Vertex Disjoint Cycles for Star Free Graphs

Guantao Chen *
Department of Mathematics
North Dakota State University
Fargo, ND 58105

Lisa R. Markus
Department of Mathematics
Furman University
Greenville SC 29613

R. H. Schelp †
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152

Abstract

A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. A graph is $K_{1,r}$-free if it does not contain $K_{1,r}$ as an induced subgraph. In this paper, we find bounds on the minimum number of edges needed to ensure a $K_{1,r}$-free graph contains k vertex disjoint cycles. The bound on claw-free graphs is sharp.

1 Introduction

Throughout all of this paper, we will let p denote the number of vertices in a graph and let q denote the number of edges. For simplicity, we will call *vertex disjoint* cycles *disjoint cycles.* The following result, due to Pósa, gives a sufficient condition for a graph to have 2 disjoint cycles.

Theorem 1 ([7]) *Let G be a graph. If $q \geq 3p-5$, then G contains 2 disjoint cycles.*

This result is sharp, since the graph $K_3 + nK_1$ has $p = n+3$, $q = 3p-6$, and does not contain 2 disjoint cycles.

For claw-free graphs, Matthews proved the following.

Theorem 2 ([6]) *If G is a claw-free graph with $q \geq p+6$, then G contains 2 disjoint cycles.*

*This research was partially funded under NSA grant number MDA 904-94-H-2060 and NSF ND-EPSCoR

†This research was partially funded under NSF grant number DMS-9400530

This is sharp since the graph K_5 with a path of length n attached to any of its vertices has $q = p + 5$ and does not contain 2 disjoint cycles. For $K_{1,r}$-free graphs, Markus and Snevily proved the following.

Theorem 3 **([5])** *If G is a $K_{1,r}$-free graph with $r \geq 4$ and $q \geq p + 2r - 1$ then G contains 2 disjoint cycles.*

This is sharp as shown by the $K_{1,r}$-free graph $K_3 + (r-1)K_1$, which is $K_{1,r}$-free with $q = p + 2r - 2$ and does not contain 2 disjoint cycles.

For the case of finding k disjoint cycles in graphs, Erdős and Pósa proved the following.

Theorem 4 **([3])** *Let $k \geq 1$ and $p \geq 24k$. Then every graph with $q \geq (2k-1)(p-k)$ contains either k disjoint cycles or $G = K_{2k-1} + (p - 2k + 1)K_1$.*

In [4], Justensen proved the following result which was conjectured to be true by Erdős and Pósa.

Theorem 5 **([4])** *Let $k \geq 1$ and $p \geq 3k$. Then every graph with*

$$q > \max\{(2k-1)(p-k), (3k-1)(3k-2)/2 + p - 3k + 1\},$$

contains k disjoint cycles.

Recently, Bodlaender [1] showed that the problem of determining whether a graph G of order p has k vertex disjoint cycles and k edge disjoint cycles can be solved in $O(p)$ times for each fixed positive integer k. In this paper, we improve the above result for claw-free graphs. The following result is obtained.

Theorem 6 *Let G be a claw-free graph and $k \geq 1$. If*

$$q \geq p + (3k-1)(3k-4)/2 + 1,$$

then G contains k disjoint cycles.

In fact, we will completely characterize the claw-free graphs with n vertices and $n + (3k-1)(3k-4)/2$ edges which do not contain k-disjoint cycles by the following theorem. To do so, the following notation is needed.

Definition 1 *For any positive integer k, we say a graph $F \in \mathcal{F}_k$ if F is obtained from the complete graph K_{3k-1} by*

- *replacing an edge by a path;*
- *attaching disjoint paths to different vertices of K_{3k-1} such that different paths are attached to different vertices.*

Theorem 7 *Let G be a claw-free graph with $q \geq p + (3k-1)(3k-4)/2$. If G does not contains k disjoint cycles, then G has exactly $p + (3k-1)(3k-4)/2$ edges and $G \in \mathcal{F}_k$.*

In fact we will prove the following stronger result.

Theorem 8 *Let G be a claw-free graph with*

$$q \geq p + (3k-1)(3k-4)/2.$$

If G does not contain k disjoint cycles for which $k-1$ of them are triangles, then $G \in \mathcal{F}_k$.

