 |

## A3. GDD64.130 expressed as a signing of $J_{16}$ over $Z_2 \times Z_2$

Note that the 8 signings of GDD64.130 over $Z_2$ can be expressed as signings of $J_{16}$ over $Z_2^3$, and the 6 final signings are equivalent to signings of $J_{16}$ over $Z_2^4$.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| 0 | 0 | 0 | 0 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 2 |
| 0 | 0 | 0 | 0 | 3 | 3 | 3 | 2 | 1 | 1 | 1 | 3 | 2 | 2 | 2 | 1 |
| 0 | 1 | 2 | 3 | 0 | 2 | 3 | 0 | 1 | 2 | 3 | 2 | 0 | 1 | 3 | 1 |
| 0 | 1 | 2 | 3 | 1 | 3 | 2 | 1 | 3 | 0 | 1 | 0 | 3 | 2 | 0 | 2 |
| 0 | 1 | 2 | 3 | 2 | 0 | 1 | 3 | 2 | 1 | 0 | 3 | 1 | 0 | 2 | 3 |
| 0 | 1 | 2 | 3 | 3 | 1 | 0 | 2 | 0 | 3 | 2 | 1 | 2 | 3 | 1 | 0 |
| 0 | 3 | 1 | 2 | 0 | 1 | 2 | 3 | 3 | 1 | 2 | 0 | 0 | 3 | 2 | 1 |
| 0 | 3 | 1 | 2 | 1 | 0 | 3 | 2 | 1 | 3 | 0 | 2 | 3 | 0 | 1 | 2 |
| 0 | 3 | 1 | 2 | 2 | 3 | 0 | 0 | 0 | 2 | 1 | 1 | 1 | 2 | 3 | 3 |
| 0 | 3 | 1 | 2 | 3 | 2 | 1 | 1 | 2 | 0 | 3 | 3 | 2 | 1 | 0 | 0 |
| 0 | 2 | 3 | 1 | 0 | 3 | 1 | 3 | 2 | 3 | 1 | 2 | 0 | 2 | 1 | 0 |
| 0 | 2 | 3 | 1 | 1 | 2 | 0 | 2 | 0 | 1 | 3 | 0 | 3 | 1 | 2 | 3 |
| 0 | 2 | 3 | 1 | 2 | 1 | 3 | 0 | 1 | 0 | 2 | 3 | 1 | 3 | 0 | 2 |
| 0 | 2 | 3 | 1 | 3 | 0 | 2 | 1 | 3 | 2 | 0 | 1 | 2 | 0 | 3 | 1 |

## A4. Self-dual signings of $J_{16}$ over $Z_2^3$

### Group table for $Z_2^3$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 |
| 3 | 2 | 1 | 0 | 7 | 6 | 5 | 4 |
| 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 5 | 4 | 7 | 6 | 1 | 0 | 3 | 2 |
| 6 | 7 | 4 | 5 | 2 | 3 | 0 | 1 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

