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ARATA Yoshiyuki
RIETI

KIMURA Yosuke
University of Tokyo

MURAKAMI Hiroki
University of Tokyo

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Macroeconomic Consequences of Lumpy Investment under Uncertainty[1]

ARATA Yoshiyuki [*]

Research Institute of Economy, Trade and Industry

KIMURA Yosuke

Graduate School of Economics, University of Tokyo

MURAKAMI Hiroki

Graduate School of Economics, University of Tokyo

Abstract

The stability of aggregate investment is central to our understanding of macroeconomic dynamics. It is also important to understand the impact of individual behaviors on aggregate patterns. Empirical studies have shown that individual investment behavior, at the plant level, is characterized by lumpiness and infrequency. The existing literature about the uncertainty-investment relationship has shown that uncertainty negatively affects investment decisions, i.e., when facing high uncertainty, firms prefer to “wait and see.” The aim of this paper is to investigate how the lumpiness of investment activity under uncertainty affects the stability of the macroeconomy. We show that heterogeneity in capital adjustment processes of micro-agents contributes to the stability at the macroscopic level, and that this macro-stability is undermined in uncertain circumstances with interacting agents. In other words, while heterogeneous behaviors always enhance the stability of aggregate investment, coordination of interacting investors weakens this stabilizing force. As a result, a slight shock is sufficient to induce instability when investors are highly sensitive to uncertainty over economic environment. This is an explanation of simultaneous reduction of investment activities under high uncertainty, leading to a prolonged economic downturn.

Keywords: Lumpy investment, Pseudo compound poisson process, Uncertainty, Collective behavior

JEL classification: E22, E32, D21

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* Corresponding author, E-mail: arata-yoshiyuki@rieti.go.jp

1 Introduction

Fluctuations of aggregate investment have been recognized as an important source of economic fluctuations such as those of GDP. The stability of aggregate investment is central to our understanding of macroeconomic dynamics. The importance of fluctuations in aggregate investment has led to the study on the relationship between individual and aggregate investment. In order to investigate how these fluctuations of aggregate investment are driven, it is a natural idea to ask how microeconomic agents (i.e., firms) make a decision on capital adjustment.

A large body of literature on investment has shown that individual investment behavior has two remarkable properties. First, investment activity at the plant level is infrequent and lumpy, that is, periods of low investment activity are followed by bursts of investment activity (e.g., Doms and Dunne (1998)). It is shown that in the presence of fixed costs of adjustment capital, the (S, s) type policy, which is characterized by an inactive range between a trigger value and a target value, is optimal (e.g., Caballero and Engel (1999)). Namely, when a firm faces a fixed cost of adjusting capital and the loss by nonadjustment is small, it is advantageous to do nothing until the profits of capital adjustment exceed the costs. Thus, firms do not immediately or continually change their capital level; their activities are characterized by inactive periods and large investment ones. The second is the relationship between uncertainty and investment. The *real option theory* (e.g., Dixit and Pindyck (1994)) provides a theoretical explanation of investment behavior when decision makers face uncertainty (e.g., uncertainty about future output prices, economic conditions, and rates of return) and their decisions are irreversible. This theory suggests that "there will exist an 'option' value to delay an investment decision in order to await the arrival of new information about market conditions"(Carruth et al., 2000, p.120). Accordingly, greater uncertainty raises the "option" value to delay investment and thus firms postpone their investment decisions.

The aim of this paper is to investigate how these microeconomic investment behaviors are related to aggregate investment dynamics. Recently, there is a growing literature discussing the impact of intermittent, lumpy investment on the aggregate investment, but there is no unanimous agreement about the aggregate significance of lumpy investment.[1] In this paper, we emphasize

[1] Caballero (1999, p.815) argues that "[i]t turns out the changes in the degree of coordination of lumpy actions play an important role in shaping the dynamic behavior of aggregate investment." On the other hand, Thomas (2002) presents the result that plant-level lumpy investment is irrelevant for aggregate activities in the general equilibrium model: "In contrast to previous partial equilibrium analyses, model results reveal that the aggregate effects of lumpy

the importance of *heterogeneity* across firms. Heterogeneous behavior across firms can be generated by, for example, idiosyncratic fixed costs of capital adjustment or heterogeneous expectations about the economic condition in the future, and the resulting heterogeneity is described by the (cross-sectional) *distribution* of capital stock across firms, which represents the macroscopic state (i.e., aggregate capital). In this paper, we analyze the evolution of the distribution underlying macroeconomic variables in order to investigate whether individual investment activities affect the fluctuation of aggregate investment dynamics. In this sense, our approach is close in spirit to the so-called *heterogeneous agents models* (see, e.g., Aoki (1996), Hommes (2006), and Kirman (2010)), where heterogeneity across economic agents plays a crucial role to explain regularity observed only at the macroscopic level and aggregate behaviors which are different from microeconomic behaviors of individual agents.

In this paper, we also take into account *interaction* among agents or a *feedback-loop*, which describes the behavior of an individual firm being affected by those of other firms. In particular, we consider the effect of uncertainty on individual investment activities as a source of the feedback-loop and the variance of the distribution of the capital stock as the realization of uncertainty of the entire economy. In uncertain circumstances, it is difficult to predict returns and losses associated with the current economic activity, and the difficulty generates disagreement among business forecasts of decision makers.[2] As a result, a large disagreement induces a large diversity of individual states across agents. Thus, uncertainty materializes as heterogeneous behaviors or states of microeconomic agents and is measured by the variance of the distribution. In addition, whenever firms decide whether to invest or not, they gather relevant information in order to forecast the unpredictable outcomes and pay attention not only to firm-specific information but also to forecasts of other firms. They may behave more cautiously after observing an increase in dispersion among other firms' expectations or actions (e.g., Bachmann et al. (2013)). This process generates a feedback loop between micro-behaviors and uncertainty in the form of the variance. This idea is close to the "macro-micro loop" emphasized by Hahn (2002), in which he argues that it is ignored in the

investment are negligible. In general equilibrium, households' preference for relatively smooth consumption profiles offsets changes in aggregate investment demand implied by the introduction of lumpy plant-level investment" (Thomas, 2002, p.508). However, Gourio and Kashyap (2007) recalibrate the Thomas's model and obtain different properties from the standard RBC model.