For general $r \geq 4$, the following result is obtained.

Theorem 9 *Let G be a $K_{1,r}$-free. If*

$$q \geq p + 16rk^2,$$

then G contains k disjoint cycles.

We believe that $p + 16rk^2$ can be improved in the above theorem. The graphs in $\mathcal{F}_k$ and the following graph show that this bound cannot be lowered to something less than

$$\min\{p + (3k-1)(3k-4)/2 + 1,\ p + (2k-1)(k+r-3) - r + 1\}.$$

Let G be a graph obtained from $K_{2k-1} + (r-1)K_1$ and a path of length $p - (2k + r - 2)$ by identifying a vertex of $(r-1)K_1$ with one of the endvertices of the path. Clearly, G is $K_{1,r}$-free and does not contain k disjoint cycles and has $p + (2k-1)(k+r-3) - r$ edges.

2 Notation and Lemmas

Let G be a graph. For any subgraph H of G, we let $E(H)$ denote the edge set of H and $e(H) = |E(H)|$. If $H_1, H_2, \cdots, H_m$ are m vertex disjoint subgraphs of G, we let $E(H_1, H_2, \cdots, H_m)$ denote the set of edges with one endvertex in H_i and the other one in H_j for $1 \leq i \neq j \leq m$. Let

$$e(H_1, H_2, \cdots, H_m) = |E(H_1, H_2, \cdots, H_m)|.$$

In general, we let $\mathcal{P}(H_1, H_2, \cdots, H_m)$ be the set of paths $P[u, v]$ from one of H_i to another H_j $(i \neq j)$ and every internal vertex of $P[u, v]$ is not in $\cup_{1 \leq t \leq m} V(H_t)$. We let $\mathcal{P}_\ell(H_1, H_2, \cdots, H_m)$ denote a subset of $\mathcal{P}(H_1, H_2, \cdots, H_m)$ for which every path has length $\leq \ell$.

For any graph G, we let $S_i(G)$ denote the set of vertices of degree i and $S_{\geq i}(G) = \cup_{j \geq i} S_j(G)$. For each vertex u in a graph G, we let $N(u)$ denote the set of vertices which are adjacent to v and $N[u] = N(u) \cup \{u\}$. Further, we let $N_m(u) = \{v : 0 < dist(u, v) \leq m\}$ and $N_m[u] = N_m(u) \cup \{u\}$. We will generally follow the notation in [2]. The following lemmas will be needed in the proofs.

Lemma 1 *Let T be a tree with at least two vertices such that T contains no vertices of degree two. Let*

$$V_1(T) = \{\{u, v\} \ : \ d(u) = d(v) = 1, \ 0 < dist(u, v) \le 4\}.$$

Then

$$|S_1(T)| \le 3|V_1(T)|.$$

Proof: We use induction on the number of vertices of T. The result is clearly true for K_2. Now we suppose that T has n vertices. If for every end-vertex u there is an end-vertex v such that $dist(u, v) \le 4$, then clearly $2|V_1(T)| \ge |S_1(T)|$. Let

$$U = \{v \ : \ d(v) = 1, \ N_4(v) \cap S_1(T) = \emptyset\}.$$

Note that for every pair of distinct vertices u and $v \in U$ we have $N_2[u] \cap N_2[v] = \emptyset$. Now we construct a new tree, T^* from T by contracting $N_2[v]$ to a new vertex v^* for each $v \in U$. Since T has no vertices of degree two, $d_{T^*}(v^*) \ge 4$. Then $|S_{\ge 4}(T^*)| \ge |U|$. It is not difficult to see that

$$V_1(T) = V_1(T^*) \quad \text{and} \quad S_1(T^*) = S_1(T) - U.$$

To prove the Lemma we only need to show that

$$3|V_1(T^*)| \ge |S_1(T^*)| + |S_{\ge 4}(T^*)|.$$

Since every vertex $v \in S_1(T^*)$ has a vertex $w \in S_1(T^*)$ which is at distance no more than 4 from v, we have $2|V_1(T^*)| \ge |S_1(T^*)|$. Thus it is sufficient to show that

$$|V_1(T^*)| \ge |S_{\ge 4}(T^*)|.$$

Let T^{**} be the tree obtained from T^* by removing all vertices of degree one. The inequality $|V_1(T^*)| \ge |S_{\ge 4}(T^*)|$ follows from the following observations.

