Codimensional Corotated Energy Note

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July 26, 2023

1 Energy

$$E(\mathbf{x}) = \sum_t \Psi(\mathbf{F}^t(\mathbf{x}))A^t \tag{1}$$

$$\begin{aligned}\Psi(\mathbf{F}(\mathbf{x})) &= \mu \operatorname{tr}\left[(\mathbf{F}-\mathbf{R}(\mathbf{F}))^T(\mathbf{F}-\mathbf{R}(\mathbf{F}))\right] + \frac{\lambda}{2}(\sqrt{\det(\mathbf{F}^T\mathbf{F})}-1)^2 \\ &= \mu \operatorname{tr}\left[\mathbf{F}^T\mathbf{F} - 2\mathbf{S} + \mathbf{I}\right] + \frac{\lambda}{2}(\sqrt{\det(\mathbf{F}^T\mathbf{F})}-1)^2\end{aligned} \tag{2}$$

where, $\mathbf{F} = \mathbf{R}(\mathbf{F})\,\mathbf{S}(\mathbf{F})$ is the polar deconposition, $\mathbf{R}^T\mathbf{R} = \mathbf{I} \in \mathbb{R}^{2\times 2}$. And $\mathbf{F}^t, \mathbf{F}, \mathbf{R} \in \mathbb{R}^{3\times 2}$, $\mathbf{S} \in \mathbb{R}^{2\times 2}$.

2 First-Piola-Kirchhoff Stress

$$\begin{aligned}\mathbf{P}(\mathbf{F}) &= \frac{\partial \Psi}{\partial \mathbf{F}}(\mathbf{F}) \\ &= \mu \frac{\partial}{\partial F_{\alpha\beta}}\left[F_{\gamma\delta}F_{\gamma\delta} - 2\mathrm{tr}(\mathbf{S})\right] + \lambda(\sqrt{\det(\mathbf{F}^T\mathbf{F})}-1)\frac{\partial \det(\mathbf{F}^T\mathbf{F})/\partial \mathbf{F}}{2\sqrt{\det(\mathbf{F}^T\mathbf{F})}} \\ &= 2\mu\mathbf{F} - 2\mu\frac{\partial \operatorname{tr}(\mathbf{S})}{\partial \mathbf{F}} + \lambda(J-1)\frac{\partial J}{\partial \mathbf{F}} \\ &= 2\mu(\mathbf{F}-\mathbf{R}) + \lambda(J-1)J\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-T}\end{aligned} \tag{3}$$

2.1 $\frac{\partial\, \mathbf{tr}(\mathbf{S})}{\partial \mathbf{F}}$

$$\begin{aligned}\delta\mathrm{tr}(\mathbf{S}) &= \operatorname{tr}(\delta\mathbf{S}) \\ &= \operatorname{tr}(\delta\mathbf{R}^T\mathbf{F}) + \operatorname{tr}(\mathbf{R}^T\delta\mathbf{F}) \\ &= \operatorname{tr}(\delta\mathbf{R}^T\mathbf{R}\cdot\mathbf{S}) + \operatorname{tr}(\mathbf{R}^T\delta\mathbf{F}) \\ &= R_{\alpha\beta}\delta F_{\alpha\beta}\end{aligned} \tag{4}$$

Thus,

$$\frac{\partial \operatorname{tr}(\mathbf{S})}{\partial \mathbf{F}} = \mathbf{R} \tag{5}$$

2.2 $\frac{\partial J}{\partial \mathbf{F}}$

$$\begin{aligned}
\frac{\partial J}{\partial \mathbf{F}} &= \frac{1}{2J}\frac{\partial J^2}{\partial \mathbf{F}^T\mathbf{F}} : \frac{\partial \mathbf{F}^T\mathbf{F}}{\partial \mathbf{F}} \\
&= \frac{1}{2J} J^2 (\mathbf{F}^T\mathbf{F})^{-1}_{\delta\gamma} : \frac{\partial (\mathbf{F}^T\mathbf{F})_{\gamma\delta}}{\partial \mathbf{F}_{\alpha\beta}} \\
&= \frac{J}{2}(\mathbf{F}^T\mathbf{F})^{-1}_{\beta\gamma} \cdot 2\mathbf{F}_{\alpha\gamma} \\
&= J\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-T}
\end{aligned} \tag{6}$$

3 Energy Denisity Hessian

$$\begin{aligned}
\mathbf{C}_{\alpha\gamma\beta\epsilon} &= \frac{\partial^2\Psi}{\partial \mathbf{F}_{\alpha\gamma}\partial \mathbf{F}_{\beta\epsilon}}(\mathbf{F}) = \frac{\partial \mathbf{P}_{\alpha\gamma}}{\partial \mathbf{F}_{\beta\epsilon}}(\mathbf{F}) \\
&= 2\mu(\delta_{\alpha\beta}\delta_{\gamma\epsilon} - \frac{\partial \mathbf{R}_{\alpha\gamma}}{\partial \mathbf{F}_{\beta\epsilon}}) + \lambda\frac{\partial J}{\partial \mathbf{F}_{\alpha\gamma}}(J\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-T})_{\beta\epsilon} + \lambda(J-1)\frac{\partial J(\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-T})_{\alpha\gamma}}{\partial \mathbf{F}_{\beta\epsilon}} \\
&= 2\mu(\mathbf{I} - \frac{\partial \mathbf{R}_{\alpha\gamma}}{\partial \mathbf{F}_{\beta\epsilon}}) + \lambda(J\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-T}) \otimes (J\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-T}) + \lambda(J-1)\frac{\partial J(\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-T})_{\alpha\gamma}}{\partial \mathbf{F}_{\beta\epsilon}}
\end{aligned} \tag{7}$$