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Agglomeration Patterns in a Long Narrow Economy of a New Economic Geography Model: Analogy to a racetrack economy*

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Abstract

Narrow industrial belts comprising a system of cities are prospering worldwide. The self-organization of a system of cities in a long narrow economy of a new economic geography model is demonstrated through a comparative study with a racetrack economy, which is an idealized uniform trading space. A spatially repeated core-periphery pattern *a la* Christaller and Lösch emerges when agglomeration forces are large. Peripheral zones of this pattern are enlarged recursively to engender agglomeration shadow en route to an atomic mono-center. A megalopolis emerges when agglomeration forces are small.

Keywords: Bifurcation, Break point, Long narrow economy, New economic geography, Racetrack economy, Spatial agglomeration, Transport cost

JEL classification: R12, R13, C65, F12

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1 Introduction

Narrow industrial belts serve as cradles of development and prosperity and engender megalopolises worldwide. A chain of cities, for example, is distributed from Boston to Washington, DC in a closed narrow zone between the Atlantic Ocean and the Appalachian Mountains. A megalopolis, such as New York in this chain, is growing as a core of the economic agglomeration.

A long narrow economy is a realistic spatial platform of a chain of cities. In the early development of NEG (new economic geography),[1] diverse agglomeration patterns of this economy were observed. A megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities* associated with lower transport costs was found by Mori (1997) [24]. A discretized highly regular central place system *a la* Christaller and Lösch[2] was observed when population size increases (Fujita and Mori, 1997 [15]). Yet not much development was made thereafter, while microeconomic structures of NEG models have been well established.

The racetrack economy has come to be employed increasingly as a spatial platform. This economy presents an idealized uniform trading space along the circumference of a circle with equal competition between places. Emergence of discrete agglomerations out of the uniformity was demonstrated. A spatial alternation of a core place with a larger population and a periphery place with zero population *a la* Christaller and Lösch (cf., Section 3) was found considering the hierarchy of different industries in Tabuchi and Thisse (2011) [30]. The emergence of this pattern was explained by recurrent bifurcations in Ikeda, Akamatsu, and Kono (2012) [17], and Akamatsu, Takayama, and Ikeda (2012) [2]. Anas (2004) [3] demonstrated the presence of balanced agglomeration, concentrated agglomeration, and de-agglomeration by removing agriculture and treating congestion and prices of land and labor as the main dispersion forces.

A comparative study of the long narrow economy and the racetrack economy was conducted by Mossay and Picard (2011) [25] for Beckman's CBD formation model (1976) [6] in a continuous space to display the difference in agglomeration patterns: a single city in a long narrow economy and multiple equilibria with an odd number of cities in the racetrack economy. This study, however, was conducted without the stability analysis of equilibria and, therefore, its validity requires further verification. In contrast, as clarified in this paper (Section 4.2), a similarity between the two economies can be observed in agglomeration patterns with high spatial regularity, in agreement with the underlying intuitive belief that both economies would tend to behave similarly as the

[1] The evolution of spatial agglomeration was studied by Krugman (1991) [21], leading to the subsequent development of NEG models various kinds (see Baldwin et al., 2003 [5] for review).

[2] For central place theory, see Christaller (1933) [8] and Lösch (1940) [23].

number of places increases.

That said, the objective of this paper is to answer the question if an NEG model in a discretized long narrow economy is capable of systematically producing the two characteristic agglomeration patterns that hitherto have been observed fragmentarily:

(1) A chain of spatially repeated core–periphery pattern *a la* Christaller and Lösch.

(2) A megalopolis which consists of large core cities that are connected by an industrial belt.

It is desirable to set forth principles underpinning the self-organization of such patterns, just as Christaller's distributions (1933) [8] were founded on geometrical principles. A study of these patterns would give a hint at the economic implication of *agglomeration shadow*.[3]

In this paper, we investigate the agglomeration patterns of long narrow economy with multiple equally spaced discrete places on a line segment. In this economy, the agglomeration progresses gradually without undergoing bifurcation and the onset of agglomeration is not clear, unlike in the two-place and racetrack economies. The *racetrack economy analogy* is introduced to describe the emergence of the chain of spatially repeated core–periphery pattern. There is a trade-off in that the racetrack economy is not realistic but analytically tractable, while the long narrow economy is realistic but is less analytically tractable. It is the basic strategy of this paper to interpret and describe agglomeration characteristics in the latter economy based on the theoretical information of the former economy.

Real economic activities allow models of various kinds that entail diverse agglomerations (e.g., Pflüger and Südekum, 2008 [27]; Berliant and Yu, 2014 [7]). This paper employs the model by Forslid and Ottaviano (2003) [12] in favor of its analytical solvability, which plays a key role in deriving a formula for the occurrence of bifurcation in the racetrack economy. There are two kinds of workers: unskilled workers are immobile and equally distributed along places, whereas skilled ones (footloose entrepreneurs) migrate between places to maximize utility. Nowadays, the immobile workers can be interpreted as a population attached to certain amenities or to traditional housing. The replicator dynamics is employed as the standard case, while the logit dynamics is used for comparison.

That question on possible agglomeration patterns is answered by demonstrating their dependence on agglomeration forces. When agglomeration forces are large, a spatially

[3] Arthur (1990) [4] stated: "Locations with large numbers of firms therefore cast an 'agglomeration shadow' in which little or no settlement takes place. This causes separation of the industry." See also Fujita, Krugman, and Venables (1999) [14], Ioannides and Overman (2004) [20], and Fujita and Mori (2005) [16].

repeated core–periphery pattern is shown to emerge, a megalopolis takes place when they are small. The value of the transport cost at the onset of bifurcation, which was called *break point* in the two-place economy (Fujita, Krugman, and Venables, 1999 [14]), is highlighted as a key concept. This point is analytically predicted in the racetrack economy and is successfully employed to index the emergence of the core–periphery pattern in the long narrow economy.

This paper is organized as follows. The analytically solvable model is presented in Section 2. Bifurcation of the racetrack economy is described in Section 3. Spatial agglomeration in the long narrow economy is investigated based on the racetrack economy analogy in Section 4. Section 5 observes the agglomeration patterns in the logit dynamics.

2 Model of spatial economy

An analytically solvable, multi-regional, new economic geography model, which replaces the production function of Krugman (1991) [21] with that of Flam and Helpman (1987) [11], is presented on the basis of Forslid and Ottaviano (2003) [12], as well as Akamatsu and Takayama (2009) [1] and Ikeda et al. (2014) [19]. Basic assumptions are presented in Section 2.1. Market equilibrium is introduced in Section 2.2 and spatial equilibrium in Section 2.3.

2.1 Basic assumptions

The economy of this model is composed of n places (labeled $i = 1, \dots, n$), two factors of production (skilled and unskilled labor) and two sectors (manufacturing, M, and agriculture, A). Both N skilled and N^{u} unskilled workers consume two types of final goods: manufacturing sector goods and agricultural sector goods. Workers supply one unit of each type of labor inelastically. Skilled workers are mobile among places, and the number of skilled workers in place i is denoted by λ_i ($\sum_{i=1}^{n} \lambda_i = N$). The total number N of skilled workers is normalized as $N = 1$. Unskilled workers are immobile and equally distributed across all places with unit density (i.e., $N^{\mathrm{u}} = 1 \times n$).

Preferences U over the M- and A-sector goods are identical across individuals. The utility of an individual in place i is

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \log C_i^{\mathrm{M}} + (1-\mu) \log C_i^{\mathrm{A}} \qquad (0 < \mu < 1), \tag{2.1}$$

where μ is a constant parameter expressing the expenditure share of manufacturing sector goods, C_i^{A} is the consumption of the A-sector product in place i, and C_i^{M} is the manufacturing aggregate in place i, which is defined as

$$C_i^{\mathrm{M}} \equiv \left(\sum_{j=1}^{n} \int_0^{n_j} q_{ji}(\ell)^{(\sigma-1)/\sigma} d\ell \right)^{\sigma/(\sigma-1)}, \tag{2.2}$$

where $q_{ji}(\ell)$ is the consumption in place i of a variety $\ell \in [0, n_j]$ produced in place j, n_j is the number of produced varieties at place j, and $\sigma > 1$ is the constant elasticity of substitution between any two varieties. The budget constraint is given as

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_{j=1}^{n} \int_0^{n_j} p_{ji}(\ell) q_{ji}(\ell) d\ell = Y_i, \tag{2.3}$$

where p_i^{A} is the price of A-sector goods in place i, $p_{ji}(\ell)$ is the price of a variety ℓ in place i produced in place j, and Y_i is the income of an individual in place i. The incomes (wages) of skilled workers and unskilled workers are represented, respectively, by w_i and w_i^{u}.

An individual in place i maximizes the utility in (2.1) subject to the budget constraint in (2.3). This yields the following demand functions:

$$C_i^{\mathrm{A}} = (1-\mu)\frac{Y_i}{p_i^{\mathrm{A}}}, \quad C_i^{\mathrm{M}} = \mu\frac{Y_i}{\rho_i}, \quad q_{ji}(\ell) = \mu\frac{\rho_i^{\sigma-1}Y_i}{p_{ji}(\ell)^{\sigma}}, \tag{2.4}$$

where ρ_i denotes the price index of the differentiated products in place i, which is

$$\rho_i = \left(\sum_{j=1}^{n}\int_0^{n_j} p_{ji}(\ell)^{1-\sigma} d\ell\right)^{1/(1-\sigma)}. \tag{2.5}$$

Since the total income in place i is $w_i\lambda_i + w_i^{\mathrm{u}}$, the total demand $Q_{ji}(\ell)$ in place i for a variety ℓ produced in place j is given by

$$Q_{ji}(\ell) = \mu\frac{\rho_i^{\sigma-1}}{p_{ji}(\ell)^{\sigma}}(w_i\lambda_i + w_i^{\mathrm{u}}). \tag{2.6}$$

The A-sector is perfectly competitive and produces homogeneous goods under constant-returns-to-scale technology, which requires one unit of unskilled labor per unit output. A-sector goods are transported without transportation cost and are chosen as the numéraire. In equilibrium, we have $p_i^{\mathrm{A}} = w_i^{\mathrm{u}} = 1$ for each i.

The M-sector output is produced under increasing-returns-to-scale technology and Dixit-Stiglitz monopolistic competition. A firm incurs a fixed input requirement of α units of skilled labor and a marginal input requirement of β units of unskilled labor. An M-sector firm located in place i chooses $(p_{ij}(\ell) \mid j = 1, \ldots, n)$ that maximizes its profit

$$\Pi_i(\ell) = \sum_{j=1}^{n} p_{ij}(\ell)Q_{ij}(\ell) - (\alpha w_i + \beta x_i(\ell)), \tag{2.7}$$

where $x_i(\ell)$ is the total supply of variety ℓ produced in place i and $(\alpha w_i + \beta x_i(\ell))$ is the cost function by Flam and Helpman (1987) [11].

The transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from place i to place j $(\neq i)$, only a fraction $1/T_{ij} < 1$ actually arrives $(T_{ii} = 1)$. Consequently, we have

$$x_i(\ell) = \sum_{j=1}^{n} T_{ij}Q_{ij}(\ell). \tag{2.8}$$

It is assumed that $T_{ij} = T_{ij}(\tau)$ is a function in a transport cost parameter τ.

Then the profit function of an M-sector firm in place i, given in (2.7) above, can be rewritten as

$$\Pi_i(\ell) = \sum_{j=1}^{n} p_{ij}(\ell)Q_{ij}(\ell) - \left(\alpha w_i + \beta \sum_{j=1}^{n} T_{ij}Q_{ij}(\ell)\right), \tag{2.9}$$

which is maximized by the firm. The first-order condition for this profit maximization yields

$$p_{ij}(\ell) = \frac{\sigma\beta}{\sigma - 1}T_{ij}. \tag{2.10}$$

This implies that $p_{ij}(\ell)$, as well as $Q_{ij}(\ell)$ and $x_i(\ell)$, does not depend on ℓ. Therefore, argument ℓ is suppressed in the sequel.