- $|S_1(T^{**})| \ge |S_{i \ge 3}(T^{**})|$ since T^{**} is a tree. Each vertex $v \in S_1(T^{**})$ has at least two vertices in $S_1(T^*)$ adjacent to it;
- Each vertex v having $d_{T^*}(v) \ge 4$ and $d_{T^{**}}(v) = 2$ is adjacent to at least two vertices in $S_1(T^*)$;
- Each vertex v having $d_{T^*}(v) \ge 4$ and $d_{T^{**}}(v) = 1$ is adjacent to at least three vertices in $S_1(T^*)$, which gives us three pairs of vertices in $V_1(T^*)$.

$\square$

Lemma 2 *Let G be a $K_{1,r}$-free graph of girth $g(G) \ge 5$. If G has two vertex disjoint cycles C_1 and C_2 such that*

$$e(C_1, C_2) \ge \max\{7, r\},$$

then G contains two vertex disjoint cycles C_1^* and C_2^* such that

$$|V(C_1^*)| + |V(C_2^*)| < |V(C_1)| + |V(C_2)|,$$

and,

$$V(C_1^* \cup C_2^*) \subseteq V(C_1 \cup C_2).$$

Proof: To the contrary, suppose that there are not two such cycles C_1^* and C_2^*. It is readily seen that both C_1 and C_2 must be induced cycles. Assume that

$$\begin{aligned} C_1: &\quad v_1v_2\cdots v_sv_1 \\ C_2: &\quad w_1w_2\cdots w_tw_1 \end{aligned}$$

Without loss of generality, let $v_1 \in C_1$ such that

$$|N(v_1)\cap C_2| = \max\{|N(v_i)\cap C_1|, |N(w_j)\cap C_2| \quad : \quad 1\le i\le s, 1\le j\le t\}.$$

Since G is $K_{1,r}$-free and girth $g(G)\ge 5$, $|N(v_1)\cap C_2|\le r-3$. We consider three cases according to the value of $|N(v_1)\cap C_2|$.

Case 1: $|N(v_1)\cap C_2| = 1$

From the maximality of $|N(v_1)\cap C_2|$, we have in this case

$$\begin{aligned} |N(v_i)\cap C_2| &\le 1 \quad \text{for every } v_i\in C_1 \text{ and} \\ |N(w_i)\cap C_1| &\le 1 \quad \text{for every } w_i\in C_2. \end{aligned}$$

Let $v_{i_1}w_{j_1}, v_{i_2}w_{j_2}, \cdots, v_{i_m}w_{j_m}$ be the edges between C_1 and C_2 with $1 = i_1 < i_2 < \cdots < i_m$ and $m\ge 7$. Without loss of generality, we may assume that $1 = w_1 < w_2$. If $w_3 > w_2$, then w_5 is either in $C_2(w_1, w_2)$, or $C(w_2, w_3)$, or $C(w_3, w_1)$. It is readily seen that in either of these cases, there are two cycles C_1^* and C_2^* with the desired properties. If $w_1 < w_3 < w_2$, then either $w_5 \in C(w_1, w_2)$ or $w_5\in C(w_2, w_1)$. In either case, it is readily seen that there are two cycles C_1^* and C_2^* with the desired properties.

Case 2: $|N(v_1)\cap C_2| = 2$

Assume $N(v_1)\cap C_2 = \{x, y\}$. By the maximality of $|N(v_1)\cap C_2|$, we have $|N(x)\cap(C_1-v_1)|\le 1$ and $|N(y)\cap(C_1-v_1)|\le 1$. Also, we have

$$\begin{aligned} e(C_1-v_1, C_2(x,y)) &\le 1, \\ e(C_1-v_1, C_2(y,x)) &\le 1, \end{aligned}$$

otherwise there are two cycles C_1^* and C_2^* with the desired properties. Thus,

$$e(C_1, C_2)\le e(C_1, C_2-\{x,y\}) + |N(x)\cap C_1| + |N(y)\cap C_2|\le 6,$$

a contradiction.

