Plain stress/ strain

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1. Relation between Lama Prarameters & Young's modules, Possion ratio

$$\mathcal{F}(\varepsilon) = \frac{1}{2}\int_\Omega \varepsilon:\sigma \, dx = \frac{1}{2}\int_\Omega \varepsilon : C : \varepsilon \, dx$$

where $\sigma = C:\varepsilon = \lambda \, tr(\varepsilon) I + 2\mu\,\varepsilon$

$$\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & & & \\ \lambda & 2\mu+\lambda & \lambda & & & \\ \lambda & \lambda & 2\mu+\lambda & & & \\ & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{01} \\ \varepsilon_{02} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{22} \\ \sigma_{01} \\ \sigma_{02} \\ \sigma_{12} \end{bmatrix}$$

use Sherman formula, $(A + uv^T) = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$

$$\Rightarrow \begin{bmatrix} a+b & b & b & & & \\ b & a+b & b & & & \\ b & b & a+b & & & \\ & & & a & & \\ & & & & a & \\ & & & & & a \end{bmatrix} \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{22} \\ \sigma_{01} \\ \sigma_{02} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{01} \\ \varepsilon_{02} \\ \varepsilon_{12} \end{bmatrix} \quad \text{where} \begin{cases} a = \frac{1}{2\mu} \\ b = \frac{-\lambda}{2\mu(2\mu+3\lambda)} \end{cases}$$

$$\text{Thus} \begin{cases} E = \frac{1}{a+b} = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda} \\ \nu = \frac{-b}{a+b} = \frac{\lambda}{2(\mu+\lambda)} \end{cases} \iff \begin{cases} \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \\ \mu = \frac{E}{2(1+\nu)} \end{cases}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & & & \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & & & \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & & & \\ & & & \frac{1}{2\mu} & & \\ & & & & \frac{1}{2\mu} & \\ & & & & & \frac{1}{2\mu} \end{bmatrix} \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{22} \\ \sigma_{01} \\ \sigma_{02} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{01} \\ \varepsilon_{02} \\ \varepsilon_{12} \end{bmatrix}$$

2D:

$$\begin{bmatrix} 2\mu+\lambda & \lambda & \\ \lambda & 2\mu+\lambda & \\ & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{01} \end{bmatrix} = \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{01} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+b & b & \\ b & a+b & \\ & & a \end{bmatrix} \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{01} \end{bmatrix} = \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{01} \end{bmatrix} \quad \text{where} \begin{cases} a = \frac{1}{2\mu} \\ b = \frac{-\lambda}{2\mu(2\mu+2\lambda)} \end{cases}$$

$$\text{Thus} \begin{cases} E = \frac{1}{a+b} = \frac{2\mu(2\mu+2\lambda)}{2\mu+\lambda} \\ \nu = \frac{-b}{a+b} = \frac{\lambda}{2\mu+\lambda} \end{cases} \iff \begin{cases} \lambda = \frac{E\nu}{(1+\nu)(1-\nu)} \\ \mu = \frac{E}{2(1+\nu)} \end{cases}$$

2. Plain Stress / Strain

$$\min_{\text{s.t. plain stress/strain}} \mathcal{F}_{3D}(\varepsilon_{3D}) = \min_{\text{s.t. plain stress/strain}} \frac{1}{2}\int_{\Omega_{3D}} \sigma_{3D} : \varepsilon_{3D} \, dx$$

$$= \min \frac{1}{2}\int_{\Omega_{2D}} \tilde{\sigma}_{2D} : \varepsilon_{2D} \, dx$$

↙ assuming no changes in z direction

$$= \min \tilde{\mathcal{F}}_{2D}(\varepsilon_{2D})$$

#1.