[2] The relationship between uncertainty and disagreement and related literature are discussed in Section 4.

literature of modern macroeconomics.[3] We show that a strong feedback loop affects the stability of the macroscopic distribution and gives rise to a remarkable phenomenon, i.e. collective behavior. In the existing literature, the mean-field effect has been intensively investigated (see, e.g., Opper and Saad (2001)), but to the best of our knowledge, the effect of the variance has been ignored in the literature mentioned above. We present an alternative method to analyze collective behavior of this type in this paper.

When analyzing the effect of heterogeneity and macro-micro feedback-loops, one may have difficulty in solving explicitly the optimization problems. In order to overcome the difficulty, we firstly show that the individual investment process is described by a *Feller process*, especially a *pseudo-compound Poisson process with drift*. This stochastic process whose sample path continuously declines and infrequently jumps is adequate for describing infrequent and lumpy activity. Our approach is justified as long as the individual investment process is a Markov process and helps us to avoid the complexity of optimization problems. Thus, we can directly deal with heterogeneity because each realization of the stochastic process reflects heterogeneity in agents' behaviors, and, what is more, we can, by analyzing the evolution of the distribution, show that heterogeneity creates the macro-stability in the sense that arbitrary initial distributions converge to a (unique) stationary distribution. It should be noted that this stabilization effect is completely different from the adjustment of market mechanisms (e.g., the change of an interest rate). Furthermore, the introduction of a jump intensity function dependent on the distribution itself enables us to investigate the effect of feedback-loops between micro-agents and the macroeconomy. Although it is an unrealistic method to solve explicitly this stochastic process because of its nonlinearity, the stability of the capital adjustment process at the aggregate level can be analyzed with the help of Gaussian approximation and numerical simulations. Consequently, we obtain the result that the stabilization effect is defeated if individual investment activity depends strongly on uncertainty (i.e., variance). Namely, under the condition that firms are highly sensitive to uncertainty, they collectively "wait and see" in spite of a large deviation from their desired levels of capital, and this

[3] Of course, the importance of a feedback mechanism is recognized in other strands of literature. For example, Caplin and Leahy (1997, pp.601-602) emphasize that "[o]ne of the most limiting aspects of these models is that they focus exclusively on the impact that microeconomic inertia has on aggregate dynamics. They ignore the feedback from aggregates onto individual behavior." Thomas (2002) constructs a dynamic stochastic general equilibrium model in order to include a feedback mechanism. We also follow the statement of Caplin and Leahy (1997) by introducing the feedback effect between micro units and macroscopic distributional properties.

collective inaction breaks down the macro-stability.

The plan of this paper is as follows. In section 2, we demonstrate that the discontinuous and lumpy adjustment process of microeconomic units possesses the Markov property. In section 3, we first introduce Feller processes and the representation result for them. This representation justifies our strategy of dispensing with an explicit optimization problem. Secondly, the lumpy behavior of microeconomic units is described by a pseudo-compound Poisson process with drift. Finally, we introduce the stabilization effect caused by randomness. Section 4 explains the effect of uncertainty and feedback loops. In section 5 we analyze the stability of the macroscopic distribution by using Gaussian approximation. In section 6, we perform some numerical simulations of our model and demonstrate the stable and the unstable behavior of the aggregate capital adjustment process. Section 7 concludes this paper.

2 Lumpy Investment

Empirical studies have shown that investment activity at the plant level is characterized by long periods of inactivity punctuated by infrequent and lumpy investment. For example, Doms and Dunne (1998) use U.S. manufacturing plants-level data and find that many plants change their capital stock in lumpy fashion, i.e., that they invest in lumps. Using Norwegian micro data, Nilsen and Schiantarelli (2003) show the intermittent and lumpy nature of investment rates. In particular, they emphasize the importance of the occurrence of zero investment episodes at the plant level. In addition, there are studies estimating plant-level hazard functions of lumpy investments. For example, Cooper et al. (1999) obtain the result that the hazard function is increasing in the time elapsed since the last lumpy investment using longitudinal data for approximately 6,900 manufacturing plants in the U.S. and a semiparametric specification.

Researchers have developed dynamic models to explain these micro-level characteristics. Within these models,[4] lumpy investment activities are typically driven by the presence of fixed costs, and characterized by the (S, s)-type models. A standard (S, s) model predicts that the level of capital stock immediately reaches the target point S if it reaches the trigger barrier s due to depreciation. Thus, a firm chooses inaction until its capital stock reaches the trigger level, and it chooses an

[4] Examples include Caballero and Engel (1991), Cooper et al. (1999), Caballero and Engel (1999) and Thomas (2002).

investment action after reaching the trigger level. However, while the standard (S, s) model is successful in characterizing the intermittent and lumpy behavior, it cannot capture the increasing hazard function of lumpy investment observed in micro data because the inactive range is fixed over time in the model. Therefore, researchers have proposed the generalized (S, s) models in which the probability of a large investment episode is increasing in the deviation between the desired and the actual stock of capital or the time elapsed since last investment. One possible strategy is to introduce randomness in the adjustment cost (see, e.g., Caballero and Engel (1999)).