Case 3: $|N(v_1)\cap C_2|\ge 3$

In this case, we have $|N(w_i)\cap(C_1-v_1)|\le 1$ for every $w_i\in V(C_2)$. Since G is $K_{1,r}$-free, v_1 has at most $r-3$ neighbors in C_2, so that $e(C_1-v, C_2)\ge 3$. Note that if x, y are two distinct neighbors of C_1-v_1 in C_2, then $|N(v_1)\cap C_2(x,y)|\le 1$ and $|N(v_1)\cap C_2(y,x)|\le 1$. Thus the inequality $|N(v_1)\cap C_2|\ge 3$ implies that $e(C_1-v_1, C_2)\le 3$. So the following two equalities hold,

$$e(C_1-v_1, C_2) = 3, \quad |N(v_1)\cap V(C_2)| = 3,$$

which shows that $e(C_1, C-2) \leq 6$, a contradiction. $\square$

From the proof of the above lemma, it is not difficult to see the following generalization holds. Since the proof is straightforward following the proof of the above lemma, we leave the proof to the reader.

Lemma 3 *Let G be a $K_{1,r}$-free graph of girth $g(G) > 4m$ and C_1 and C_2 be two disjoint cycles of G. Let $\mathcal{P}^*_m[C_1, C_2]$ be a set of paths of length $\leq m$ with one end vertex in C_1 and the other one in C_2 and which are internally disjoint from $C_1 \cup C_2$. Then, if*

$$|\mathcal{P}^*_m(C_1, C_2)| \geq \max\{7, r\},$$

G contains two vertex disjoint cycles C_1^* and C_2^* such that

$$|V(C_1^*)| + |V(C_2^*)| < |V(C_1)| + |V(C_2)|.$$

Further,

$$V(C_1^* \cup C_2^*) \subseteq V(C_1 \cup C_2) \cup_{P \in \mathcal{P}^*_m(C_1, C_2)} V(P).$$

3 Proof of Theorem 8

We use induction on k. When $k = 1$, G has p vertices and at least $p-1$ edges. Then, if G contains no cycles, it is a tree, in fact a path, so that the result holds. Assume that the above theorem holds for $k-1$ and $k \geq 2$. Let G be a claw-free graph with p vertices and at least $p + (3k-1)(3k-4)/2$ edges and that G fails to contain k disjoint cycles of which $k-1$ of them are triangles. Without loss of generality, we assume that G is connected. Suppose that $G \notin \mathcal{F}_k$.

Since $k \geq 2$, G must contain a vertex of degree at least three. Therefore G contains a triangle.

Claim 1 *Let T be a triangle in G and $H = G - V(T)$. Then,*

$$\begin{aligned} &(1) \qquad & e(H) &\leq p - 3 + (3k-4)(3k-7)/2, \\ &(2) \qquad & e(T, H) &\geq 3(3k-4) = 9k - 12. \end{aligned}$$

Proof: To the contrary, suppose that $e(H) > p - 3 + (3k-4)(3k-7)/2$. By our induction hypothesis, H contains $k-1$ disjoint cycles for which $k-2$ of them are triangles. Thus G contains k disjoint cycles for which $k-1$ of them are triangles, a contradiction. $\square$

Claim 2 *Let $\Delta(G)$ denote the maximum degree of G. Then*

$$3k - 2 \leq \Delta(G) \leq 3k - 1.$$

In particular on each triangle of G there is a vertex u such that $d(u) \geq 3k - 2$.

Proof: Let $T = uvw$ be a triangle of G such that $d(u) \geq d(v) \geq d(w)$. Then by Claim 1

$$\Delta(G) \geq d(u) \geq 2 + \frac{1}{3}e(T, G - V(T)) \geq 3k - 2.$$

On the other hand suppose $d(x) \geq 3k$ for some vertex $x \in V(G)$. Recall, the Ramsey number $r(3,3) = 6$ and $G(N(x))$ contains no three independent vertices. Take any 6 vertices of $N(x)$. There must be a triangle T_1 here. Take any 6 vertices not including any vertex of T_1, there is another triangle T_2. Continuing in this way, we see that $G(N(x))$ contains $k-1$ vertex disjoint triangles T_1, T_2, $\cdots$, T_{k-1}. Since $d(x) \geq 3k$, $N(x) - \cup_{i=1}^{k-1} V(T_i)$ has at least three vertices. Since G is claw-free, there is an edge in $N(x) - \cup_{i=1}^{k-1} V(T_i)$, say yz. Let $T_k = xyz$. Then G has k disjoint triangles T_1, T_2, $\cdots$, T_k, a contradiction. □