2.2 Market equilibrium

In the short run, skilled workers are immobile between places, i.e., their spatial distribution $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is assumed to be given. The market equilibrium conditions consist of three conditions: the M-sector goods market clearing condition, the zero-profit condition due to the free entry and exit of firms, and the skilled labor market clearing condition. The first condition is written as (2.8) above. The second requires that the operating profit of a firm, given in (2.7), be absorbed entirely by the wage bill of its skilled workers. This gives

$$w_i = \frac{1}{\alpha}\left\{\sum_{j=1}^{n} p_{ij}Q_{ij} - \beta x_i\right\}. \tag{2.11}$$

The third condition is expressed as

$$\alpha n_i = \lambda_i. \tag{2.12}$$

The price index ρ_i in (2.5) can be rewritten as

$$\rho_i = \frac{\sigma\beta}{\sigma - 1}\left(\frac{1}{\alpha}\sum_{j=1}^{n} \lambda_j d_{ji}\right)^{1/(1-\sigma)} \tag{2.13}$$

by (2.10) and (2.12). Here d_{ji} is a spatial discounting factor between places j and i, being defined by

$$d_{ji} = T_{ji}^{1-\sigma}. \tag{2.14}$$

The market equilibrium wage w_i in (2.11) can be represented as

$$w_i = \frac{\mu}{\sigma} \sum_{j=1}^{n} \frac{d_{ij}}{\Delta_j}(w_j \lambda_j + 1) \tag{2.15}$$

with the use of (2.6), (2.8), (2.10), (2.13), and (2.14). Here, $\Delta_j = \sum_{k=1}^{n} d_{kj}\lambda_k$ denotes the market size of the M-sector in place j. The indirect utility v_i is obtained as

$$v_i = \frac{\mu}{\sigma - 1} \log \Delta_i + \log w_i. \tag{2.16}$$

The equation (2.15) is solvable for w_i as follows. With the notation

$$\begin{cases} \boldsymbol{w} = (w_i), \quad D = (d_{ij}), \quad \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_n), \\ \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \quad \mathbf{1} = (1, \ldots, 1)^\top, \quad I = \mathrm{diag}(1, \ldots, 1), \end{cases}$$

the equation (2.15) can be written as

$$\boldsymbol{w} = \frac{\mu}{\sigma} D\Delta^{-1}(\Lambda \boldsymbol{w} + \mathbf{1}), \tag{2.17}$$

which is solved for $\boldsymbol{w}$ as

$$\boldsymbol{w} = \frac{\mu}{\sigma}\left(I - \frac{\mu}{\sigma} D\Delta^{-1}\Lambda\right)^{-1} D\Delta^{-1}\mathbf{1}. \tag{2.18}$$

The use of (2.18) in (2.16) gives the indirect utility function vector $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\lambda}, \tau) = (v_1(\boldsymbol{\lambda}, \tau), \ldots, v_n(\boldsymbol{\lambda}, \tau))^\top$. Note that the values of some parameters, such as α and β, are not influential on $\boldsymbol{v}$.

2.3 Spatial equilibrium and stability

We introduce the spatial equilibrium, for which high skilled workers are allowed to migrate among cities. A customary way to define such an equilibrium is to consider the following problem: Find $(\boldsymbol{\lambda}^*, \hat{v})$ satisfying

$$\begin{cases} (v_i - \hat{v})\lambda_i^* = 0, \quad \lambda_i^* \geq 0, \quad v_i - \hat{v} \leq 0, \quad (i = 1, \ldots, n), \\ \sum_{i=1}^{n} \lambda_i^* = 1. \end{cases} \tag{2.19}$$

For the solution of this problem, $\hat{v}$ serves as the highest (indirect) utility. When the system is in a *spatial equilibrium*, no individual can improve his/her utility by changing his/her location unilaterally.

We march on to consider the stability of the spatial equilibrium satisfying (2.19) with the use of the replicator dynamics (Sandholm, 2010 [28]):

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \tau), \tag{2.20}$$

where $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = (F_i(\boldsymbol{\lambda}, \tau) \mid i = 1, \ldots, n)$, and

$$F_i(\boldsymbol{\lambda}, \tau) = (v_i(\boldsymbol{\lambda}, \tau) - \bar{v}(\boldsymbol{\lambda}, \tau))\lambda_i, \quad (i = 1, \ldots, n). \tag{2.21}$$

Here $\bar{v} = \sum_{i=1}^{n} \lambda_i v_i$ is the average utility.

In this paper, we would like to replace the problem to obtain a set of stable spatial equilibria by another problem to find a set of stable stationary points of the replicator dynamics, as guaranteed in Sandholm (2010) [28]. Stationary points (rest points) $\boldsymbol{\lambda}^*(\tau)$ of the replicator dynamics (2.20) are defined as those points which satisfy the static equilibrium equation

$$\boldsymbol{F}(\boldsymbol{\lambda}^*, \tau) = \mathbf{0}. \tag{2.22}$$

The conservation law of population $\sum_{i=1}^{n} \lambda_i^* = 1$ is always satisfied.

To define stability of the stationary points, we consider the Jacobian matrix

$$J(\boldsymbol{\lambda}^*, \tau) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*, \tau) \tag{2.23}$$

of the equilibrium equation.[4] We classify stability using the eigenvalues of this matrix:

$$\begin{cases} \text{linearly stable:} & \text{every eigenvalue has negative real part,} \\ \text{linearly unstable:} & \text{at least one eigenvalue has positive real part.} \end{cases}$$

If an equilibrium is linearly stable it is asymptotically stable, and if it is linearly unstable it is asymptotically unstable. Bifurcation takes place when the Jacobian matrix has one or more zero eigenvalues.

Remark 1. The logit dynamics (e.g., Fudenberg and Levine, 1998 [13]) is used for comparison later in Section 5. The skilled workers are assumed to be heterogeneous in their preferences for their location choice (Tabuchi and Thisse, 2002 [29]; Murata, 2003 [26]; and Akamatsu, Takayama, and Ikeda, 2012 [2]). A specific functional form

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = N\boldsymbol{P}(\boldsymbol{v}(\boldsymbol{\lambda}, \tau)) - \boldsymbol{\lambda} \tag{2.24}$$

is employed. Here $\boldsymbol{P}(\boldsymbol{v}) = (P_1, \ldots, P_n)^\top$ is the choice function vector and $N = 1$ by

[4] The spatial equilibrium $\boldsymbol{\lambda}^*$ in (2.19) is defined on $(n-1)$-dimensional simplex. The extendibility of the Jacobian matrix to the full n-dimension is presented in Sandholm (2010, Chapter 3) [28].

normalization. The conservation law $\sum_{i=1}^{n} \lambda_i = 1$ is satisfied. The logit choice function

$$P_i(\boldsymbol{v}) = \frac{\exp(\theta v_i)}{\sum_{j=1}^{n} \exp(\theta v_j)}, \tag{2.25}$$

is employed, where $\theta \in (0, \infty)$ denotes a parameter related to the magnitude of the heterogeneity. When $\theta \to \infty$, the skilled workers decide their location only by v_i, which corresponds to the case without heterogeneity.

3 Racetrack economy: idealistic spatial platform

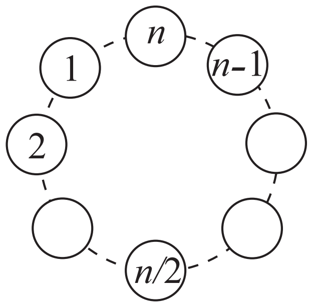


Figure 1: Racetrack economy as a spatial platform of economic activities

The *racetrack economy* shown in Fig. 1 is introduced as an idealized uniform space for economic activities, in which all places are given identical economic environments (Section 3.1). This economy with such uniformity is suitable for theoretical treatment by bifurcation theory, unlike the long narrow economy lacking in the uniformity. The bifurcation rules[5] for the racetrack economy are presented as a tool to describe spatial agglomerations in Section 3.2. Moreover, formulas for the break point, which gives the transport cost at the onset of bifurcation, are presented in Section 3.3 by the method in Akamatsu, Takayama, and Ikeda (2012) [2] for the logit dynamics adapted to the replicator dynamics. These rules and formulas play a pivotal role in the study of the long narrow economy based on the racetrack economy analogy in Section 4.2.

3.1 Racetrack economy

A series of n places (labeled $i = 1, \ldots, n$) is spread equally on the circumference of the circle in Fig. 1. Neighboring places are connected by a road of a length $d = 1/n$ and the total length $\mathcal{L}$ of roads is $\mathcal{L} = 1$. To be consistent with numerical examples in Section 4.2, the number n of places is assumed as below:

Assumption 1. *The number n of places is $n = 2^k$ $(k = 3, 4, \ldots)$.*

The transport cost T_{ij} between places i and j is defined as

$$T_{ij} = \exp\left(\tau\, m(i, j)\, \frac{\mathcal{L}}{n}\right), \tag{3.1}$$

where $\tau(> 0)$ is the transport cost parameter and $m(i, j) = \min\{|i - j|,\ n - |i - j|\}$.

[5]These rules are given as a summary of Ikeda, Akamatsu, and Kono (2012) [17], Akamatsu, Takayama, and Ikeda (2012) [2], and Ikeda and Murota (2014) [18].

The flat earth equilibrium

$$\boldsymbol{\lambda}^* = \frac{1}{n}(1, \dots, 1)^\top \tag{3.2}$$

exists for any values of the transport cost parameter τ. The spatial period L between neighboring places is equal to $L = \mathcal{L}/n = 1/n$, which is normalized as $L/d = 1$. The agglomeration from the flat earth equilibrium proceeds only via bifurcation.

3.2 Bifurcation rules

At a pitchfork bifurcation point,[6] a bifurcating equilibrium path exists in the direction of the eigenvector

$$\boldsymbol{\eta}^* = (1, -1, \dots, 1, -1)^\top \tag{3.3}$$

of the Jacobian matrix in (2.23). A bifurcating state has a population distribution of the form:

$$\boldsymbol{\lambda} = (1/n + a, 1/n - a, \dots, 1/n + a, 1/n - a)^\top, \quad -1/n \leq a \leq 1/n. \tag{3.4}$$

This represents a state in which concentrated places and extinguished places alternate along the circle leading to a spatially repeated core–periphery pattern *a la* Christaller and Lösch. This pattern is associated with the $k = 4$ system in one-dimension (cf., Section 4.1, Footnote 10 in particular). The spatial period between agglomerated place becomes $L/d = 2$ and is double to $L/d = 1$ of the flat earth equilibrium, as depicted in the left side of Fig. 2.

In the limit of agglomeration to core places, this state becomes

$$\hat{\boldsymbol{\lambda}} = \frac{1}{n}(2, 0, 2, 0, \dots, 2, 0, 2, 0)^\top, \tag{3.5}$$

from which a bifurcated equilibrium exists in the direction of the eigenvector

$$\hat{\boldsymbol{\eta}} = (1, 0, -1, 0, \dots, 1, 0, -1, 0)^\top. \tag{3.6}$$

Thereafter the racetrack economy with $n = 2^k$ ($k = 3, 4, \dots$) places can undergo a

[6] A pitchfork bifurcation point is either subcritical or supercritical. The bifurcation for the two-place economy is subcritical and is called tomahawk (Fujita, Krugman, and Venables, 1999 [14]), whereas those in the present study in Section 4.2 are supercritical.

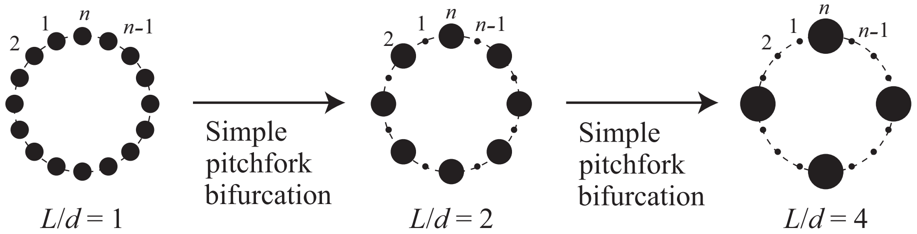


Figure 2: Spatial period doubling bifurcation cascade in the racetrack economy for 16 places (size of an area represents population size)

cascade of pitchfork bifurcations[7] leading to successive doublings of the spatial period:

$$\frac{L}{d} = 1 \longrightarrow 2 \longrightarrow 4 \longrightarrow \cdots \longrightarrow n. \tag{3.7}$$

This is called *spatial period doubling bifurcation cascade*. Figure 2 depicts this cascade for 16 places.