While the existing literature adopts a stochastic trigger in order to describe the empirical features, one can consider a more general situation in which there is heterogeneity in desired levels S across firms and over time. Thus, we take the differences in target levels across firms into account as an additional source of heterogeneity. However, we do not employ a strategy to formulate an optimization problem for a firm (allowing for variable S) for the derivation of an optimal investment behavior. Rather, in this paper, we show that, whatever the optimization problem is, the resulting investment process turns out to be a Feller process. Using the representation of Feller processes, we can choose an appropriate process consistent with empirical data (i.e., lumpiness) among them (see the next section). The prerequisite for this strategy is that the optimal process has the Markov property, which, as explained below, is the common feature of the existing models and a quite natural assumption. For this reason, our model not only entails the existing (generalized) (S, s) models, but also expresses more general situations.

In the rest of this section, we explain that the generalized (S, s) model has the Markov property. Suppose an economy that each firm faces an optimization problem to choose whether to undertake an investment action after observing the realizations of random adjustment cost (like the generalized (S, s) model). At any point of time, each firm has the opportunity to adjust the capital stock but the adjustment can be done only when the firm pays the fixed cost. It is optimal for each firm to choose an investment action if the present value is greater when it adjusts capital than when it does not. It should be noted that both values depend on the current level of its capital stock and the expectations of future profits, and the former value depends on a fixed cost at the current time. The history of capital stock levels is irrelevant to the calculation of the value because that of the present value is forward looking. When comparing the values if the firm adjusts its capital stock and if it does not adjust for any given capital stock levels, the current fixed cost determines whether to invest. If the current fixed cost is small and hence the former value is greater than the latter value, it chooses to invest. Otherwise, it chooses to do nothing and its capital depreciates. As a result, the firm's optimal policy depends only on the current states of the firm.

Formally, the above discussion can be described as follows. Let $\{X(t), t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}$, and let $\mathcal{F}_t^X = \sigma(X(s) : s \leq t)$, i.e., the information set generated by $\{X(s) : s \leq t\}$. If the state of a firm (i.e., the level of capital stock) at time t is represented by $X(t) \in \mathbb{R}$, an optimal investment policy discussed above implies that the conditional probability of $X(t)$ satisfies the following property;

$$P\{X(t+s) \in B | \mathcal{F}_t^X\} = P\{X(t+s) \in B | X(t)\}, \tag{1}$$

for all s, $t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is Borel sets of $\mathbb{R}$. This is the definition of Markov processes, and means that the probability distribution of the future level of its capital stock depends only on the current level of its capital stock. An optimal investment process, which depends only on the current state, derived as a solution of an optimization problem must be a Markov process. In the next section, we discuss how to characterize a Markov process representing an optimal investment policy, without explicitly solving an optimization problem of a firm's investment decision.

3 Feller process

3.1 Courrege's theorem

In the remaining of this paper, we investigate the stochastic process concerning the logarithm of a firm's capital $X_t \equiv \log(k_t)$. As discussed in Section 2, individual investment activities are characterized by lumpiness and discontinuity. When investment projects involve construction of new plants or the purchase of large equipment, small adjustments of the capital stock are infeasible. In order to represent the empirically observed feature of individual investment behavior, it is natural to assume that the investment process is characterized by *jump* processes, rather than continuous movements. The existing literature of lumpy investment at the plant-level employs the (S, s) model or its modifications in order to describe the character of discontinuous jumps. In this paper, we consider a more generalized one called a *pseudo-compound Poisson process with drift* to represent the capital adjustment process.

Before discussing how to characterize the capital adjustment process by a particular stochastic process, we begin with an explanation of Feller processes, which are Markov processes satisfying the properties below. As we discussed in the previous section, the stochastic process of firms' capital is described by a Markov process. If this process satisfies additional properties, we can conclude that the capital adjustment process of individual firms is a Feller process. Then, using the representation of Feller processes, we can choose an appropriate Markov (Feller) process exhibiting the empirically observed behavior (i.e., lumpy investment). This is advantageous if our focus is on the aspects of the economy that heterogeneity and interaction across firms have a great impact on macroeconomy, because it helps us to avoid solving difficult and complex optimization problems with heterogeneous interacting agents.

To begin with, we introduce some mathematical notations necessary for the following analysis. Let $(T_t)_{t\geq 0}$ be a Markov semigroup defined by

$$(T_t f)(x) = E(f(X(t))|X(0) = x) \quad \text{for each } f \in B_b(\mathbb{R}), x \in \mathbb{R},$$

where we denote the bounded Borel measurable functions by $B_b(\mathbb{R})$ and the real line by $\mathbb{R}$. If we set $f = \mathbf{1}_B$ (the indicator function, i.e., $\mathbf{1}_B(x) = 1$ if $x \in B$ and 0 otherwise), the transition function is obtained:

$$p_t(x, B) = P(X(t) \in B|X(0) = x) \quad \text{for } B \in \mathcal{B}(\mathbb{R}), x \in \mathbb{R}.$$

Markov processes are characterized by the corresponding transition function (or, equivalently, the Markov semigroup).

To proceed further, we need an additional assumption: The Markov semigroup $(T_t)_{t\geq 0}$ representing the individual investment process satisfies the following properties.

Assumption 1

$$T_t u \in C_\infty(\mathbb{R}), \quad \forall u \in C_\infty(\mathbb{R}), \quad t > 0 \quad \text{(the Feller property)},$$

$$\lim_{t\to 0} ||T_t u - u|| = 0, \quad \forall u \in C_\infty(\mathbb{R}) \quad \text{(the strong continuity)}.$$

Here, we denote by $C_\infty(\mathbb{R})$ the space of continuous functions vanishing at infinity. A Markov semigroup $(T_t)_{t\geq 0}$ satisfying these properties is called a Feller semigroup. In Appendix A.1, we discuss the economic meaning of this assumption, concluding that this is a plausible assumption in an economic sense. In short, the Feller property is satisfied if there exists a physical (or financial) upper bound of the size of lumpy investment, and the strong continuity requires only that the *exact timing* when each firm invests in lumps is unknown to outside observers (note that we consider the continuous time).

Next, we define a Feller generator that characterizes the structure of the corresponding Feller semigroup.