Claim 3 *Let w be a vertex of G. If w is on a triangle, then $d(w) \geq 3k - 4$.*

Proof: Suppose w is on a triangle T. Then this claim follows from $e(T, G - V(T)) \geq 9k - 12$ and $\Delta(G) \leq 3k - 1$. □

In what follows, we will break the remainder of the proof into two cases depending on the value of $\Delta(G)$.

3.1 The maximum degree $\Delta(G) = 3k - 1$

Let x be a vertex of G such that $d(x) = \Delta(G) = 3k - 1$. Since $G(N(x))$ contains no three independent vertices and $r(3,3) = 6$, it follows as argued above, that $G(N(x))$ contains $k-2$ disjoint triangles T_1, T_2, $\cdots$, T_{k-2}. Let $W = N(x) - \cup_{1 \leq i \leq k-2} V(T_i)$. Then $|W| = 5$. Assume that $W = \{w_1, w_2, w_3, w_4, w_5\}$. If W contains a triangle, say $T = w_1w_2w_3$, and an edge w_4w_5 vertex disjoint from the triangle T, then G has k disjoint cycles

$$T_1, T_2, \cdots, T_{k-2}, T_{k-1} = w_1w_2w_3, T_k = xw_4w_5,$$

a contradiction.

It is not difficult to check that a graph of order 5 containing neither $K_3 \cup K_2$ nor three independent vertices is either a C_5 or $K_4 \cup K_1$.

If $W = C_5$, without loss of generality, we assume that $G(W) = w_1w_2w_3w_4w_5w_1$. If there is a vertex $v \notin N[x]$ such that $vw_1 \in E(G)$, then either $vw_5 \in E(G)$ or $vw_2 \in E(G)$ since G is claw-free. Without loss of generality, we assume that $vw_2 \in E(G)$. Then G has k disjoint cycles

$$T_1, T_2, \cdots, T_{k-1} = vw_1w_2, T_k = xw_3w_4,$$

a contradiction.

Thus $N(w_1) \subseteq N[x]$ which implies that $d(w_1) \leq 3k-3$. Similarly, $d(w_2) \leq 3k-3$. Let $T = xw_1w_2$. By Claims 1 and 2,

$$9k-12 \leq e(T, G-V(T)) \leq d(x)+d(w_1)+d(w_2)-6 \leq (3k-1)+2(3k-3)-6 = 9k-13,$$

a contradiction.

Thus $G(W) = K_4 \cup K_1$. Without loss of generality, we assume that w_5 is the isolated vertex in $G(W)$. If there are two vertices $y,\ z \in N(w_5) - N[x]$, then there is a triangle T_{k-1} with $V(T_{k-1}) \subseteq \{w_5,\ x,\ y,\ z\}$ since G is claw-free. Then G has k disjoint triangles

$$T_1,\ T_2,\ \cdots,T_{k-1},\ T_k = w_1w_2w_3,$$

a contradiction. Thus $|N(w_5) - N[x_5]| \leq 1$. Hence

$$d(w_5) \leq 1 + ((3k-1) - 4) = 3k - 4.$$

If w_5 is not on a triangle, then $d(w_5) = 2$. Since $G(N(x))$ contains no three independent vertices, $G(N(x) - w_5) = K_{3k-2}$. Thus $G(N[x] - w_5) = K_{3k-1}$. Since G contains no k disjoint cycles of which $k-1$ of them are triangles, $G - (N[x] - \{w_5\})$ is a forest and each tree of the forest is attached to $N[x] - w_5$ by one and only one vertex. It is readily seen that $G \in \mathcal{F}_k$, a contradiction.