3.3 Law for break points

A decrease of the transport cost to the *break point*, at which symmetric places change catastrophically into a core–periphery pattern, is highlighted as a key concept in the two-place economy (Fujita, Krugman, and Venables, 1999 [14]). In this paper, this concept is extended to the period doubling cascade in (3.7) engendering core–periphery patterns, and the value of τ for the mth period doubling for n places is denoted by $\tau_{m,n}$. As for the secondary bifurcation, we employ the following assumption that is in agreement with the numerical results in Section 4.

Assumption 2. *The secondary bifurcation takes place from the state of* $\hat{\boldsymbol{\lambda}} = \frac{1}{n}(2, 0, 2, 0, \ldots, 2, 0, 2, 0)^{\top}$ *in* (3.5).

The eigenanalysis of the Jacobian matrix $J = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ is conducted analytically in A to derive Lemma 1 and Propositions 1 and 2 below.

Lemma 1. *The break point* $\tau_{m,n}$ *for the direct* ($m = 1$) *or the secondary* ($m = 2$) *pitchfork bifurcation exists when*

$$\frac{\mu}{\sigma - 1} < \frac{1}{m}, \quad m = 1, 2 \tag{3.8}$$

is satisfied.

[7]See Tabuchi and Thisse (2011) [30], Ikeda, Akamatsu, and Kono (2012) [17], and Akamatsu, Takayama, and Ikeda (2012) [2].

Proof. See (A.21) and (A.42) in the Appendix. □

These values of break points are analytically predictable up to the second period doubling by the formulas in the following proposition, while the associated formulas for the logit dynamics are presented in Remark 2.

Proposition 1. *When* (3.8) *with* $m = 2$ *is satisfied, the break points of the first and the second period doublings are given by*

$$\tau_{m,n} = c_m n, \quad m = 1, 2 \tag{3.9}$$

with constants

$$c_m \quad = \quad \frac{1}{\mathcal{L}m(\sigma - 1)} \log\left(\frac{1 + \sqrt{\epsilon_m^*}}{1 - \sqrt{\epsilon_m^*}}\right), \quad m = 1, 2. \tag{3.10}$$

Here $\mathcal{L} = 1$ *and*

$$\epsilon_1 = \frac{\kappa + \kappa'}{\kappa\kappa' + 1}, \quad \epsilon_2 = \frac{\kappa + \kappa'}{\kappa\kappa' + (1 + \kappa)/2}, \quad \kappa = \frac{\mu}{\sigma}, \quad \kappa' = \frac{\mu}{\sigma - 1}. \tag{3.11}$$

Proof. See (A.23) and (A.44) in the Appendix. □

By Assumption 1, we exclude the case of two place ($n = 2$); therefore, the formulas in Proposition 1 are not applicable to the two place. See (A.27) for the formula for the two place.

Since the theoretical formula (3.9) with (3.10) are complicated, the following approximate formula would be more pertinent in the discussion of economic implications of parameter dependence of break points.

Proposition 2. *Under the condition*

$$\frac{\sigma}{\mu} \gg 1, \tag{3.12}$$

we have a simplified formula for an approximate value $\hat{\tau}_{m,n}$ *of* $\tau_{m,n}$ *as*

$$\hat{\tau}_{m,n} = \frac{2^{2-m/2}\mu^{1/2}}{\mathcal{L}(\sigma - 1)^{3/2}}\, n, \quad m = 1, 2. \tag{3.13}$$

Proof. See (A.26) and (A.46) in the Appendix. □

Parameter dependence of break points $\tau_{m,n}$ can be seen from (3.13). A larger $\tau_{m,n}$ causes an earlier break bifurcation triggering agglomeration. Such agglomeration is accelerated by a smaller σ leading to greater differentiation of the variety of products or a larger μ enhancing the relative influence of the role of manufactured goods.

For higher bifurcations for $m \geq 3$, for which no analytical formulas are applicable, the following law for a semi-empirical value $\tilde{\tau}_{m,n}$ of $\tau_{m,n}$ is suggested for use.

$$\tilde{\tau}_{m,n} = \sqrt{\frac{8}{m!}} \frac{\mu^{1/2}}{(\sigma - 1)^{3/2}} n, \quad m = 1, 2, 3, 4. \tag{3.14}$$

This law encompasses theoretical law (3.13) for $m = 1$ and $m = 2$ and turns out to be in good agreement with the numerical analysis in Section 4.2.

Remark 2. For the logit dynamics, two break points exist conditionally. Between these two break points, a larger one is related to the breaking of uniformity. The theoretical formula (3.9) and the approximate formula (3.13) for this larger break point are obtained by replacing the parameters ϵ_1 and ϵ_2 in (3.11) by

$$\epsilon_m = \frac{b + \sqrt{b^2 - 4a_m\theta^{-1}}}{2a_m}, \quad m = 1, 2 \tag{3.15}$$

with

$$a_1 = \kappa\kappa' + 1, \quad a_2 = \kappa\kappa' + \frac{1+\kappa}{2}, \quad b = \kappa + \kappa' + \theta^{-1}\kappa. \tag{3.16}$$

The break points $\tau_{m,n}$ $(m = 1, 2)$ exist when the following conditions are satisfied:

$$\frac{\mu}{\sigma - 1} < \frac{1}{m} + \frac{1}{\theta}, \tag{3.17a}$$

$$b^2 - \frac{4a_m}{\theta} > 0. \tag{3.17b}$$

4 Long narrow economy: realistic spatial platform

A *long narrow economy* in Fig. 3 is employed as a realistic spatial platform, in which a series of $n + 1$ places (labeled $i = 0, 1, \ldots, n$) is evenly spread in a line segment with a length of 1 and neighboring places are connected by an inter-road of a length $d = 1/n$. This economy has the same length 1 and the same number n of inter-place roads as the racetrack economy in Section 3. The economy, unlike the racetrack economy, has spatial inhomogeneity due to the influence of borders because a border place (0 or n), which is connected only to one place, has an inferior transportation environment than places inside (1 through $n - 1$).

Figure 3: Long narrow economy for a spatial platform of economic activities

The spatial agglomeration patterns of this economy are observed through extensive numerical comparative static analyses with respect to the transport cost by solving non-linear static equilibrium equation (2.22) in Section 2. In order to investigate the influence of the borders, the number of places is chosen as 17 with a stronger influence and 65 with a weaker one. Spatial period enlargement in agglomeration engendering several patterns of interest is observed for the 17 places in Section 4.1. In Section 4.2, the emergence of a spatially repeated core–periphery pattern *a la* Christaller and Lösch (Fig. 2) for the 65 places is explained by the racetrack economy analogy. The onset of enlargement of spatial period between agglomerated places in the long narrow economy is investigated by the formulas of break points in the racetrack economy (Section 3.3).

The standard values[8] of the expenditure share μ of manufacturing sector goods and the constant elasticity σ of substitution between any two varieties are chosen as $(\mu, \sigma) = (0.4, 10.0)$, whereas the dependence of the agglomeration on these values is investigated from time to time. These parameter values satisfy the conditions (3.8) for the existence of the direct and the secondary bifurcations.

4.1 Spatial period enlargement: 17 places

Emergence of several agglomeration patterns of interest through spatial period enlargement is observed for 17 places.

[8]The values of $(\mu, \sigma) = (0.4, 10.0)$ were chosen with reference to $(\mu, \sigma) = (0.4, 5.0)$ used in Fujita et al. (1999) [14]. A larger value of σ was employed to strengthen dispersion force.

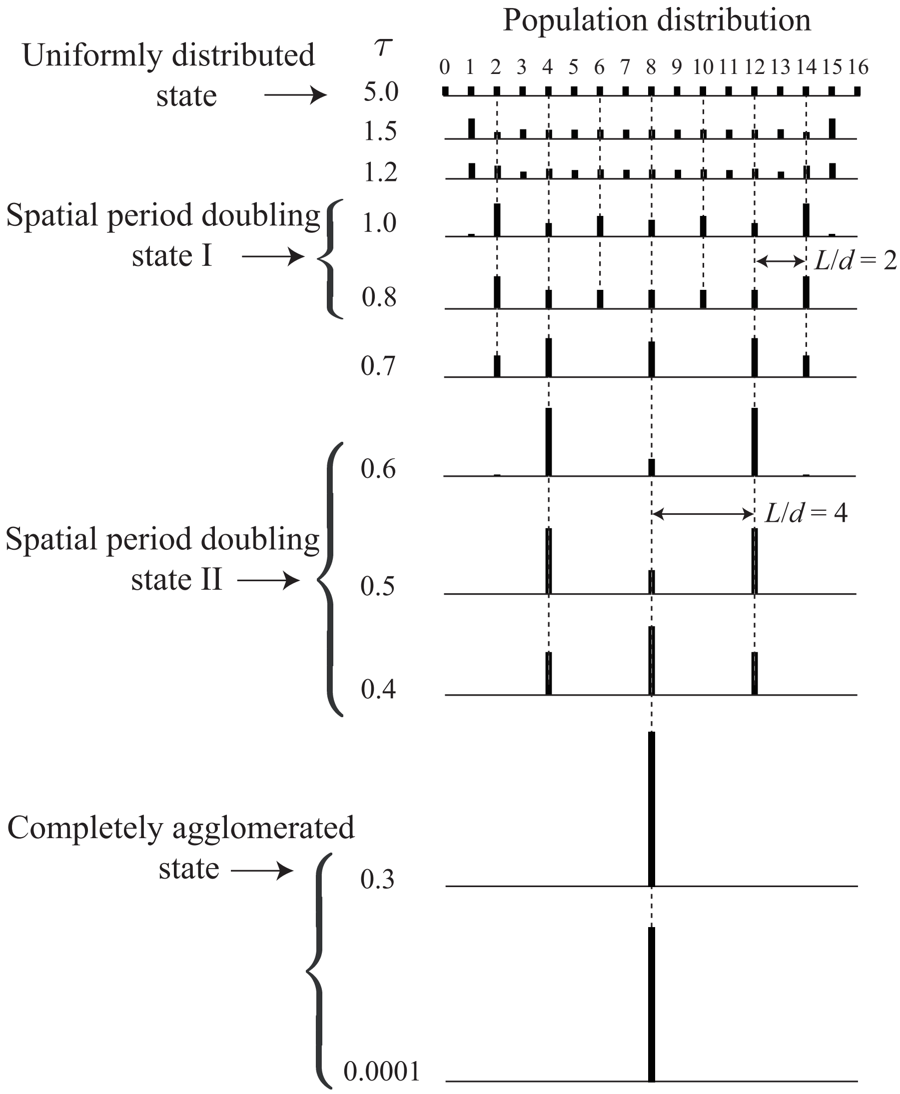


Figure 4: Spatial agglomeration of the long narrow economy with 17 places observed in association with the decrease of transport cost parameter τ

4.1.1 Spatially repeated core–periphery patterns

The evolution of population distributions observed with a decrease of τ is expressed in the bar charts in Fig. 4. *Almost uniformly distributed state* prevails until $\tau \approx 1.5$, at which border effects start to take effect and the populations at the border places 0 and 16 decrease and those in the neighboring places 1 and 15 increase. As τ decreases, the tradeoff between transport cost and scale economies engenders increased competition between places leading to the growth of several discretized places and the decay at neighboring places developing into agglomeration shadows. The number of agglomerated places decreases recurrently as $17 \rightarrow 7 \rightarrow 5 \rightarrow 3 \rightarrow 1$ en route to an atomic mono-center as explained below.