Definition 2 *A Feller generator of a Feller semigroup $(T_t)_{t\geq 0}$ is a linear operator $(A, \mathcal{D}(A))$ defined by*

$$\begin{aligned}\mathcal{D}(A) &\equiv \{u \in C_\infty(\mathbb{R}) \ : \lim_{t\to 0} \frac{T_t u - u}{t} \ \text{ exists as a uniform limit}\},\\ Au &\equiv \lim_{t\to 0} \frac{T_t u - u}{t}, \quad \forall u \in \mathcal{D}(A).\end{aligned}$$

The following theorem proven by Courrège (see, Courrège (1965-1966) and Böttcher et al. (2013)) characterizes the structure of the Feller generators.

Theorem 3 (**Courrège**) *Let $(A, \mathcal{D}(A))$ be a Feller generator. If $C_c^\infty(\mathbb{R}) \subset \mathcal{D}(A)$,*[5] *then, A is of the following form*

$$(Au)(x) = -e(x)u(x) + l(x)\frac{du(x)}{dx} + \frac{1}{2}Q(x)\frac{d^2u(x)}{dx^2} + \int_{\mathbb{R}-\{0\}} [u(x+y) - u(x) - \frac{du(x)}{dx} y \mathbf{1}_{B_1}(y)] N(x, dy) \quad (2)$$

where $e(x) > 0$, $(l(x), Q(x), N(x, \cdot))$ is, for fixed $x \in \mathbb{R}$, a Lévy triplet: $l(x) \in \mathbb{R}$, $Q(x) \geq 0$, and $N(x, \cdot)$ is a Borel measure on $\mathbb{R}$ (called Lévy measure) such that $\int_{\mathbb{R}\setminus\{0\}} \min(|y|^2, 1) N(x, dy) < \infty$ and $N(x, 0) = 0$.

When $e \equiv 0$ and neither l, Q nor N depends on x, it corresponds to the generator of Lévy processes. Namely, a Feller process determined by a Feller semigroup behaves locally like a Levy process (with killing when $e(x) \neq 0$) in the sense that the generator is given by the x-dependent

[5] Here, $C_c^\infty(\mathbb{R})$ denote the set of smooth functions on $\mathbb{R}$ with compact support. We simply assume $C_c^\infty(\mathbb{R}) \subset \mathcal{D}(A)$, that is, A is defined on the set of test functions that is large enough.

Lévy triplet (l, Q, N). It is possible to interpret probabilistically l, Q, and N as a drift term, a diffusion term, and a jump term with variable coefficients.[6]

It means that, whatever an optimization problem is, the stochastic behavior of firms' capital is described by (2), especially l, Q and N. In other words, even if we begin with the formulation of the optimization problems that firms choose whether and when to invest, the optimal behavior turns out to be the stochastic process characterized by l, Q and N. Therefore, we choose an appropriate Feller process exhibiting empirically observed behavior (i.e., lumpy investment) bypassing the formulation of optimization problems.

3.2 Pseudo-Poisson (Compound Poisson) process

In order to represent lumpiness and infrequency of individual investment behavior, it is natural to assume that the investment process is characterized by a *jump* process.[7] Namely, discrete movements of the jump processes correspond to lumpy investment of the capital adjustment processes and intervals between jumps correspond to periods of inactivity. Among the Feller processes represented by equation (2), the processes displaying these properties are pseudo-compound Poisson processes with drift, where the drift term represents capital depreciation and stochastic realization of jumps represents lumpy investment.

More precisely, a pseudo-compound Poisson process with drift is characterized by the following generator

$$(Bf)(x) = -\delta \frac{df(x)}{dx} + \int_{\mathbb{R}} [f(x+y) - f(x)]\lambda(x)\nu(x, dy), \quad \text{for each } x \in \mathbb{R}. \tag{3}$$

Here, $\delta > 0$ represents depreciation rate. The second term in the right-hand side represents a generalization of a compound Poisson process where the intensity rate $\lambda(x)$ determines the number of jumps, and $\nu(x, dy)$ determines the size of independent jumps. Note that they depend on x. Compared with the generalized (S, s) models discussed in Section 2, the upper bound (the target level) S to which the capital level jumps is also a stochastic random variable. In this sense,

[6]In fact, this interpretation is justified because the x-dependent Lévy triplet of Feller processes and the differential characteristics of Ito processes coincides with each other. For more details, see Section 2.5 in Böttcher et al. (2013)

[7]In general, a conservative Markov semigroup (i.e., firms do not disappear) does not imply that the killing rate e is equal to 0. However, if jump size is bounded as discussed in A.1, conservativeness implies $e = 0$ by Lemmas 2.28 and 2.32 in Böttcher et al. (2013). Therefore, we assume $e = 0$ in the following.

our formalization presents a generalization of (S, s) and "generalized (S, s)" models. Because we consider the cyclical movement of the capital adjustment process, $\lambda(x)$ is assumed to be a nonincreasing function. Namely, when X_t is small, the probability of a jump (i.e., lumpy investment) becomes high. A sample path of this process is shown in Figure 1 (for the functional forms of $\lambda(x)$ and $\nu(x, dy)$, and parameters, see Sections 5 and 6). The path declines continuously due to depreciation (the first term in equation (3)) and infrequently jumps (the second term in equation (3)) when lumpy investment occurs. It clearly shows a saw tooth pattern representing lumpy behavior of a firm.