$$\sigma_{3D} = \begin{bmatrix} \sigma_{2D} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix}$$ ← plain stress constrain

$$\begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & & & \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & & & \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & & & \\ & & & \frac{1}{2\mu} & & \\ & & & & \frac{1}{2\mu} & \\ & & & & & \frac{1}{2\mu} \end{bmatrix} \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \cancel{\sigma_{22}} \\ \sigma_{01} \\ \cancel{\sigma_{02}} \\ \cancel{\sigma_{12}} \end{bmatrix} = \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{01} \\ \cancel{\varepsilon_{02}} \\ \cancel{\varepsilon_{12}} \end{bmatrix} \rightarrow \varepsilon_{22} = \frac{-\nu}{E}(\sigma_{00} + \sigma_{11})$$

so, $\varepsilon_{3D} = \begin{bmatrix} \varepsilon_{2D} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \varepsilon_{22} \end{bmatrix}$.

noticing that

$$\begin{bmatrix} 2\tilde{\mu}+\tilde{\lambda} & \tilde{\lambda} & \\ \tilde{\lambda} & 2\tilde{\mu}+\tilde{\lambda} & \\ & & 2\tilde{\mu} \end{bmatrix} \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{01} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{01} \end{bmatrix} = \begin{bmatrix} (2\mu+\lambda)\varepsilon_{00} + \lambda\varepsilon_{11} - \frac{\nu}{E}(\sigma_{00}+\sigma_{11}) \\ (2\mu+\lambda)\varepsilon_{11} + \lambda\varepsilon_{00} - \frac{\nu}{E}(\sigma_{00}+\sigma_{11}) \\ 2\mu\,\varepsilon_{01} \end{bmatrix}}_{\Downarrow}$$

$$\begin{bmatrix} 1+\frac{\nu}{E} & \frac{\nu}{E} & \\ \frac{\nu}{E} & 1+\frac{\nu}{E} & \\ & & 1 \end{bmatrix} \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{01} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \\ \lambda & 2\mu+\lambda & \\ & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{01} \end{bmatrix}$$

using sherman :

$$\Rightarrow \begin{bmatrix} 1-\frac{\nu}{E+2\nu} & \frac{-\nu}{E+2\nu} & \\ \frac{-\nu}{E+2\nu} & 1-\frac{\nu}{E+2\nu} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 2\mu+\lambda & \lambda & \\ \lambda & 2\mu+\lambda & \\ & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{01} \end{bmatrix} = \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{01} \end{bmatrix}$$

$$= \begin{bmatrix} 2\mu+\lambda-\frac{2\nu}{E+2\nu}(\mu+\lambda) & \lambda-\frac{2\nu}{E+2\nu}(\mu+\lambda) & \\ \lambda-\frac{2\nu}{E+2\nu}(\mu+\lambda) & 2\mu+\lambda-\frac{2\nu}{E+2\nu}(\mu+\lambda) & \\ & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{00} \\ \varepsilon_{11} \\ \varepsilon_{01} \end{bmatrix} = \begin{bmatrix} \sigma_{00} \\ \sigma_{11} \\ \sigma_{01} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \tilde{\mu} = \mu = \frac{E}{2(1+\nu)} \\ \tilde{\lambda} = \lambda - \frac{2\nu(\mu+\lambda)}{E+2\nu} = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{E}{E+2\nu} - \frac{E}{2(1+\nu)}\frac{2\nu}{E+2\nu} \\ \quad = \frac{2E\nu(E+2\nu-1)}{2(1+\nu)(1-2\nu)(E+2\nu)} = \lambda - \frac{E\nu}{(1+\nu)(1-2\nu)(E+2\nu)} \end{cases}$$

After solving $\min \tilde{\mathcal{F}}_{2D}(\varepsilon_{2D})$,

we should solve a $\min_{\varepsilon_{22}} \mathcal{F}_{3D}(\varepsilon_{3D})$ to get ε_{22}.

#2.

$$\varepsilon_{3D} = \begin{bmatrix} \varepsilon_{2D} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix}$$ ← plain strain constrain.

just solve $\min \tilde{\mathcal{F}}_{2D}(\varepsilon_{2D})$ is enough

and this σ_{2D} is not related to $\varepsilon_{2j}, \varepsilon_{i2}$ term

so it's really the same as 2D case!

Noticing that $\begin{cases} \tilde{\lambda} = \lambda_{3D} \\ \tilde{\mu} = \mu_{3D} \end{cases}$ is not the same as formula directly for 2D!