Spatial period doubling state I ($1.0 \geq \tau \geq 0.8$): The populations at odd numbered places $1, 3, \ldots, 15$ almost disappear; a core place and a periphery place alternate spatially to engender a spatially repeated core–periphery pattern. The spatial period between the agglomerated places is doubled to $L/d = 2$. Since an agglomerated place is surrounded by two periphery places, this pattern serves as a one-dimensional counterpart[9] of Christaller's $k = 4$ system (Christaller, 1933 [8]). Since the places are located on a straight road, the emergence of this system based on *traffic principle*[10] has a logical sequel.

Spatial period doubling state II ($0.6 \geq \tau \geq 0.4$): The population is agglomerated to three places 4, 8, and 12, thereby displaying another spatially repeated core–periphery pattern with enlarged peripheral zones. The spatial period between agglomerated places is doubled again to $L/d = 4$.

Spatially repeated core–periphery patterns thus observed are in line with Fujita and Mori (1997) [15], in which "a highly regular system *a la* Christaller" was found for a different microeconomic model. This observation, however, is in sharp contrast with the study of Mossay and Picard (2011) [25], in which the existence of multiple cities was denied. Such contrast might be due to the difference in working forces: interplay between spatial interaction externalities and competition in the land market.

Atomic mono-center ($\tau \leq 0.353$) is an extreme case of the formation of the single core city surrounded by a series of periphery cities. The existence of this mono-center is robust against the number of places as the mono-center exists in $\tau \leq 0.356$ for 33 places and in $\tau \leq 0.358$ for 65 places.

A question on the spatial agglomeration, set forth above, is "How and when do spa-

[9]This system was seen clearly in a two-dimensional economic space (Ikeda and Murota, 2014 [18]; Ikeda et al., 2014 [19]).

[10]Christaller (1933) [8] wrote: "The traffic principle states that the distribution of central places is most favorable when as many important places as possible lie on one traffic route between two important towns, the route being as straightly and as cheaply as possible."

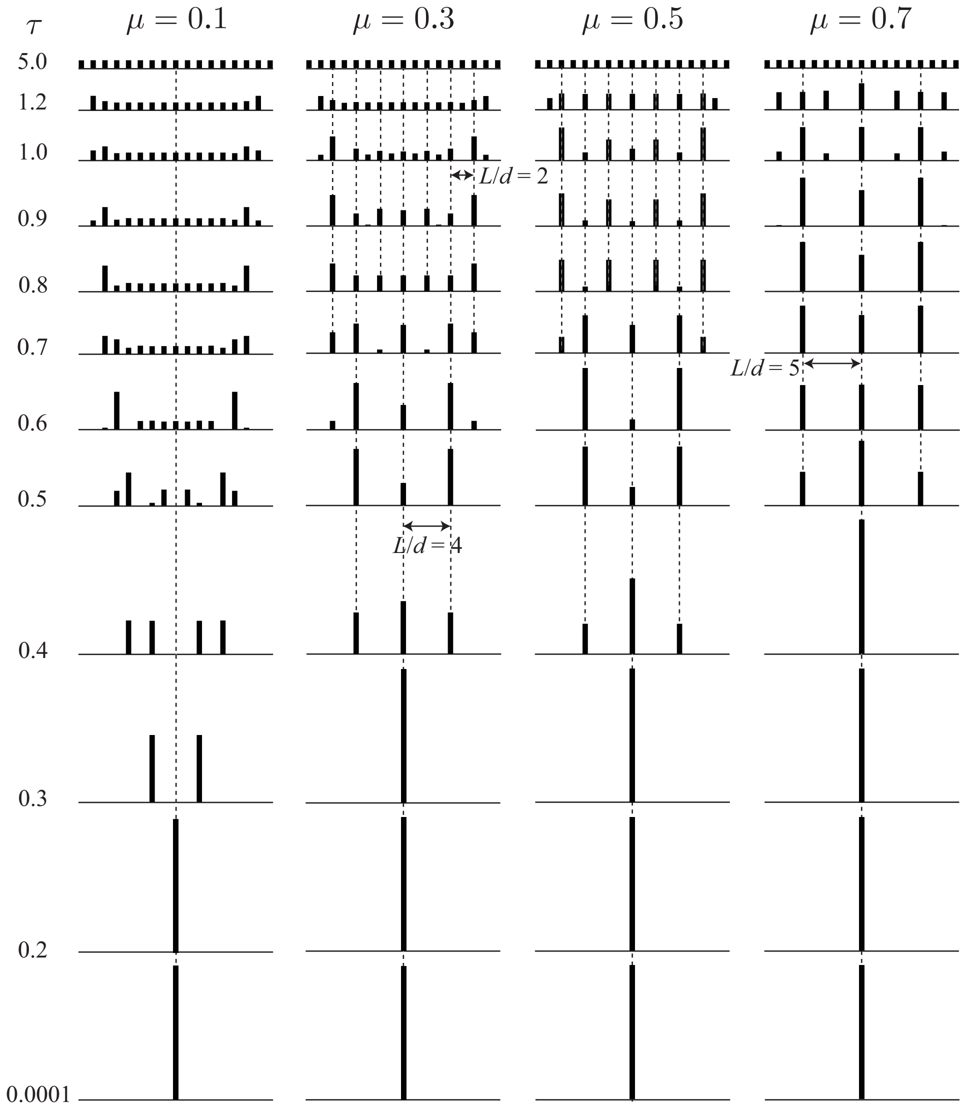


Figure 5: Influence of μ on the spatial agglomeration of the long narrow economy (σ = 10.0)

tially repeated core–periphery patterns emerge?" This question is answered in Section 4.2 in the light of the racetrack economy analogy.

4.1.2 Other patterns

The relative predominance of centripetal forces promoting agglomeration and centrifugal forces engendering dispersion varies with parameter values in core–periphery models (Section 2). To make clear parameter dependence of agglomeration patterns, the progress of agglomerations is depicted in the bar charts of the population distributions in Fig. 5 for several values μ = 0.1, 0.3, 0.5, and 0.7 of the expenditure share of manufactured goods (σ = 10.0).

For all values of μ, as τ decreases and spatial period between agglomerated places is enlarged, highly regular systems *a la* Christaller and Lösch are observed. The onset of the enlargement of the spatial period, which takes place for a larger τ as μ increases, is in accordance with the break points $\tau_{1,16} = 1.44,\ 1.20,\ 0.91,\ 0.52$ for $\mu = 0.7, 0.5, 0.3$, and 0.1, respectively, evaluated analytically by (3.9). The atomic mono-center also emerges earlier (larger τ) for a larger μ.

For $\mu = 0.1$ with a small agglomeration force, a distribution with twin peak places near the borders connected by a chain of industrial belt with smaller populations are observed for $1.0 \geq \tau \geq 0.6$. This distribution is close to the one found by Mori (1997) [24]: "formation of a megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities*."

For $\mu = 0.3$, 0.5, and 0.7, agglomerated places are discrete, and the number of these places decreases as $7 \longrightarrow 5 \longrightarrow 3 \longrightarrow 1$ as τ decreases. Such decrease is a robust feature for $0.3 \leq \mu \leq 0.7$.

4.2 Racetrack economy analogy: 65 places

The geometrical configurations of the long narrow economy and the racetrack economy are alike in that places are located equidistantly. In order to demonstrate that both economies display similar agglomerations, a comparative study is conducted for these economies.

Stable[11] spatial agglomerations of the long narrow economy with 65 places and the racetrack economy with 64 places are illustrated comparatively for several values of the transport cost parameter τ in Fig. 6, which depicts the equilibrium curve between population and transport cost and associated population distributions.

The thin curve displays a stable course OGHIJKL of agglomeration in the racetrack economy via a spatial period doubling bifurcation cascade occurring at pitchfork bifurcation points G, H, I, J, and K (Section 3.2). This cascade is followed by a dynamical shift to a stable state at point L to point M leading to complete agglomeration at N. In the long narrow economy, border effects are weakened in comparison with the 17 places in Section 4.1; accordingly, the cascade can be seen clearer as explained below.

There are three major stages. In the early stage of agglomeration with a large $\tau (> 0.94)$, the bold curve of the long narrow economy accurately traces the thin curve OGHIJ of the racetrack economy. In the intermediate stage ($0.94 > \tau > 0.1$), the curves of these two economies behave differently. In the last stage of formation of a mono-center

[11]The stability of bifurcating equilibria of the racetrack economy is model dependent. In the present study on the Forslid and Ottaviano model, the first bifurcation is stable but, for the Krugman model, the first bifurcation was unstable (Ikeda, Akamatsu, and Kono, 2012 [17]).

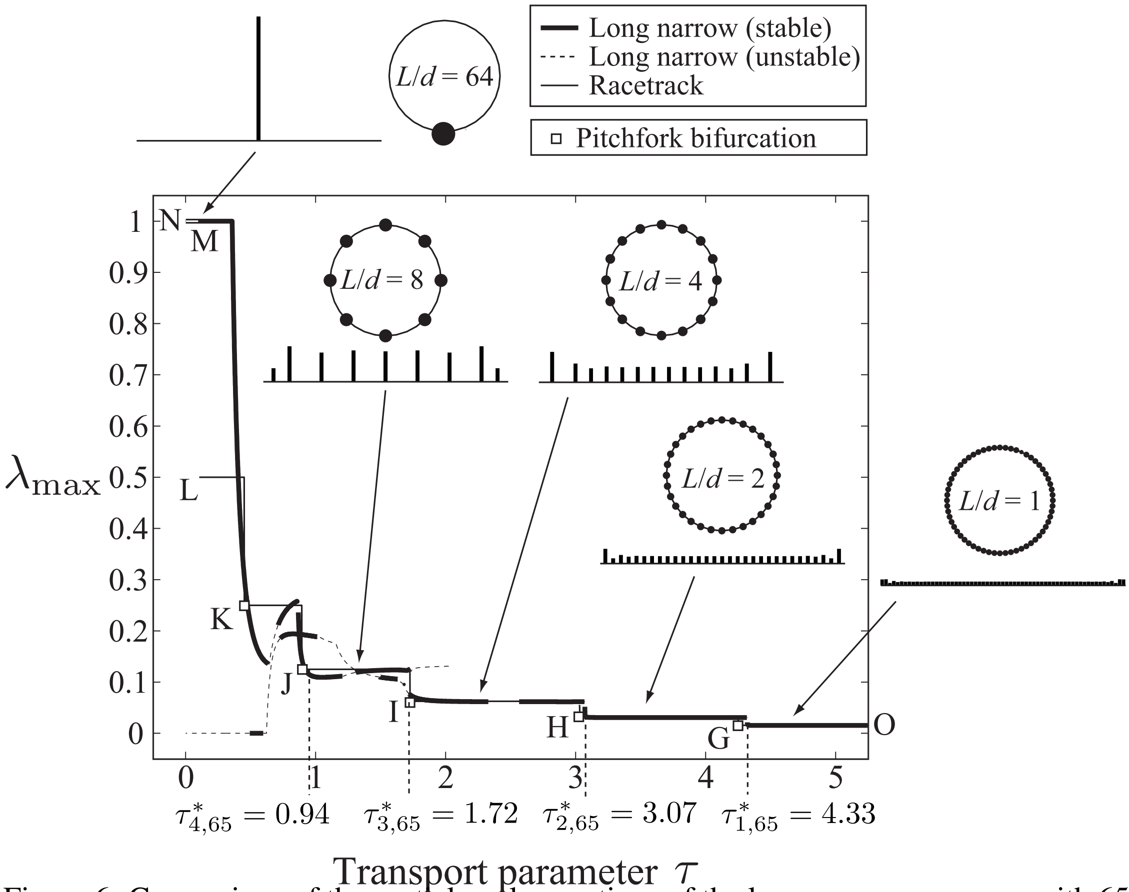


Figure 6: Comparison of the spatial agglomerations of the long narrow economy with 65 places and those of the racetrack economy with 64 places ($\lambda_{\max}$ is the maximum population in places in the middle defined by $\lambda_{\max} = \max_{28 \le i \le 38} \lambda_i$)

($0.1 > \tau > 0.0$), the curves are identical (MN).