Thus w_5 is on a triangle. By Claim 3, $d(w_5) = 3k - 4$. In particular, we have $N(w_5) \supseteq N[x] - W$. If $k = 2$, $d(w_5) = 2$. There is a triangle containing x and w_5. This triangle and a triangle in $G(W - w_5)$ shows that G contains two triangles, a contradiction. Thus $k \geq 3$, which gives $k - 2 \geq 1$. Let $T_{k-2} = u_1u_2u_3$. Considering the triangle xu_1w_5, we have $d(u_1) = \Delta(G) = 3k - 1$.

If $|N(u_1) - N[x]| \geq 3$, say $v_1,\ v_2,\ v_3 \in N(u_1) - N[x]$. Then there is a triangle T^*_{k-2} in $G(\{u_1,\ v_1,\ v_2,\ v_3\})$, which shows that G has k disjoint triangles

$$T_1,\ T_2,\ \cdots,T_{k-3},\ T^*_{k-2},\ T_{k-1} = xu_2u_3,\ T_k = w_1w_2w_3,$$

a contradiction.

Hence $|N(u_1) - N[x]| \leq 2$, which gives us $|N(u_1) \cap N[x]| \geq 3k - 3$. In particular, $|N(u_1) \cap \{w_1,\ w_2,\ w_3,\ w_4\}| \geq 2$. Without loss of generality, assume $w_1,\ w_2 \in N(u_1)$. Then G has k disjoint triangles

$$T_1,\ T_2,\ \cdots,T_{k-3},T_{k-2} = u_1w_1w_2,\ T_{k-1} = xw_3w_4,\ T_k = w_5u_2u_3,$$

a contradiction.

3.2 The maximum degree $\Delta(G) = 3k - 2$

In this case, every vertex on a triangle has degree $\Delta(G) = 3k - 2$ by Claim 1. In particular, every vertex of degree > 2 has degree $3k-2$. Let x be an arbitrary vertex of G with $d(x) = \Delta(G) = 3k - 2$

Suppose that there is a vertex $y \in N(x)$ with $d(y) \leq 2$. Let $S = N[x] - y$. Then $G(S) = K_{3k-2}$ and $|N(v) - S| = 1$ for each $v \in S$. Since G is claw-free and $G(S)$ contains $k-1$ disjoint triangles, $G - S$ is a union of paths. Let s denote the number of components of $G - S$. Using the fact $(3k-2)(3k-3)/2 = (3k-1)(3k-4)/2 + 1$, we have

$$\begin{aligned} p + (3k-1)(3k-4)/2 \leq e(G) &= e(G(S)) + e(S, V - S) + e(G - S) \\ &\leq (3k-2)(3k-3)/2 + (3k-2) + (p - (3k-2) - 5) \\ &= p + (3k-1)(3k-4)/2 - (s-1). \end{aligned}$$

Thus $s = 1$. Hence $G - S$ is a path. Assume

$$G - S = v_1 v_2 \cdots v_m,$$

where $m = p - (3k - 2)$.

Since G is claw-free and every vertex in S has exactly one neighbor outside S, then $N(v_i) \cap S = \emptyset$ for each $2 \le i \le m - 1$. Thus $d(v_i) = 2$ for $i = 2, 3, \cdots, m - 1$. Further

$$N(v_1) \cup N(v_m) \supseteq S, \quad \text{and} \quad N(v_1) \cap N(v_m) \cap S = \emptyset.$$

Without loss of generality, assume that $|N(v_1) \cap S| \ge |N(v_m) \cap S|$. If $|N(v_m) \cap S| \ge 2$, say $N(v_1) \cap S \supseteq \{x_1, x_2\}$ and $N(v_m) \cap S \supseteq \{x_3, x_4\}$, then G contains triangles $v_1 x_1 x_2$, $v_m x_3 x_4$, and $k-2$ triangles in the complete subgraph $G(S - \{x_1, x_2, x_3, x_4\}) = K_{3k-6}$, a contradiction.

Thus we have either $N(v_1) \supseteq S$ and $N(v_m) \cap S = \emptyset$ or $|N(v_1) \cap S| = |S| - 1$ and $|N(v_m) \cap S| = 1$. It is readily seen that in either of these cases, we have $G \in \mathcal{F}_k$, a contradiction.