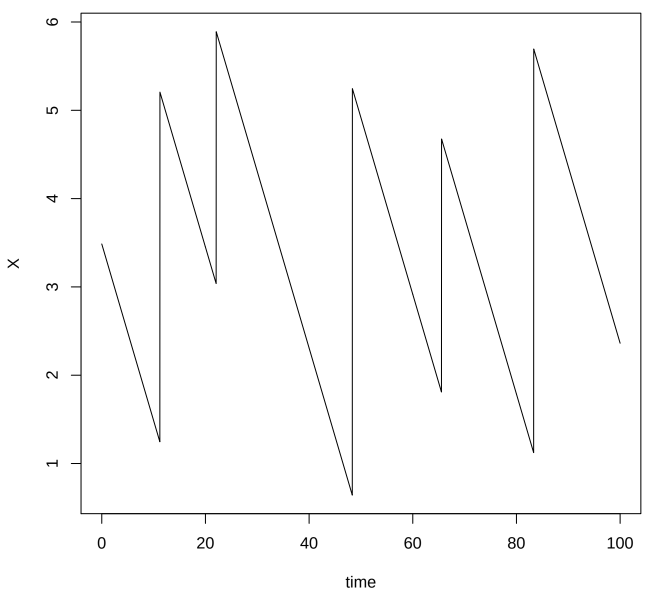

Figure 1: A sample path.

3.3 Stabilization Effect Caused by Randomness

Our next concern is the evolution of the probability distribution of X_t because we are interested in the stability of aggregate variables which are determined by the distribution. Let $\mathcal{M}(\mathbb{R})$ be a family of probability measures on $\mathbb{R}$ and $\mu_0 \in \mathcal{M}(\mathbb{R})$ an initial one. The probability distribution of

X_t, μ_t, with the initial distribution μ_0 and transition function $p_t(x, B)$ is given by

$$\mu_t(B) = \int_{\mathbb{R}} p_t(x, B)\mu_0(dx),\ \ B \in \mathcal{B}(\mathbb{R}).$$

A probability measure $\mu^* \in \mathcal{M}(\mathbb{R})$ is said to be invariant (or stationary) if μ^* satisfies the following property:

$$\int_{\mathbb{R}} p_t(x, B)\mu^*(dx) = \mu^*(B),\ \ t \geq 0,\ \ B \in \mathcal{B}(\mathbb{R}).$$

This means that if the initial distribution is an invariant measure, the distribution after time t is also given by the same invariant measure. Intuitively speaking, even if individual agents behave heterogeneously and their states change over time, the economic environment represented by the distribution μ^* does not change at the macroscopic level. By the Krylov-Bogolyubov theorem (see, e.g., Corollary 3.1.2 in Da Prato and Zabczyk (1996)), it can be shown that our stochastic process characterized by (3) has an invariant measure.[8]

Because the timing of investment and its size are stochastic, the states of firms disperse over time even if they start at the same initial point. It is expected that by this randomness, an arbitrary initial distribution converges to an invariant distribution μ^* as $t \to \infty$. In fact, by Doob's theorem (see, for instance, Section 4.2 in Da Prato and Zabczyk (1996)), it can be shown that the invariant measure μ^* is unique and the transition function converges to μ^* as $t \to \infty$:[9]

$$\lim_{t \to +\infty} p_t(x, B) = \mu^*(B), \quad B \in \mathcal{B}(\mathbb{R}). \tag{4}$$

This implies that the probability distribution μ_t at time t converges to the unique probability measure μ^* as $t \to \infty$. Because aggregate variables (e.g., capital per capita $k_t \equiv \int_{\mathbb{R}} \exp(x)\mu_t(dx)$)

[8]Da Prato and Zabczyk (1996) employ a different definition of the Feller semigroup: the C_b-Feller semigroup. However, if we consider a compact set $[-E, E]$ instead of $\mathbb{R}$ as discussed in the next footnote, these two definitions coincide. See also the next footnote.

[9]More precisely, in order to apply the Krylov-Bogolyubov theorem and Doob's theorem, we need to modify the stochastic behavior at large $|X_t|$. For example, we have to assume that the depreciation rate $\delta(x)$, which was simply assumed to be constant, becomes 0 at $x \ll 0$ and the measure $\nu(x, \cdot)$ has a compact support, in order that $\int_{[-E,E]} p_t(x, y)dy = 1,\ \forall t \geq 0,\ \forall x \in [-E, E]$ for some $E > 0$. Then, we redefine our stochastic process on $[-E, E]$ and apply the two theorems. Although these procedures are artificial, it is unreasonable to expect that they affect the macroscopic behavior of our system. In fact, as we will see later, the well-posedness (i.e., the existence and uniqueness of the solution) and convergence are confirmed by the following stability analysis and numerical simulations.

are uniquely determined by μ_t, the convergence of probability distributions also implies that of aggregate variables (capital per capita $k^* \equiv \int_{\mathbb{R}} \exp(x)\mu^*(dx)$). Thus, the macroeconomic system represented by the aggregate variables is stable. In addition, the paths of convergence of aggregate variables can differ even with the same initial values of aggregate variables because the path depends on the underlying distribution. It is worth noting that this stability of the system is achieved by *randomness* (or heterogeneity), instead of market mechanisms (e.g., the adjustment of interest rate). We call this effect the *stabilization effect caused by randomness*

This stabilizing effect is reminiscent of the concept of "stochastic macro-equilibrium", which is originally advanced by Tobin (1972) in his attempt to explain the observed Phillips curve.[10] He argues that "a theory of stochastic macro-equilibrium: stochastic, because random inter-sectional shocks keep individual labor markets in diverse states of disequilibrium; macro-equilibrium, perpetual flux of particular markets produces fairly definite aggregate outcomes..." (Tobin, 1972, p.9). In other words, heterogeneity across individual markets incessantly exists because of random shocks, and continuous transitions over states of markets create the stable outcomes at the aggregate level. One of our contributions of this paper is to provide a theoretical foundation of Tobin's stochastic macro-equilibrium.

As we show in the following sections, however, this stochastic macro-equilibrium can be undermined in the presence of uncertainty.