Hereafter, we specifically examine the early stage OGHIJ ($\tau > 0.94$), which is the major interest of this paper. Population distributions depicted in Fig. 6 in both economies, especially near the middle (place $n/2$), display an amazing resemblance: recurrent emergence of spatially repeated core–periphery patterns with increasing peripheral zones as τ decreases. This shows the validity of the racetrack economy analogy presented in this paper.

By virtue of this analogy, several indexes for agglomerations in the long narrow economy, which are of great assistance in a policy recommendation, are set forth as follows:

Distance between agglomerated places doubles recurrently as $L/d = 1 \to 2 \to 4 \to 8$ (see (3.7)), thereby enlarging agglomeration shadows.

Populations at agglomerated places away from boundaries double recurrently and

Table 1: The values of break points $\tau_{m,n}$ for racetrack economy obtained analytically

(a) $\tau_{m,n}$ by (3.9)

n	16	32	64
$\tau_{1,n}$	1.06	2.13	4.25
$\tau_{2,n}$	0.76	1.52	3.03
$\tau_{3,n}$	—	—	—
$\tau_{4,n}$	—	—	—

(b) $\hat{\tau}_{m,n}$ by (3.13) and $\tilde{\tau}_{m,n}$ by (3.14)

n	16	32	64
$\hat{\tau}_{1,n}$	1.06	2.12	4.24
$\hat{\tau}_{2,n}$	0.75	1.50	3.00
$\tilde{\tau}_{3,n}$	0.43	0.87	1.73
$\tilde{\tau}_{4,n}$	0.22	0.43	0.87

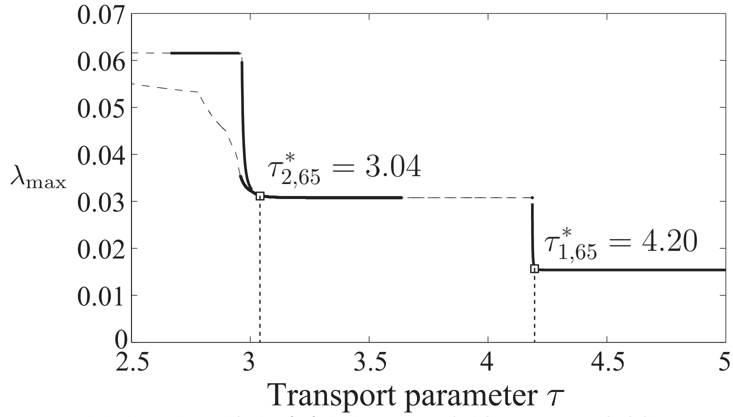
can be predicted qualitatively as a function of the transport cost parameter τ as

$$\begin{cases} 1/64 & (\tau > \tau^*_{1,65} = 4.33), \\ 1/32 & (3.07 = \tau^*_{2,65} < \tau < \tau^*_{1,65} = 4.33), \\ 1/16 & (1.72 = \tau^*_{3,65} < \tau < \tau^*_{2,65} = 3.07), \\ 1/8 & (0.94 = \tau^*_{4,65} < \tau < \tau^*_{3,65} = 1.72). \end{cases}$$

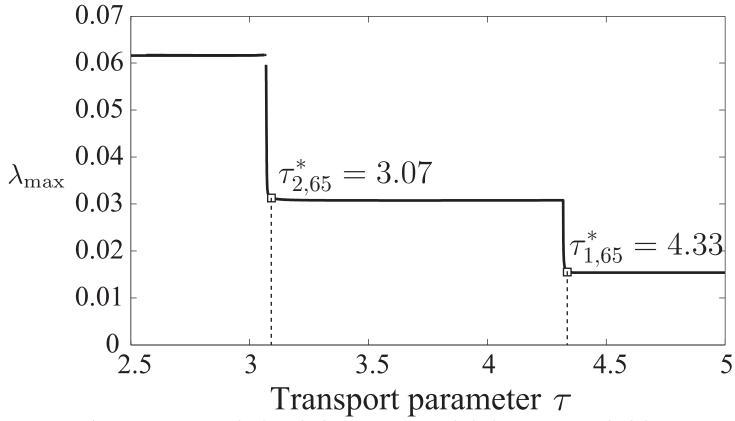
Here $\tau^*_{m,65}$ $(m = 1, \ldots, 4)$ are break points of the long narrow economy with 65 places that are read from Fig. 6.

Break points $\tau^*_{1,65} = 4.33$, $\tau^*_{2,65} = 3.07$, $\tau^*_{3,65} = 1.72$, and $\tau^*_{4,65} = 0.94$ serve as excellent indexes for agglomeration pattern changes. The laws presented in Section 3.3 realize an accurate analytical prediction of these break points: The break points $\tau_{1,64} = 4.25$ and $\tau_{2,64} = 3.03$ in Table 1(a) computed by the formula (3.9) are in fair agreement with $\tau^*_{1,65} = 4.33$ and $\tau^*_{2,65} = 3.07$, thereby ensuring the validity of this formula. The break points $\tilde{\tau}_{3,64} = 1.73$ and $\tilde{\tau}_{4,64} = 0.87$ in Table 1(b) estimated by the semi-empirical formula (3.14) are in fair agreement with $\tau^*_{3,65} = 1.72$ and $\tau^*_{4,65} = 0.94$, thereby showing its usefulness.

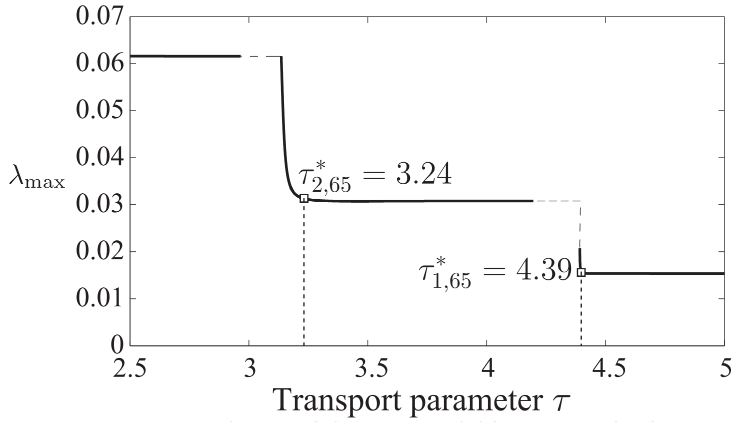
We note that the formula (3.13) of $\hat{\tau}_{m,n}$ ($\approx \tau_{m,n}$) gives an insight into parameter dependence. There exists a series of (μ, σ) which has the same value of $\mu/(\sigma - 1)^3$ and, hence, has the same value of $\hat{\tau}_{m,n}$. Agglomeration patterns are observed for three different sets of $(\mu, \sigma) = (0.1, 6.67)$, $(0.4, 10.0)$ and $(0.7, 11.85)$, which have approximately the same values of $\mu/(\sigma - 1)^3 \approx 5.5 \times 10^{-4}$, $\hat{\tau}_{1,64} = 4.24$, and $\hat{\tau}_{2,64} = 3.00$. As shown in Fig. 7, these different sets of (μ, σ) display similar spatial period doubling behaviors, especially for the direct bifurcation. This suffices to demonstrate the insight and usefulness of the formula (3.13) for $\hat{\tau}_{m,n}$.



(a) $(\mu, \sigma) = (0.1, 6.67)$ ($\tau_{1,64} = 4.12,\ \tau_{2,64} = 2.93$)



(b) $(\mu, \sigma) = (0.4, 10.0)$ ($\tau_{1,64} = 4.25,\ \tau_{2,64} = 3.03$)



(c) $(\mu, \sigma) = (0.7, 11.85)$ ($\tau_{1,64} = 4.32,\ \tau_{2,64} = 3.10$)

Figure 7: Agglomeration patterns are observed for 65 places with three different sets of $(\mu, \sigma) = (0.1, 6.67)$, $(0.4, 10.0)$ and $(0.7, 11.85)$ with approximately the same values of $\mu/(\sigma - 1)^3 \approx 5.5 \times 10^{-4}$, $\hat{\tau}_{1,64} = 4.24$, and $\hat{\tau}_{2,64} = 3.00$ ($\lambda_{\max}$ is the maximum population in places in the middle defined by $\lambda_{\max} = \max_{28 \le i \le 38} \lambda_i$)

5 Comparison with the logit dynamics

Agglomeration patterns in the logit dynamics are investigated to study the similarity and difference of agglomeration patterns in comparison with the replicator dynamics (Section 4). The redistribution of population for a very small transport cost is characteristic in the logit dynamics. Such redistribution is inherent in several economic activities, such as trade costs in the agricultural sector, congestion costs caused by spatial concentration of activities, and the heterogeneity in individual migration behaviors (Combes, Mayer, Thisse, 2008 [9]). Agglomeration patterns observed herein, accordingly, are of great economic interest. We consider 17 places and set $\sigma = 10.0$ throughout this section.

First, spatial agglomerations are investigated by changing the expenditure share as $\mu = 0.1$, 0.4, and 0.7 to arrive at the bar chart of the population distributions shown in Fig. 8. Here the parameter θ in (2.25) is chosen as $\theta = 10000$. Again the onset of the enlargement of the spatial period takes place for a larger τ as μ increases in accordance with the break points $\tau_{1,16} =$ 0.42, 1.05, 1.43 for $\mu = 0.1, 0.4, 0.7$, respectively, evaluated analytically by (3.9). Since θ is large, these points are almost the same as those for the replicator dynamics. When the transport cost τ is very small ($\tau \leq 0.001$), the redistribution from the atomic mono-center ($\tau = 0.02$) takes place to engender a *hump-shaped population distribution* for all values of μ. This distribution is interpreted as another kind of megalopolis formation. When the transport cost τ is large ($\tau \geq 0.6$), the agglomeration patterns are very close to those observed in the replicator dynamics (Fig. 5).

Next, spatial agglomerations are investigated by changing $\theta = 100$, 600, and 1000 to arrive at Fig. 9 ($\mu = 0.4$). The break points are $\tau_{1,16} = 0.83$ and 0.96 for $\theta = 600$ and 1000, but do not exist for $\theta = 100$ because the condition (3.17b) for the existence of break point is violated. When the transport cost τ is very small ($\tau \leq 0.02$), similarly in Fig. 5, a *hump-shaped population distribution* is observed for all values of θ. This distribution emerges for a larger τ as θ decreases. For $\theta = 100$, a megalopolis with twin peaks appears in $0.6 \geq \tau \geq 0.4$. For $\theta = 600$, more than two humps are observed at $\tau = 0.9$ and 0.8, and a megalopolis with three core-cities connected by two industrial belts emerges at $\tau = 0.6$.