Thus every vertex of degree $3k - 2$ is not adjacent to the vertices of degree at most 2. Since we assume that G is connected, G is a $3k - 2$ regular graph. Let x be an arbitrary vertex of G and $y \in N(x)$ such that $t = |N(y) - N[x]|$ is maximum over all neighbors of x. Clearly, $G \in \mathcal{F}_k$ if $t = 0$. If $t \ge 2$, let v, w be two vertices in $N(y) - N[x]$. Since G is claw-free and the vertex x is also in $N(y)$, we have $vw \in E(G)$. Thus G has a triangle $T = yvw$. In the same manner as above, we see that G has $k - 1$ disjoint triangles such the vertices in $N[x] - \{y\}$, which implies that G has k disjoint triangles, a contradiction. Therefore we have $t = 1$. So there is at most a matching missed in the induced subgraph $G(N(x))$. Let w be the vertex in $N(y)$ but not in $N[x]$. If $|N(w) - N[x]| \ge 2$, let u, $v \in N(w) - N[x]$. Since y is also in $N(w)$ and y is not adjacent to either u or v, $uv \in E(G)$. Then G has a triangle uvw. In the same manner as before, it is not difficult to show that $N[x]$ contains $k - 1$ disjoint triangles, so G has k disjoint triangles, a contradiction. Thus $|N(w) - N[x]| \le 1$. Note that $wx \notin E(G)$, we have $|N(w) \cap N(x)| \ge |N(x)| - 1$ since $d(x) = 3k - 2$. Since $t = 1$ and G is $3k - 2$ regular, each vertex $z \in N(w) \cap N(x)$ has exactly one nonadjacent vertex in $N(x)$.

If $|N(w) - N[x]| = 1$, then the vertex set $N(w) \cap N(x)$ induces a clique since G is claw-free, a contradiction. Thus $N(w) = N(x)$ and there is exactly a 1-factor missed in the subgraph induced by $N(x)$. Using the Ramsey number $r(3,3) = 6$ in the same manner as before, we can show that $N(x)$ can be partitioned into $k - 2$ disjoint triangles $T_1, T_2, \cdots, T_{k-2}$ and a set of 4 vertices, say $\{z_1, z_2, z_3, z_4\}$. Since there are at most two independent edges missed in the subgraph induced by $\{z_1, z_2, z_3, z_4\}$, the induced subgraph $G(\{z_1, z_2, z_3, z_4\})$ contains two independent edges, say $z_1 z_2$ and $z_3 z_4$. Thus G contains k disjoint triangles

$$T_1, T_2, \cdots, T_{k-2}, T_{k-1} = x z_1 z_2, T_k = w z_3 z_4,$$

a contradiction. □

3.3 Proof of Theorem 9

We will prove Theorem 9 by induction on k. It is clear that the result holds for $k = 1$. Assume that $k \geq 2$ and the result holds for $k-1$. To the contrary, we suppose that the result fails. Let G be a $K_{1,r}$-free graph on p vertices and at least $p + 16rk^2$ edges and not contain k disjoint cycles. Further we assume that the p is minimum with this property.

Claim 4 *G contains no vertices of degree ≤ 1. Every vertex of degree two lies on a triangle.*

Proof: If G contains a vertex of degree ≤ 1, we simply remove the vertex, which contradicts the minimality of p. If v is a vertex of degree 2, let u and w be two neighbors of v. If $uw \notin E(G)$, we remove the vertex v and add a new edge uw. The resulting graph contradicts the minimality of p. Thus v lies in a triangle uvw. □

Let v be an arbitrary vertex in G and S be a subset of $N(v)$. Since G is $K_{1,r}$-free, then the subgraph $G(S)$ induced by S must contain a cycle if $|S| \geq 2r-1$. Thus for every $S \subseteq N(v)$ containing no less than $(2r-1)m$ vertices, $G(S)$ contains at least m disjoint cycles. The following proof is broken into two parts.

3.4 G contains a cycle of length ≤ 16

Let C_1 be a cycle of G with the minimum length. Then $|V(C_1)| \leq 16$ and C_1 contains no chords. Let $H = G - V(C_1)$. Since G does not contain k disjoint cycles, $|N(v) \cap H| \leq (2r-1)(k-1) - 1$ otherwise $N(v) \cap H$ contains $k-1$ disjoint cycles, so G contains k disjoint cycles, a contradiction. Thus

$$e(H) \geq p + 16rk^2 - (|V(C_1)|(2r-1)(k-1) + |V(C_1)|) \geq (p - |V(C_1)|) + 16r(k-1)^2,$$

since $|V(C_1)| \leq 16$. By induction, H contains $k-1$ disjoint cycles, so G contains k disjoint cycles, a contradiction.