4 Uncertainty and Feedback-loop

In this section, we explicitly take into account the effect of uncertainty. As discussed in Introduction, uncertainty is an important factor that affects investment decisions. Facing high uncertainty, "the firm prefers to 'wait and see' rather than undertaking a costly action with uncertain consequences"(Bloom et al. (2007), p.391). The "wait and see" effect has been highlighted since the Great Recession in the late 2000s because economists have tried to find the reason for the prolonged recession in uncertainty. Business leaders tend to behave cautiously when it is difficult to predict the future outcomes of investment projects. If the entire economy faces uncertain environment, a large part of firms may abandon their investment projects or "wait and see" until the arrival of

[10] Foley (1994) also coined the concept of "statistical equilibrium" in the field of the Walrasian equilibrium theory by borrowing the concept of entropy from physics.

new information. This simultaneous reduction of investment activities may lead to an economic downturn.

In this regard, one can raise a natural question: How can we measure uncertainty in the real economy? Recent empirical studies have constructed and examined various measures of time-varying uncertainty; for example, the volatility of stock market returns (VIX index is widely used, see e.g., Bloom (2014)), the variance of asset returns (Leahy and Whited (1996)), the conditional volatility of the component of the economic time series purely unforecastable from models (Jurado et al. (2015)), and the frequency of newspaper references to uncertainty (Baker et al. (2013)).[11] In this paper, uncertainty is assumed to be measured by the variance of X_t, σ_t^2. In terms of the literature mentioned above, our measure of uncertainty is close to *disagreement* of firm expectations, which is one of the most commonly used measures of uncertainty (see Bachmann et al. (2013) in which they conclude that forecast disagreement among firms is a good measure of uncertainty).[12] Under a more uncertain circumstance, the difficulty of prediction over the future consequences generates a disagreement among business forecasts of business leaders. Moreover, this disagreement leads to a dispersion of investment actions because investment decisions are based on predictions of future economic conditions. Some of firms may make an investment in plants and equipment if they predict an economic boom. The others may refrain from investment if they predict that it would cause a loss because of an economic downturn. As a result, a large disagreement generates a large diversity of individual capital stocks across firms. Thus, uncertainty materializes as heterogeneous behaviors or states of microeconomic agents and, therefore, is measured by the variance of the distribution.

On the other hand, when an increase in the dispersion among other firms is observed, that is, when no consensus on the economic condition in the future has been formed, firms may become more uncertain themselves and put off investment until the arrival of new information. This process generates a feedback-loop between micro-behaviors and the variance as a measure of uncertainty. It is worth noting that we do not assume that uncertainty is simply given exogenously, as in the existing literature including Bloom (2009). We emphasize that, whereas the macroeconomic

[11] Bloom (2014) surveys both empirical and theoretical literatures that study the relationship between uncertainty and economic activity.

[12] Papers in the literature that use a measure of uncertainty close to the one employed here include Zarnowitz and Lambros (1987), Bomberger (1996), Giordani and Söderlind (2003), Fuss and Vermeulen (2008), and Clements (2008), to name a few.

environment affects the firms' behaviors, an aggregation of firms' behaviors generates the environment in which firms engage in business activity. This feedback mechanism is closely related to the "macro-micro loop" emphasized by Hahn (2002), where a macro variable acts as an externality.

In the following, we introduce these effects into our model developed in the previous section. In the presence of the "wait and see" effect, the generator of the pseudo-compound Poisson process (3) is replaced with

$$(B_{\mu_t} f)(x) = -\delta \frac{df(x)}{dx} + \int_{\mathbb{R}} [f(x+y) - f(x)] \lambda(x, \sigma_t^2) \nu(x, dy), \quad \text{for } x \in \mathbb{R}. \tag{5}$$

where the intensity function $\lambda(x, \sigma_t^2)$ is low when uncertainty measured by σ_t^2 is high. It means that, for a given capital level x, the probability of lumpy investment becomes small when uncertainty σ_t^2 increases. It can describe a situation in which managers put off investment (the "wait and see" effect).

Because the process characterized by (5) is affected by the probability distribution of the process itself, it can be considered nonlinear with respect to μ_t. This semigroup generated by (5) is called a nonlinear Markov semigroup. Kolokoltsov (2011) shows that the evolution of the probability measure μ_t is governed by

$$\frac{d}{dt}(f, \mu_t) = (B_{\mu_t} f, \mu_t), \quad \mu_t \in \mathcal{M}(\mathbb{R}), \quad \text{for all } f \in C^1(\mathbb{R}) \tag{6}$$

under suitable conditions (for more details, see Appendix A.2)[13,14]. Here, $(f, \mu) \equiv \int_{\mathbb{R}} f(x) \mu(dx)$.

In the following sections, we investigate the behavior of μ_t. However, because the probability measure μ_t is generally an infinite dimensional variable, it is an unrealistic method to solve equation (6) explicitly. Thus, we use an approximation method (Gaussian approximation). We show that, when $\lambda(x, \sigma_t^2)$ strongly depends on uncertainty σ_t^2, the stability caused by randomness is lost and *collective behavior* appears through the feedback–loop mechanism.

[13] This equation is a natural generalization of linear Markov processes. In fact, if the generator is given by $Bf(x) = b(x)\frac{df(x)}{dx} + \frac{1}{2}\sigma^2(x)\frac{d^2 f(x)}{dx^2}$ corresponding to an Ito diffusion satisfying a stochastic differential equation of the form $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ and the adjoint operator B^* is defined by $(Bf, \mu_t) = (f, B^* \mu_t)$, equation (6) is reduced to the well-known Fokker-Planck equation.

[14] In this paper, $C^k(\mathbb{R})$ denotes the space of k times bounded continuously differentiable functions on $\mathbb{R}$ with bounded derivatives.