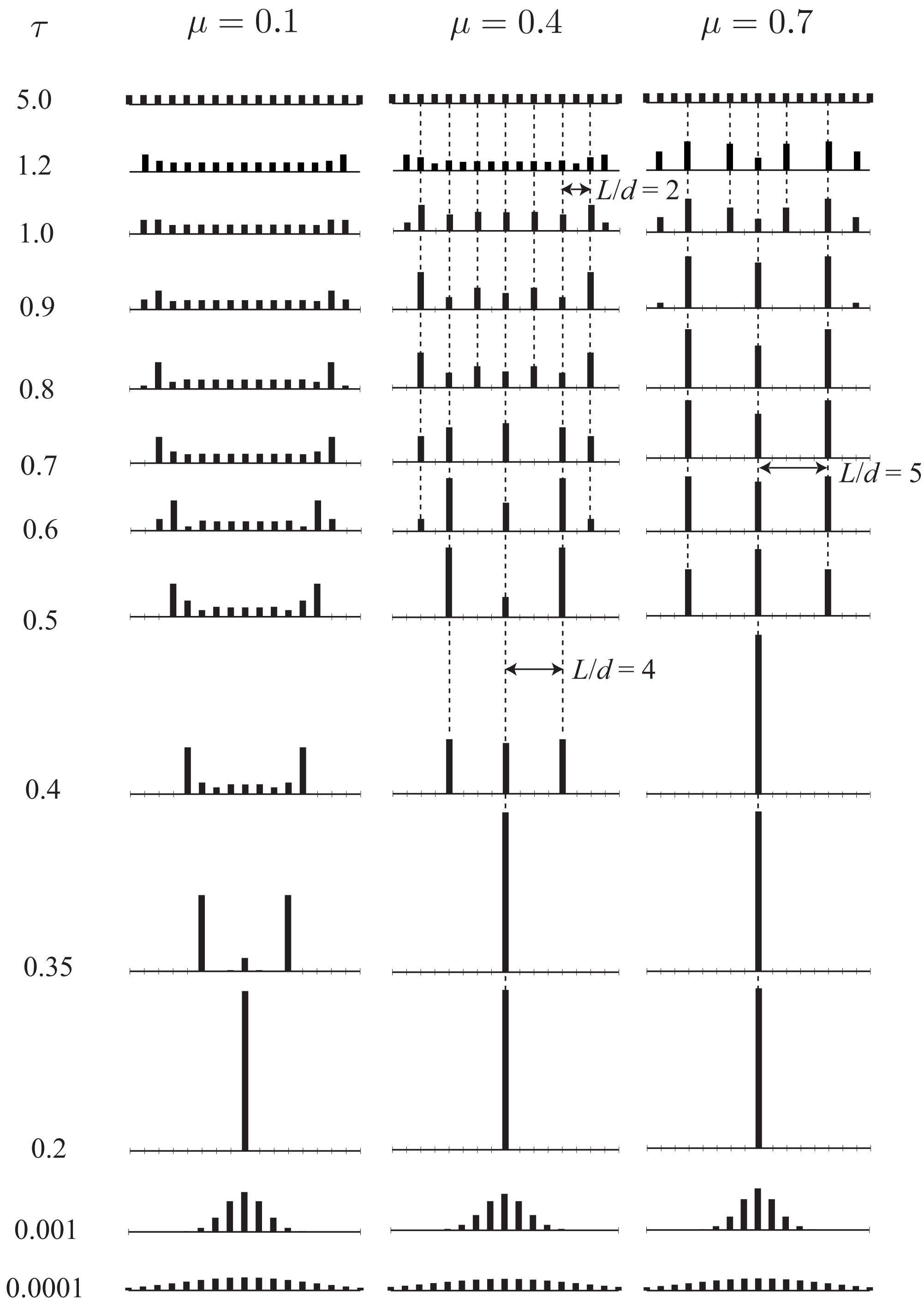


Figure 8: Influence of μ on the spatial agglomeration of the long narrow economy (σ = 10.0, θ = 10000)

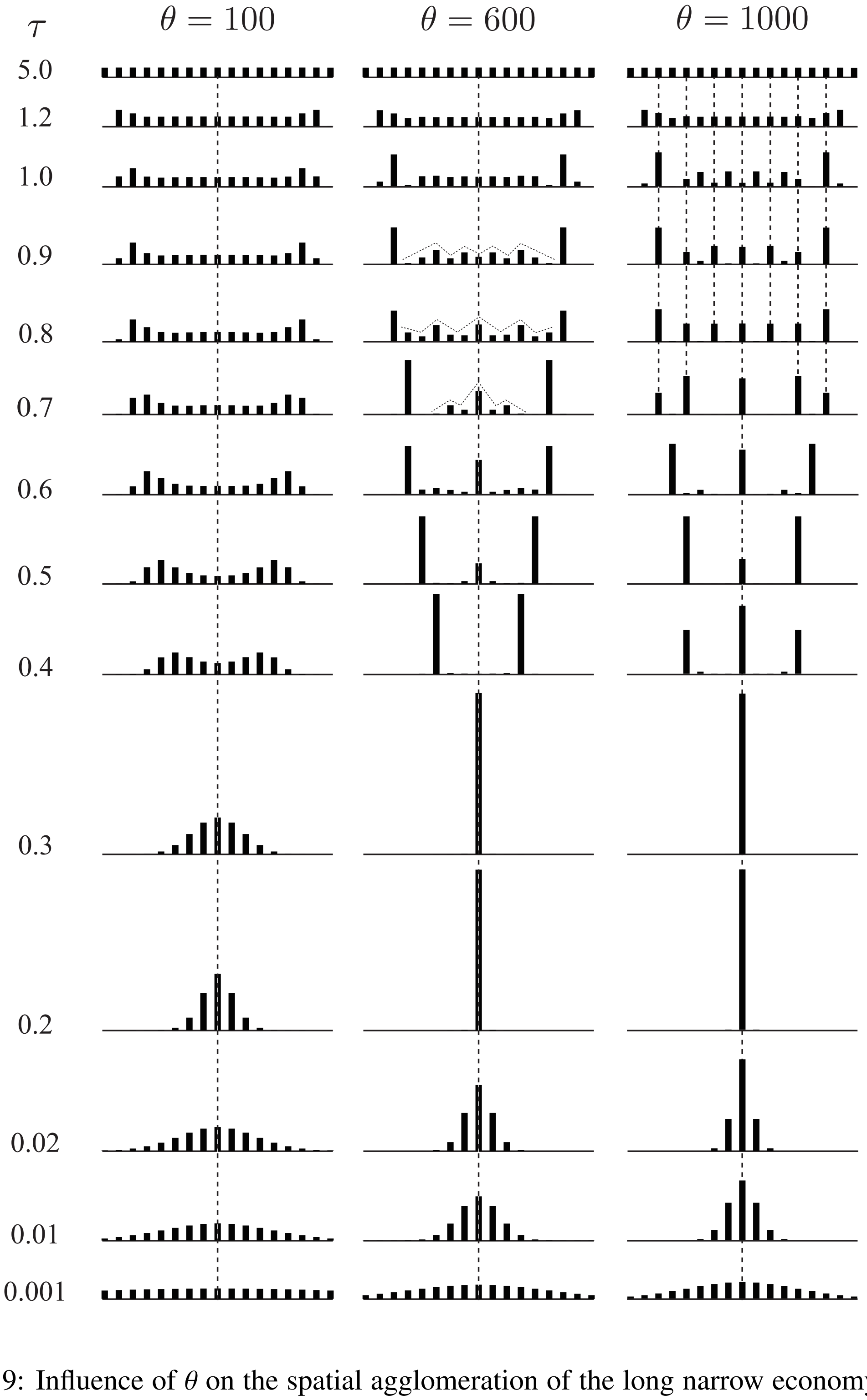


Figure 9: Influence of θ on the spatial agglomeration of the long narrow economy (σ = 10.0, μ = 0.4)

6 Conclusion

Agglomeration patterns in the long narrow economy were found somewhat fragmentarily in the literature (see Introduction). In this paper, it is demonstrated that a new economic geography model in a discretized long narrow economy accommodates several characteristic agglomeration patterns.

When the agglomeration force is large, the spatial agglomeration is close to the highly regular central place system *a la* Christaller and Lösch (Fujita and Mori, 1997 [15]). When the transport cost was decreased, spatially repeated core–periphery patterns emerged recurrently with increased peripheral zones, engendering agglomeration shadows.

When this force is small, a distribution with twin peaks near the borders connected by a chain of industrial belt with smaller populations is observed. This distribution is close to the one found by Mori (1997) [24]: "formation of a megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities*."

Another kind of megalopolis formation with a hump-shaped population distribution was observed for small transport cost in the logit dynamics. This paper, in comparison with the literature, thus extends the horizon of the study of the evolution and of the formation of central places, megalopolis, and industrial belt.

In search of the underlying mechanism which produces the highly regular central place system, the concept of *racetrack economy analogy* was presented to import the theoretical results in the racetrack economy in the description of the agglomeration patterns in the long narrow economy. An amazing resemblance was found in the agglomeration characteristics of these two economies, thereby supporting the validity of the racetrack economy analogy. By virtue of this resemblance, population distributions and the values of the transport cost at the evolution of spatially repeated core–periphery patterns in the long narrow economy can be predicted analytically. The long narrow economy, with the aid of this analogy, has thus been reinforced with much desired analytical tractability.

The long narrow economy yields rich economic implications on spatial agglomerations, which cannot be given by the conventional two-place economy with an overly simple spatial structure. The diversity and complexity of the agglomeration behaviors of this economy, which we encountered, are essential ones that are to be meshed into the study of economic geography models. The robustness of the spatial agglomeration properties observed herein should be studied in the future for spatial models of various kinds.

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A Proof of the formula for break points in the racetrack economy

The formula (3.9) for break points for the pitchfork bifurcation in the racetrack economy with n even can be derived by the method in Akamatsu, Takayama, and Ikeda (2012) [2] for the logit dynamics adapted to the replicator dynamics. Although the discussion on the main text is restricted to the case of $n = 2^k$ $(k = 3, 4, , \ldots)$ by Assumption 1 in Section 3, we consider herewith a more general case of n even.

From the governing equation $\boldsymbol{F}$ in (2.22) with (2.21), we have

$$\frac{\partial F_i}{\partial \lambda_j} = \left(v_i - \sum_{k=1}^{n} \lambda_k v_k \right) \delta_{ij} + \lambda_i \left(\frac{\partial v_i}{\partial \lambda_j} - v_j - \sum_{k=1}^{n} \lambda_k \frac{\partial v_k}{\partial \lambda_j} \right), \tag{A.1}$$

where δ_{ij} is the Kronecker delta. This shows that the Jacobian matrices

$$J(\lambda) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}} = \left(\frac{\partial F_i}{\partial \lambda_j} \right), \quad V(\lambda) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\lambda}} = \left(\frac{\partial v_i}{\partial \lambda_j} \right)$$

are related as

$$J(\lambda) = \mathrm{diag}(v_1 - \bar{v}, \ldots, v_n - \bar{v}) + (\Lambda - \boldsymbol{\lambda}\boldsymbol{\lambda}^\top) V - \boldsymbol{\lambda}\boldsymbol{v}^\top, \tag{A.2}$$

where $\bar{v} = \sum_{i=1}^{n} \lambda_i v_i$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

In regard to $V(\lambda)$ we recall (2.16):

$$v_i = \frac{\mu}{\sigma - 1} \log \Delta_i + \log w_i \tag{A.3}$$

as well as (2.15):

$$w_i = \frac{\mu}{\sigma} \sum_{k=1}^{n} \frac{d_{ik}}{\Delta_k} (w_k \lambda_k + 1), \tag{A.4}$$

where

$$\Delta_k = \sum_{j=1}^{n} d_{jk} \lambda_j.$$

The differentiations of (A.3) and (A.4) with respect to λ_j yield respectively

$$\frac{\partial v_i}{\partial \lambda_j} = \kappa' \frac{d_{ji}}{\Delta_i} + \frac{1}{w_i} \frac{\partial w_i}{\partial \lambda_j}, \tag{A.5}$$

$$\frac{\partial w_i}{\partial \lambda_j} = \kappa \sum_{k=1}^{n} \frac{d_{ik}}{{\Delta_k}^2} \left[\left(\frac{\partial w_k}{\partial \lambda_j} \lambda_k + w_k \delta_{kj} \right) \Delta_k - (w_k \lambda_k + 1) d_{jk} \right], \tag{A.6}$$

where

$$\kappa = \frac{\mu}{\sigma}, \quad \kappa' = \frac{\mu}{\sigma - 1}. \tag{A.7}$$

We have $0 < \kappa < 1$ and $0 < \kappa' < 1$ because $\sigma > 1$, $0 < \mu < 1$, and $\mu/(\sigma - 1) < 1$ by (3.8) with $m = 1$ (no-black-hole condition).

With the use of

$$r = \exp\left[-\tau(\sigma - 1)\,\frac{\mathcal{L}}{n}\right] \tag{A.8}$$

($0 < r < 1$ for $\tau > 0$) in the transport cost T_{ij} in (A.8), the spatial discounting factor $d_{ij} = T_{ij}^{1-\sigma}$ in (2.14) is rewritten as

$$d_{ij} = r^{\,m(i,j)}, \tag{A.9}$$

representing distance decaying friction between pairs of places.