### The 4 signings

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 |
| 0 | 1 | 0 | 1 | 4 | 5 | 4 | 5 | 6 | 7 | 6 | 7 | 2 | 3 | 2 | 3 |
| 0 | 1 | 1 | 0 | 6 | 7 | 7 | 6 | 2 | 3 | 3 | 2 | 4 | 5 | 5 | 4 |
| 0 | 2 | 4 | 6 | 2 | 0 | 6 | 4 | 5 | 7 | 1 | 3 | 7 | 5 | 3 | 1 |
| 0 | 2 | 5 | 7 | 0 | 2 | 5 | 7 | 1 | 3 | 4 | 6 | 1 | 3 | 4 | 6 |
| 0 | 3 | 4 | 7 | 6 | 5 | 2 | 1 | 3 | 0 | 7 | 4 | 5 | 6 | 1 | 2 |
| 0 | 3 | 5 | 6 | 4 | 7 | 1 | 2 | 7 | 4 | 2 | 1 | 3 | 0 | 6 | 5 |
| 0 | 4 | 6 | 2 | 5 | 1 | 3 | 7 | 5 | 1 | 3 | 7 | 0 | 4 | 6 | 2 |
| 0 | 4 | 7 | 3 | 7 | 3 | 0 | 4 | 1 | 5 | 6 | 2 | 6 | 2 | 1 | 5 |
| 0 | 5 | 6 | 3 | 1 | 4 | 7 | 2 | 3 | 6 | 5 | 0 | 2 | 7 | 4 | 1 |
| 0 | 5 | 7 | 2 | 3 | 6 | 4 | 1 | 7 | 2 | 0 | 5 | 4 | 1 | 3 | 6 |
| 0 | 6 | 2 | 4 | 7 | 1 | 5 | 3 | 0 | 6 | 2 | 4 | 7 | 1 | 5 | 3 |
| 0 | 6 | 3 | 5 | 5 | 3 | 6 | 0 | 4 | 2 | 7 | 1 | 1 | 7 | 2 | 4 |
| 0 | 7 | 2 | 5 | 3 | 4 | 1 | 6 | 6 | 1 | 4 | 3 | 5 | 2 | 7 | 0 |
| 0 | 7 | 3 | 4 | 1 | 6 | 2 | 5 | 2 | 5 | 1 | 6 | 3 | 4 | 0 | 7 |

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 |
| 0 | 1 | 0 | 1 | 4 | 5 | 4 | 5 | 6 | 7 | 6 | 7 | 2 | 3 | 2 | 3 |
| 0 | 1 | 1 | 0 | 6 | 7 | 7 | 6 | 2 | 3 | 3 | 2 | 4 | 5 | 5 | 4 |
| 0 | 2 | 4 | 6 | 2 | 0 | 6 | 4 | 5 | 7 | 1 | 3 | 7 | 5 | 3 | 1 |
| 0 | 2 | 5 | 7 | 0 | 2 | 5 | 7 | 1 | 3 | 4 | 6 | 1 | 3 | 4 | 6 |
| 0 | 3 | 4 | 7 | 6 | 5 | 2 | 1 | 3 | 0 | 7 | 4 | 5 | 6 | 1 | 2 |
| 0 | 3 | 5 | 6 | 4 | 7 | 1 | 2 | 7 | 4 | 2 | 1 | 3 | 0 | 6 | 5 |
| 0 | 4 | 6 | 2 | 5 | 1 | 3 | 7 | 6 | 2 | 0 | 4 | 3 | 7 | 5 | 1 |
| 0 | 4 | 7 | 3 | 7 | 3 | 0 | 4 | 2 | 6 | 5 | 1 | 5 | 1 | 2 | 6 |
| 0 | 5 | 6 | 3 | 1 | 4 | 7 | 2 | 0 | 5 | 6 | 3 | 1 | 4 | 7 | 2 |
| 0 | 5 | 7 | 2 | 3 | 6 | 4 | 1 | 4 | 1 | 3 | 6 | 7 | 2 | 0 | 5 |
| 0 | 6 | 2 | 4 | 7 | 1 | 5 | 3 | 3 | 5 | 1 | 7 | 4 | 2 | 6 | 0 |
| 0 | 6 | 3 | 5 | 5 | 3 | 6 | 0 | 7 | 1 | 4 | 2 | 2 | 4 | 1 | 7 |
| 0 | 7 | 2 | 5 | 3 | 4 | 1 | 6 | 5 | 2 | 7 | 0 | 6 | 1 | 4 | 3 |
| 0 | 7 | 3 | 4 | 1 | 6 | 2 | 5 | 1 | 6 | 2 | 5 | 0 | 7 | 3 | 4 |