5 Stability Analysis

In this section, we study the behavior of the probability measure μ_t by analyzing lower moments of μ_t, instead of studying μ_t directly. First, we substitute $f(x) = f_1(x) \equiv x$ into equation (6) in order to obtain an ordinary differential equation for the mean (denoted by m_t). Similarly, substituting $f(x) = f_2(x) \equiv (x - m_t)^2$ into equation (6), we obtain an ordinary differential equation for the variance σ_t^2.[15] However, these equations are not closed because they depend on higher moments of μ_t in general. Thus, in order to obtain a closed-form expression for the dynamical system of m_t and σ_t^2, we employ an approximation method called *Gaussian approximation*. Gaussian approximation means that we approximate the probability measure μ_t by a Gaussian distribution with time varying parameters. Because Gaussian distributions are characterized by two parameters, the mean and variance, we can obtain a closed-form expression as seen in the following.

Under Gaussian approximation with m_t and σ_t^2, the moment-generating function of X, $\phi(\theta)$, is explicitly written:

$$\phi(\theta) = E[e^{\theta X}] = (\mu_t, e^{\theta X}) = e^{m_t\theta + \frac{\sigma_t^2\theta^2}{2}}. \tag{7}$$

To give a concrete example, we set $\lambda(x, \sigma_t^2) = e^{-x-c(\sigma_t^2 - \sigma^{*2})}$. The intensity function $\lambda(x, \sigma_t^2)$ is a decreasing function of both x and σ_t^2. The former relation expresses that the probability of investment is larger as capital depreciates more, and the latter is the "wait and see" effect. m^* and σ^{*2} are the mean and variance of the invariant measure μ^*. Thus, the variance term of the intensity function represents a negative effect on the investment frequency as the actual variance exceeds and deviates from the stationary one. In addition, we assume that the mean and variance of the distribution of jump (i.e., the size of investment), $\eta(x, dy)$, are given by $m_j(x) = a - x$ and $\sigma_j^2 > 0$, respectively.[16]

Combining these equations together, we have

$$\begin{aligned} B_{\mu_t} f_1(x) &= -\delta + \int (x + y - x)\lambda(x, \sigma_t^2)\eta(x, dy) \\ &= -\delta + e^{-x-c(\sigma_t^2 - \sigma^{*2})}(a - x), \end{aligned}$$

[15] Strictly speaking, f_1, f_2 are not included in $C^1(\mathbb{R})$. However, if m_t and σ_t^2 are bounded for $t \in [0, T]$ for some $T > 0$, we can approximate m_t and σ_t^2 with arbitrary precision by functions in $C^1(\mathbb{R})$.

[16] As will be shown later, we do not need to specify $\eta(x, dy)$ other than $m_j(x)$ and σ_j^2 under Gaussian approximation.

$$\begin{aligned}
B_{\mu_t} f_2(x) &= -2\delta(x-m_t) + \int ((x+y-m_t)^2 - (x-m_t)^2)\lambda(x,\sigma_t^2)\eta(x,dy) \\
&= -2\delta(x-m_t) + e^{-x-c(\sigma_t^2-\sigma^{*2})}\big(-x^2 + 2m_t x - 2am_t + a^2 + \sigma_j^2\big).
\end{aligned}$$

Therefore, using (7), the (closed-formed) dynamical system of m_t and σ_t^2 is obtained:

$$\frac{dm_t}{dt} = (B_{\mu_t} f_1, \mu_t) = -\delta + e^{-c(\sigma_t^2-\sigma^{*2})} e^{-m_t + \frac{\sigma_t^2}{2}} (a - m_t + \sigma_t^2), \tag{8}$$

$$\frac{d\sigma_t^2}{dt} = (B_{\mu_t} f_2, \mu_t) = e^{-c(\sigma_t^2-\sigma^{*2})} e^{-m_t + \frac{\sigma_t^2}{2}} \big(m_t^2 - 2am_t - \sigma_t^2 - \sigma_t^4 + a^2 + \sigma_j^2\big). \tag{9}$$

Regarding the well-posedness of the dynamical system, we have the following theorem.

Theorem 4 *There uniquely exists an equilibrium point $(m_{GA}, \sigma_{GA}^2) \in \mathbb{R} \times \mathbb{R}_+$ of the dynamical system.*

Proof. See Appendix A.3. ■

Setting the left hand side of equations equal to 0, the unique stationary solution (m_{GA} and σ_{GA}^2) satisfies the following equations:

$$0 = -\delta + e^{-m_{GA} + \frac{\sigma_{GA}^2}{2}} (a - m_{GA} + \sigma_{GA}^2), \tag{10}$$

$$0 = m_{GA}^2 - 2am_{GA} - \sigma_{GA}^2 - \sigma_{GA}^4 + a^2 + \sigma_j^2. \tag{11}$$

Next, we investigate the linear stability of this unique equilibrium point. Considering $m_{GA} = m^*$ and $\sigma_{GA}^2 = \sigma^{*2}$,[17] we calculate the Jacobian matrix of the dynamical system ((8) and (9)) at m^* and σ^{*2}. Using equations (10) and (11), the Jacobian matrix is given as follows:

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

[17] Of course, because we use the approximation method, m_{GA} and σ_{GA}^2 derived from Gaussian approximation deviate from their counterparts m^* and σ^{*2}. However, as we will see later, Gaussian approximation well describes the evolution of m^* and σ^{*2}, i.e. that of the probability distribution qualitatively and quantitatively.

$$
\begin{aligned}
J_{11} &= -e^{-m^*+\frac{\sigma^{*2}}{2}} - \delta, \\
J_{12} &= e^{-m^*+\frac{\sigma^{*2}}{2}} + \frac{1}{2}\delta - c\delta, \\
J_{21} &= e^{-m^*+\frac{\sigma^{*2}}{2}}(-2a + 2m^*), \\
J_{22} &= e^{-m^*+\frac{\sigma^{*2}}{2}}(-1 - 2\sigma^{*2}).
\end{aligned}
$$

The trace and determinant of the Jacobian matrix are calculated as follows:

$$
\begin{aligned}
trace &\equiv J_{11} + J_{22} = e^{-m^*+\frac{\sigma^{*2}}{2}}(-2 - 2\sigma^{*2}) - \delta = Const. < 0, \qquad (12) \\
det &\equiv J_{11}J_{22} - J_{12}J_{21} = -2\delta e^{-m^*+\frac{\sigma^{*2}}{2}}(a - m^*)c + Const. \qquad (13)
\end{aligned}
$$

The trace is a negative constant and does not depend on c. Because $a - m^* > 0$ (see Corollary 13 in Appendix), the determinant is a decreasing function of c and positive when $c = 0$. In particular, because the eigenvalues of a 2×2 matrix are given by

$$
\lambda_\pm = \frac{1}{2}\Big(trace \pm \sqrt{trace^2 - 4det}\Big),
$$

one of the eigenvalues λ_+ becomes positive if c is sufficiently large. This implies that the originally stable point m_{GA} and σ^2_{GA} becomes unstable.