A.1 Direct bifurcation

The direct bifurcation from the flat earth equilibrium $\boldsymbol{\lambda}^* = \frac{1}{n}(1, \ldots, 1)^\top$ in the direction of the critical vector

$$\boldsymbol{\eta} = (1, -1, \ldots, 1, -1)^\top \tag{A.10}$$

in (3.3) is investigated. We consider this vector $\boldsymbol{\eta}$ since it is the relevant critical eigenvector of $J(\boldsymbol{\lambda}^*)$ for the spatial period doubling bifurcation.

The vector $\boldsymbol{\eta}$ is also an eigenvector of the spatial discounting matrix $D = (d_{ij})$. Indeed, the odd-numbered components of $D\boldsymbol{\eta}$ are equal to

$$\begin{aligned}\sum_{j=1}^{n}(-1)^{j-1} r^{\,m(i,j)} &= 1 + 2\left(-r + r^2 - \cdots - (-1)^{n/2} r^{n/2-1}\right) + (-1)^{n/2} r^{n/2} \\ &= \frac{1-r}{1+r}\left(1 - (-r)^{n/2}\right)\end{aligned}$$

and the even-numbered components are the negative of this number. Hence

$$D\boldsymbol{\eta} = \tilde{\epsilon}\boldsymbol{\eta} \tag{A.11}$$

with

$$\tilde{\epsilon} = \frac{1-r}{1+r}\left(1 - (-r)^{n/2}\right). \tag{A.12}$$

At the flat earth equilibrium $\boldsymbol{\lambda}^* = \frac{1}{n}(1, \ldots, 1)^\top$, (A.2) gives

$$J(\boldsymbol{\lambda}^*) = \left(\frac{1}{n}I - \frac{1}{n^2}\mathbf{1}\mathbf{1}^\top\right)V(\boldsymbol{\lambda}^*) \tag{A.13}$$

where $v = v_1 = \cdots = v_n$. The matrix $V(\boldsymbol{\lambda}^*)$ in (A.13) can be evaluated as follows. At $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, we have

$$\Delta_j = \sum_{k=1}^{n} d_{kj}\lambda_k = \frac{d}{n},$$

where d denotes the sum of the entries of a column of D, which is given by

$$d = \sum_{i=1}^{n} r^{m(i,j)} = 1 + 2(r + r^2 + \cdots + r^{n/2-1}) + r^{n/2} = \frac{1+r}{1-r}\left(1 - r^{n/2}\right). \tag{A.14}$$

Since w_j is independent of j, we may put $w_j = w$, and then (A.4) becomes

$$w = \kappa \sum_{j=1}^{n} \frac{n}{d}\, d_{ij}\left(\frac{w}{n} + 1\right) = \kappa\,(w + n)\,,$$

which gives

$$w = \frac{\kappa n}{1-\kappa}. \tag{A.15}$$

At $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, (A.6) becomes

$$\frac{\partial w_i}{\partial \lambda_j} = \kappa \sum_{k=1}^{n} \frac{n^2}{d^2}\, d_{ik}\left[\left(\frac{1}{n}\frac{\partial w_k}{\partial \lambda_j} + w\delta_{kj}\right)\frac{d}{n} - \left(\frac{w}{n} + 1\right)d_{jk}\right],$$

which in a matrix form reads

$$W = \kappa\frac{n^2}{d^2}\, D\left[\frac{d}{n}\left(\frac{1}{n}W + wI\right) - \frac{w+n}{n}D\right]$$

with $W = (\partial w_i/\partial \lambda_j)$. With the use of (A.15), this equation can be rewritten as

$$\left(I - \kappa\frac{D}{d}\right)W = nw\,\frac{D}{d}\left(\kappa I - \frac{D}{d}\right),$$

which is further rewritten as

$$W = nw\left(I - \kappa\frac{D}{d}\right)^{-1} \cdot \frac{D}{d}\left(\kappa I - \frac{D}{d}\right).$$

Then the partial derivatives in (A.5) can be evaluated in a matrix form as

$$V(\boldsymbol{\lambda}^*) = n\left[\kappa'\frac{D}{d} + \left(I - \kappa\frac{D}{d}\right)^{-1} \cdot \frac{D}{d}\left(\kappa I - \frac{D}{d}\right)\right]. \tag{A.16}$$

For the vector $\boldsymbol{\eta}$ in (A.10), we have $(D/d)\boldsymbol{\eta} = \epsilon\boldsymbol{\eta}$ with

$$\begin{aligned} \epsilon = \frac{\tilde{\epsilon}}{d} &= \left(\frac{1-r}{1+r}\right)^2 \frac{1-(-1)^{n/2}r^{n/2}}{1-r^{n/2}} \\ &= \begin{cases} \left(\dfrac{1-r}{1+r}\right)^2 & \text{(if } n \text{ is a multiple of 4),} \\ \left(\dfrac{1-r}{1+r}\right)^2 \dfrac{1+r^{n/2}}{1-r^{n/2}} & \text{(if } n \text{ is even, not a multiple of 4),} \end{cases} \end{aligned} \tag{A.17}$$

where (A.11), (A.12) and (A.14) are used. Since $0 < r < 1$, we have

$$0 < \epsilon < 1. \tag{A.18}$$

Then (A.16) shows that

$$V(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \gamma\boldsymbol{\eta}$$

with

$$\gamma = n[\kappa'\epsilon + (1-\kappa\epsilon)^{-1} \cdot \epsilon(\kappa - \epsilon)].$$

Multiplying (A.13) by the vector $\boldsymbol{\eta}$ in (A.10) from the right and using

$$\mathbf{1}^\top V(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \gamma\mathbf{1}^\top\boldsymbol{\eta} = 0,$$

we obtain

$$J(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \frac{\gamma}{n}\boldsymbol{\eta}.$$

Then the eigenvalue β of the Jacobian matrix $J(\boldsymbol{\lambda}^*)$ for the eigenvector $\boldsymbol{\eta}$ is expressed in terms of ϵ as

$$\beta = \Psi(\epsilon)$$

with a function Ψ defined as

$$\Psi(x) = \kappa' x + \frac{x(\kappa - x)}{1-\kappa x} = \frac{(\kappa+\kappa')x - (\kappa\kappa'+1)x^2}{1-\kappa x}. \tag{A.19}$$

The break point $\tau_{1,n}$ is determined from the condition that the eigenvalue β for $\tau = \tau_{1,n}$

vanishes. The value ϵ^* satisfying $\Psi(\epsilon^*) = 0$ is given by

$$\epsilon^* = \epsilon_1^* = \frac{\kappa + \kappa'}{\kappa\kappa' + 1}, \tag{A.20}$$

which is independent of n. Since $0 < \epsilon < 1$ by (A.18), this gives the no-black-hole condition:

$$\frac{\mu}{\sigma - 1} < 1. \tag{A.21}$$

The value of $r^* = r(\tau_{1,n})$ corresponding to $\epsilon = \epsilon^*$ can be determined from (A.17), and $\tau_{1,n}$ is given as

$$\tau_{1,n} = \frac{n}{\mathcal{L}(\sigma - 1)}(-\log r^*) \tag{A.22}$$

from (A.8). In accordance with (A.17), which relate r to ϵ, we divide into three cases of n to derive a concrete form of (A.22).

When $n = 4\ell$ for an integer $\ell \geq 1$, (A.17) yields

$$r^* = \frac{1 - \sqrt{\epsilon^*}}{1 + \sqrt{\epsilon^*}}.$$

The substitution of this into (A.22) yields

$$\tau_{1,n} = c_1 n \tag{A.23}$$

with

$$c_1 = \frac{1}{\mathcal{L}(\sigma - 1)} \log\left(\frac{1 + \sqrt{\epsilon^*}}{1 - \sqrt{\epsilon^*}}\right). \tag{A.24}$$

Note that c_1 is a constant independent of n. Under the condition

$$\frac{\sigma}{\mu} \gg 1, \tag{A.25}$$

we have

$$\epsilon^* \approx \frac{2\mu}{\sigma - 1} \ll 1,$$

and hence c_1 in (A.24) and, in turn, $\tau_{1,n}$ in (A.23) can be approximated as

$$\tau_{1,n} \approx \frac{2\sqrt{\epsilon^*}}{\mathcal{L}(\sigma - 1)} \approx \frac{2^{3/2}\mu^{1/2}}{\mathcal{L}(\sigma - 1)^{3/2}}. \tag{A.26}$$

Thus we have proved (3.9) for $m = 1$ with (3.10) and (3.13) in Section 3.3.

When $n = 2$, which is nothing but a two place, (A.17) yields

$$r^* = \frac{1 - \epsilon^*}{1 + \epsilon^*}$$

and the substitution of this into (A.22) yields

$$\tau_{1,2} = \frac{2}{\mathcal{L}(\sigma - 1)} \log\left(\frac{1 + \epsilon^*}{1 - \epsilon^*}\right), \tag{A.27}$$

which is approximated under (A.25) as

$$\tau_{1,2} \approx \frac{8\mu}{\mathcal{L}(\sigma - 1)^2}. \tag{A.28}$$

The formula (A.27) is in line with that was obtained for the same model (Forslid and Ottaviano, 2003 [12]).

When $n = 4\ell + 2$ for an integer $\ell \geq 1$, an approximate formula is not available.

A.2 Secondary bifurcation

The bifurcation from the spatial doubling state

$$\hat{\boldsymbol{\lambda}} = \frac{1}{n}(2, 0, 2, 0, \dots, 2, 0, 2, 0)^\top \tag{A.29}$$

in the direction of the critical vector

$$\hat{\boldsymbol{\eta}} = (1, 0, -1, 0, \dots, 1, 0, -1, 0)^\top \tag{A.30}$$

is investigated when n is a multiple of 4. We consider this vector $\hat{\boldsymbol{\eta}}$ since it is the relevant critical eigenvector of $J(\hat{\boldsymbol{\lambda}})$ for the secondary spatial period doubling bifurcation.

The vector $\hat{\boldsymbol{\eta}}$ is also an eigenvector of the spatial discounting matrix D. That is,

$$D\hat{\boldsymbol{\eta}} = \tilde{\epsilon}\hat{\boldsymbol{\eta}} \tag{A.31}$$

with

$$\tilde{\epsilon} = \frac{1 - r^2}{1 + r^2}\left(1 - (-r^2)^{n/4}\right), \tag{A.32}$$

where $\tilde{\epsilon}$ is equal to the first component of $D\hat{\boldsymbol{\eta}}$. Let D_{11} be the $(n/2) \times (n/2)$ submatrix of D with odd indices of rows and columns, i.e.,

$$D_{11} = (d_{2i-1,2j-1} \mid i, j = 1, 2, \dots, n/2),$$

and similarly let

$$D_{21} = (d_{2i,2j-1} \mid i, j = 1, 2, \ldots, n/2), \quad D_{12} = (d_{2i-1,2j} \mid i, j = 1, 2, \ldots, n/2).$$

Also define an $n/2$-dimensional vector

$$\boldsymbol{\eta} = (1, -1, \ldots, 1, -1)^\top, \tag{A.33}$$

which is the subvector of $\hat{\boldsymbol{\eta}}$ consisting of the odd indexed components. Then $D\hat{\boldsymbol{\eta}} = \tilde{\varepsilon}\hat{\boldsymbol{\eta}}$ in (A.31) can be rewritten as

$$D_{11}\boldsymbol{\eta} = \tilde{\varepsilon}\boldsymbol{\eta}, \quad D_{21}\boldsymbol{\eta} = \mathbf{0}. \tag{A.34}$$

For the Jacobian matrices $J(\lambda)$ and $V(\lambda)$ we similarly denote the $(n/2) \times (n/2)$ submatrices with odd row and column indices by $J_{11}(\lambda)$ and $V_{11}(\lambda)$, respectively. At the equilibrium $\hat{\boldsymbol{\lambda}} = \frac{1}{n}(2, 0, 2, 0, \ldots, 2, 0, 2, 0)^\top$, for which we have $v_1 = v_3 = \cdots = v_{n-1}$ and $v_2 = v_4 = \cdots = v_n$, the corresponding submatrix of (A.2) gives

$$J_{11}(\hat{\boldsymbol{\lambda}}) = \left(\frac{2}{n}I - \frac{4}{n^2}\mathbf{1}\mathbf{1}^\top\right)V_{11}(\hat{\boldsymbol{\lambda}}) - \frac{2v_1}{n}\mathbf{1}\mathbf{1}^\top, \tag{A.35}$$

where $\mathbf{1} = (1, \ldots, 1)^\top$ is an $n/2$-dimensional vector and I denotes the identity matrix of order $n/2$. The matrix $V_{11}(\hat{\boldsymbol{\lambda}})$ in (A.35) can be evaluated as follows.