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 |
| 0 | 1 | 0 | 1 | 4 | 5 | 4 | 5 | 6 | 7 | 6 | 7 | 2 | 3 | 2 | 3 |
| 0 | 1 | 1 | 0 | 6 | 7 | 7 | 6 | 3 | 2 | 2 | 3 | 5 | 4 | 4 | 5 |
| 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 | 7 | 5 | 3 | 1 | 7 | 5 | 3 | 1 |
| 0 | 2 | 5 | 7 | 2 | 0 | 7 | 5 | 1 | 3 | 4 | 6 | 3 | 1 | 6 | 4 |
| 0 | 3 | 4 | 7 | 4 | 7 | 0 | 3 | 5 | 6 | 1 | 2 | 1 | 2 | 5 | 6 |
| 0 | 3 | 5 | 6 | 6 | 5 | 3 | 0 | 2 | 1 | 7 | 4 | 4 | 7 | 1 | 2 |
| 0 | 4 | 6 | 3 | 7 | 1 | 5 | 2 | 0 | 4 | 6 | 3 | 7 | 1 | 5 | 2 |
| 0 | 4 | 7 | 2 | 5 | 3 | 6 | 1 | 4 | 0 | 3 | 6 | 1 | 7 | 2 | 5 |
| 0 | 5 | 6 | 2 | 3 | 4 | 1 | 7 | 6 | 3 | 0 | 4 | 5 | 2 | 7 | 1 |
| 0 | 5 | 7 | 3 | 1 | 6 | 2 | 4 | 3 | 6 | 4 | 0 | 2 | 5 | 1 | 7 |
| 0 | 6 | 2 | 5 | 7 | 3 | 1 | 4 | 7 | 1 | 5 | 2 | 0 | 4 | 6 | 3 |
| 0 | 6 | 3 | 4 | 5 | 1 | 2 | 7 | 1 | 7 | 2 | 5 | 4 | 0 | 3 | 6 |
| 0 | 7 | 2 | 4 | 3 | 6 | 5 | 1 | 5 | 2 | 7 | 1 | 6 | 3 | 0 | 4 |
| 0 | 7 | 3 | 5 | 1 | 4 | 6 | 2 | 2 | 5 | 1 | 7 | 3 | 6 | 4 | 0 |

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 |
| 0 | 1 | 0 | 1 | 4 | 5 | 4 | 5 | 6 | 7 | 6 | 7 | 2 | 3 | 2 | 3 |
| 0 | 1 | 1 | 0 | 6 | 7 | 7 | 6 | 3 | 2 | 2 | 3 | 5 | 4 | 4 | 5 |
| 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 | 7 | 5 | 3 | 1 | 7 | 5 | 3 | 1 |
| 0 | 2 | 5 | 7 | 2 | 0 | 7 | 5 | 1 | 3 | 4 | 6 | 3 | 1 | 6 | 4 |
| 0 | 3 | 4 | 7 | 4 | 7 | 0 | 3 | 5 | 6 | 1 | 2 | 1 | 2 | 5 | 6 |
| 0 | 3 | 5 | 6 | 6 | 5 | 3 | 0 | 2 | 1 | 7 | 4 | 4 | 7 | 1 | 2 |
| 0 | 4 | 6 | 3 | 7 | 1 | 5 | 2 | 1 | 4 | 6 | 2 | 0 | 7 | 3 | 5 |
| 0 | 4 | 7 | 2 | 5 | 3 | 6 | 1 | 4 | 1 | 2 | 6 | 7 | 0 | 5 | 3 |
| 0 | 5 | 6 | 2 | 3 | 4 | 1 | 7 | 6 | 2 | 1 | 4 | 3 | 5 | 0 | 7 |
| 0 | 5 | 7 | 3 | 1 | 6 | 2 | 4 | 2 | 6 | 4 | 1 | 5 | 3 | 7 | 0 |
| 0 | 6 | 2 | 5 | 7 | 3 | 1 | 4 | 0 | 7 | 3 | 5 | 1 | 4 | 6 | 2 |
| 0 | 6 | 3 | 4 | 5 | 1 | 2 | 7 | 7 | 0 | 5 | 3 | 4 | 1 | 2 | 6 |
| 0 | 7 | 2 | 4 | 3 | 6 | 5 | 1 | 3 | 5 | 0 | 7 | 6 | 2 | 1 | 4 |
| 0 | 7 | 3 | 5 | 1 | 4 | 6 | 2 | 5 | 3 | 7 | 0 | 2 | 6 | 4 | 1 |

*(Received 11/2/94)*