3.5 The girth $g(G) \geq 17$

By the induction hypotheses on k, we know G contains $k-1$ disjoint cycles. Let C_1, $C_2, \cdots, C_{k-1}$ be $k-1$ such cycles with

$$|V(C_1)| + |V(C_2)| + \cdots + |V(C_{k-1})|$$

minimum. Let $H = G - \cup_{1 \leq i \leq k-1} V(C_i)$. Then H is a forest. Since G contains no triangle, the minimum degree $\delta(G) \geq 3$. Consider a component F induced by H and the edges of $E(\cup_{1 \leq i \leq k-1} C_i, H)$. Form a tree T^* from F as follows: if u is a vertex of both F and C_i (for some i) and $d_F(u) > 1$, then replace u in F by $d_F(u)$ new vertices of degree 1, each of them adjacent to a different vertex in

$N_F(u)$. Note that tree T^* does not contains a vertex of degree two. By Lemma 1, $|V_1(T^*)| \geq \frac{1}{3}|E(\cup_{1\leq i\leq k-1} C_i,\, T)|$. Hence

$$\begin{aligned} |\mathcal{P}_4(C_1,\, C_2,\, \cdots, C_{k-1})| &= |V_1(T^*)| + |E(C_1, C_2, \cdots, C_{k-1})| \\ &\geq \frac{1}{3}(e(G) - (\sum_{1\leq i\leq k-1} |V(C_i)| + e(H)) \geq \frac{16}{3} r k^2. \end{aligned}$$

Let

$$\mathcal{P}_4^*(C_i,\, C_j) = \mathcal{P}_4(C_1, C_2,\, \cdots, C_{k-1}) \cap \mathcal{P}_4(C_i,\, C_j),$$

that is, the subset of $\mathcal{P}_4(C_1, C_2, \cdots, C_{k-1})$ for which every path has an end vertex in C_i and the other one in C_j. By the Pigeonhole Principle, we can assume, without loss of generality, that

$$|P_4^*(C_1,\, C_2)| \geq \frac{32}{3} r \geq \max\{10,\, r\}.$$

By Lemma 3 for the case $m = 4$, since $g(G) > 16$, G contains two disjoint cycles C_1^* and C_2^* such that

$$|V(C_1^*)| + |V(C_2^*)| < |V(C_1)| + |V(C_2)|,$$

and

$$V(C_1^*) \cup V(C_2^*) \subset V(C_1) \cup V(C_2) \cup_{P \in P_4(C_1, C_2)} V(P),$$

which contradicts the minimality of

$$|V(C_1)| + |V(C_2)| + \cdots + |V(C_{k-1})|.$$

□

References

[1] Hans L. Bodlaender, *On disjoint cycles*, Graph-theoretic concepts in computer science (Fischbachau, 1991), 230-238, Lecture Notes in Comput. Sci., 570, Springer, Berlin, 1992.

[2] Bollobás B., *Graph Theory – An Introductory Course*, Springer-Verlag New York Inc. 1979.

[3] P. Erdős and L. Pósa, *On the maximal number of disjoint circuits of a graph*, Publ. Math. Debrecen 9(1962), 3-12.

[4] P. Justensen, *On independent circuits in finite graphs and a conjecture of Erdős and Pósa*, Graph Theory in Memory of G.A. Dirac (Sandbjerg, 1985), 299-305, Ann. Discrete Math., 41, North-Holland, Amsterdam-New York, 1989.

[5] L.R. Markus and H.S. Snevily, *Disjoint cycles in star-free graphs*, submitted.

[6] M. M. Matthews, *Extremal results for $K_{1,3}$-free graphs*, Congree. Numer. 49(1985) 49-55.

[7] L. Pósa, *On the circuits of finite graphs*, Magyar Tud. Akad. Mat. Kutato Int. Kozl. 8(1963), 355-361.

(Received 14/6/94)