At $\boldsymbol{\lambda} = \hat{\boldsymbol{\lambda}}$, we have

$$\Delta_1 = \Delta_3 = \cdots = \Delta_{n-1} = \frac{2d_1}{n}, \quad \Delta_2 = \Delta_4 = \cdots = \Delta_n = \frac{2d_2}{n}$$

with

$$\begin{aligned} d_1 &= \sum_{k=1}^{n/2} d_{2k-1,1} = \frac{1+r^2}{1-r^2}\left(1 - r^{n/2}\right), \\ d_2 &= \sum_{k=1}^{n/2} d_{2k,1} = \frac{2r}{1-r^2}\left(1 - r^{n/2}\right). \end{aligned}$$

The wage w_j is determined by the parity of j, and therefore, we have $w_{2k-1} = w_1$ and

$w_{2k} = w_2$. Then (A.4) becomes

$$\begin{aligned} w_1 &= \kappa \sum_{j=1}^{n} \left(\frac{d_{1j}}{\Delta_j} w_j \lambda_j + \frac{d_{1j}}{\Delta_j} \right) \\ &= \kappa \left(\sum_{k=1}^{n/2} \frac{d_{1,2k-1}}{\Delta_1} w_1 \frac{2}{n} + \sum_{k=1}^{n/2} \left(\frac{d_{1,2k-1}}{\Delta_1} + \frac{d_{1,2k}}{\Delta_2} \right) \right) \\ &= \kappa \left(\frac{n}{2d_1} d_1 w_1 \frac{2}{n} + \frac{n}{2d_1} d_1 + \frac{n}{2d_2} d_2 \right) \\ &= \kappa(w_1 + n), \end{aligned}$$

which gives

$$w_1 = \frac{\kappa n}{1-\kappa}. \tag{A.36}$$

At $\boldsymbol{\lambda} = \hat{\boldsymbol{\lambda}}$, (A.6) becomes

$$\begin{aligned} \frac{\partial w_i}{\partial \lambda_j} &= \kappa \sum_{k=1}^{n} \frac{d_{ik}}{{\Delta_k}^2} \left[\left(\frac{\partial w_k}{\partial \lambda_j} \lambda_k + w_k \delta_{kj} \right) \Delta_k - (w_k \lambda_k + 1) d_{jk} \right] \\ &= \kappa \sum_{k=1}^{n/2} \frac{d_{i,2k-1}}{{\Delta_1}^2} \left[\left(\frac{\partial w_{2k-1}}{\partial \lambda_j} \frac{2}{n} + w_1 \delta_{2k-1,j} \right) \Delta_1 - \left(w_1 \frac{2}{n} + 1 \right) d_{j,2k-1} \right] \\ &\quad + \kappa \sum_{k=1}^{n/2} \frac{d_{i,2k}}{{\Delta_2}^2} (w_2 \delta_{2k,j} \Delta_2 - d_{j,2k}) \\ &= \kappa \sum_{k=1}^{n/2} \frac{d_{i,2k-1}}{d_1} \left[\frac{\partial w_{2k-1}}{\partial \lambda_j} + \frac{n}{2} w_1 \delta_{2k-1,j} - \frac{n}{4} (2w_1 + n) \frac{d_{j,2k-1}}{d_1} \right] \\ &\quad + \kappa \sum_{k=1}^{n/2} \frac{d_{i,2k}}{d_2} \left(\frac{n}{2} w_2 \delta_{2k,j} - \frac{n^2}{4} \frac{d_{j,2k}}{d_2} \right). \end{aligned} \tag{A.37}$$

By setting

$$W_{11} = \left(\frac{\partial w_{2i-1}}{\partial \lambda_{2j-1}} \mid i, j = 1, \ldots, n/2 \right)$$

and using

$$\kappa(2w_1 + n) = w_1(1+\kappa),$$

which follows from (A.36), we can rewrite (A.37) in a matrix form as

$$W_{11} = \kappa \frac{D_{11}}{d_1} W_{11} + \frac{n w_1}{2} \frac{D_{11}}{d_1} \left(\kappa I - \frac{1+\kappa}{2} \frac{D_{11}}{d_1} \right) - \frac{\kappa n^2}{4} \frac{D_{12}}{d_2} \frac{{D_{12}}^\top}{d_2},$$

where $D_{12}{}^{\top} = D_{21}$ by $D^{\top} = D$. The above equation can be solved for W_{11} as

$$\begin{aligned} W_{11} &= \frac{nw_1}{2}\left(I - \kappa\frac{D_{11}}{d_1}\right)^{-1}\frac{D_{11}}{d_1}\left(\kappa I - \frac{1+\kappa}{2}\frac{D_{11}}{d_1}\right) \\ &\quad - \frac{\kappa n^2}{4}\left(I - \kappa\frac{D_{11}}{d_1}\right)^{-1}\frac{D_{12}}{d_2}\frac{D_{21}}{d_2}. \end{aligned}$$

Then using this equation and (A.5), we can obtain

$$\begin{aligned} V_{11}(\hat{\boldsymbol{\lambda}}) &= \frac{n}{2}\left[\kappa'\frac{D_{11}}{d_1} + \left(I - \kappa\frac{D_{11}}{d_1}\right)^{-1}\frac{D_{11}}{d_1}\left(\kappa I - \frac{1+\kappa}{2}\frac{D_{11}}{d_1}\right)\right] \\ &\quad - \frac{\kappa n^2}{4w_1}\left(I - \kappa\frac{D_{11}}{d_1}\right)^{-1}\frac{D_{12}}{d_2}\frac{D_{21}}{d_2}. \end{aligned} \tag{A.38}$$

For the eigenvector $\boldsymbol{\eta}$ in (A.33), we have $D_{11}\boldsymbol{\eta} = \tilde{\epsilon}\boldsymbol{\eta}$ and $D_{21}\boldsymbol{\eta} = \mathbf{0}$ in (A.34), the former of which is rewritten as $(D_{11}/d_1)\boldsymbol{\eta} = \epsilon\boldsymbol{\eta}$ with

$$\begin{aligned} \epsilon = \frac{\tilde{\epsilon}}{d_1} &= \left(\frac{1-r^2}{1+r^2}\right)^2 \frac{1-(-1)^{n/4}r^{n/2}}{1-r^{n/2}} \\ &= \begin{cases} \left(\dfrac{1-r^2}{1+r^2}\right)^2 & \text{(if } n \text{ is a multiple of 8),} \\ \left(\dfrac{1-r^2}{1+r^2}\right)^2 \dfrac{1+r^{n/2}}{1-r^{n/2}} & \text{(if } n \text{ is a multiple of 4, not a multiple of 8).} \end{cases} \end{aligned} \tag{A.39}$$

Therefore, (A.38) gives

$$V_{11}(\hat{\boldsymbol{\lambda}}) \cdot \boldsymbol{\eta} = \gamma\boldsymbol{\eta}$$

with

$$\gamma = \frac{n}{2}\left[\kappa'\epsilon + (1-\kappa\epsilon)^{-1}\cdot\epsilon\left\{\kappa - \frac{1+\kappa}{2}\epsilon\right\}\right].$$

Multiplying (A.35) by the vector $\boldsymbol{\eta}$ from the right and using

$$\mathbf{1}^{\top}\boldsymbol{\eta} = 0, \quad \mathbf{1}^{\top}V_{11}(\hat{\boldsymbol{\lambda}})\cdot\boldsymbol{\eta} = \gamma\mathbf{1}^{\top}\boldsymbol{\eta} = 0,$$

we obtain

$$J_{11}(\hat{\boldsymbol{\lambda}})\cdot\boldsymbol{\eta} = \frac{2\gamma}{n}\boldsymbol{\eta}.$$

Then the eigenvalue β of the Jacobian matrix $J_{11}(\hat{\boldsymbol{\lambda}})$ for the eigenvector $\boldsymbol{\eta}$ is expressed in terms of ϵ as

$$\beta = \Psi(\epsilon)$$

with a function Ψ defined as

$$\Psi(x) = \kappa' x + \frac{x\{\kappa - [(1+\kappa)/2]x\}}{1-\kappa x} = \frac{(\kappa+\kappa')x + \left(\kappa\kappa' + \dfrac{1+\kappa}{2}\right)x^2}{1-\kappa x}. \tag{A.40}$$

It should be clear that $J_{11}(\hat{\boldsymbol{\lambda}})\boldsymbol{\eta} = \beta\boldsymbol{\eta}$ implies $J(\hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\eta}} = \beta\hat{\boldsymbol{\eta}}$, and hence β is the eigenvalue of $J(\hat{\boldsymbol{\lambda}})$ for $\hat{\boldsymbol{\eta}}$ in (A.30).

The break point $\tau_{2,n}$ is determined from the condition that the eigenvalue β for $\tau = \tau_{2,n}$ vanishes. The value ϵ^* satisfying $\Psi(\epsilon^*) = 0$ is given by

$$\epsilon^* = \epsilon_2^* = \frac{\kappa+\kappa'}{\kappa\kappa' + \dfrac{1+\kappa}{2}}, \tag{A.41}$$

which is independent of n. Since $0 < \epsilon < 1$, the use of (A.7) in (A.41) gives a condition:

$$\kappa\kappa' + \frac{1+\kappa}{2} - (\kappa+\kappa') = \frac{(\mu-\sigma)(2\mu-\sigma+1)}{2\sigma(\sigma-1)} > 0,$$

which, together with $\sigma > 1 > \mu > 0$, yields

$$\frac{\mu}{\sigma-1} < \frac{1}{2}. \tag{A.42}$$

The value of $r^* = r(\tau_{2,n})$ corresponding to $\epsilon = \epsilon^*$ can be determined from (A.39), and $\tau_{2,n}$ is given as

$$\tau_{2,n} = \frac{n}{\mathcal{L}(\sigma-1)}(-\log r^*) \tag{A.43}$$

from (A.8).

When $n = 8\ell$ for an integer $\ell \geq 1$, (A.39) yields

$$r^* = \sqrt{\frac{1-\sqrt{\epsilon^*}}{1+\sqrt{\epsilon^*}}}.$$

The substitution of this into (A.43) yields

$$\tau_{2,n} = c_2 n \tag{A.44}$$

with

$$c_2 = \frac{1}{2\mathcal{L}(\sigma-1)} \log\left(\frac{1+\sqrt{\epsilon^*}}{1-\sqrt{\epsilon^*}}\right). \tag{A.45}$$

Note that c_2 is a constant independent of n. Under the condition (A.25), we have

$$\epsilon^* \approx \frac{4\mu}{\sigma - 1} \ll 1,$$

and hence c_2 in (A.45) and, in turn, $\tau_{2,n}$ in (A.44) can be approximated as

$$\tau_{2,n} \approx \frac{\sqrt{\epsilon^*}}{\mathcal{L}(\sigma - 1)} n \approx \frac{2\mu^{1/2}}{\mathcal{L}(\sigma - 1)^{3/2}} n. \tag{A.46}$$

Thus we have proved (3.9) for $m = 2$ with (3.10) and (3.13) in Section 3.3.

When $n = 8\ell - 4$ for an integer $\ell \geq 1$, an approximate formula is not available.