In the remaining of this section, we put $a = 5, \delta = 0.2, \sigma_j^2 = 3/4$ and seek numerical solutions. First, from equations (10) and (11), $m_{GA} = 3.40$ and $\sigma^2_{GA} = 1.39$ are obtained. Substituting these values into (12) and (13), we have

$$
f(c) \equiv \sqrt{trace^2 - 4det} = \sqrt{-0.142632 + 0.171188c}. \qquad (14)
$$

The graph of $f(c)$ is shown in Figure 2. Then, we calculate numerically the critical value $c^* = 2.41$ at which λ_+ alternates its sign, that is, the stability is lost. With our parameter values, $\lambda_+ = 0$ when $c \equiv c^*$ and its corresponding eigenvector is $(m_+, \sigma_+^2) = (-0.763, 0.646)$. Namely, the stability analysis suggests that, in the neighborhood of the equilibrium point, the probability distribution μ_t representing our economy becomes unstable in this direction when $c > c^*$. Similarly, the other eigenvalue λ_- and its corresponding eigenvector are given by $\lambda_- = trace = -0.51964$

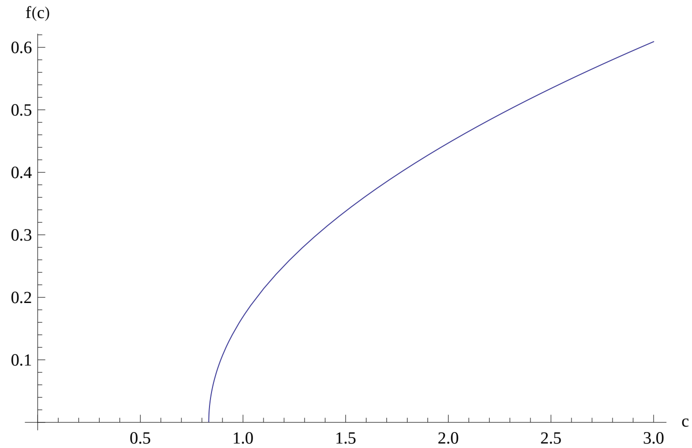

Figure 2: Graph of $f(c)$.

and $(m_-, \sigma_-^2) = (0.780, 0.626)$, respectively. Thus, we obtain the phase diagram shown in Figure 3 when c is (slightly) larger than c^*. [18]

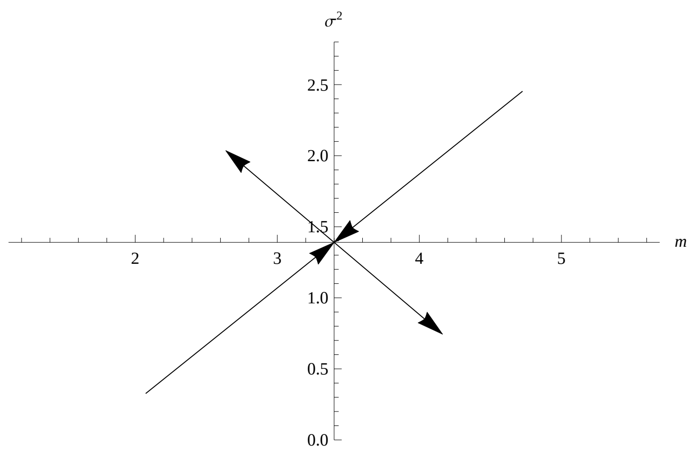

Figure 3: Phase diagram.

In the next section, we carry out simulations and show that the stability analysis well describe the behavior of probability distribution μ_t.

6 Simulations

To perform our numerical simulations, it should be noted that the nonlinear Markov process described by equation (5) behaves locally like a Levy process. Namely, an increment of the nonlinear

[18] Note that the eigenvectors depend on the value of c.

Markov process within a small interval can be approximated by the family of Lévy processes. Let $Y_\tau(z, \mu)$ be a Lévy process given by the following generator:

$$B[z, \mu]f(x) = b(z, \mu)\frac{df(x)}{dx} + \int_{\mathbb{R}} (f(x + y) - f(x))\lambda(z, \sigma^2)\nu(z, dy), \text{ for } x, z \in \mathbb{R}$$

where z, μ, $m = \int_{\mathbb{R}} x\mu(dx)$, and $\sigma^2 = \int_{\mathbb{R}}(x - m)^2\mu(dx)$ are fixed. Kolokoltsov (2011) shows that the Euler type approximation defined by

$$X^\tau(t) = X^\tau(l\tau) + Y^l_{t-l\tau}(X^\tau(l\tau), \mathcal{L}(X^\tau(l\tau))), \quad \mathcal{L}(X^\tau(0)) = \mu \tag{15}$$

where $l\tau < t \leq (l + 1)\tau$, $l = 0, 1, 2, ...,$, $\mathcal{L}(X)$ denotes the law (i.e., the probability distribution) of X, and $Y^l_\tau(x, \mu)$ are a collection of independent Lévy processes $Y_\tau(z, \mu)$ indexed by l, has its limit X_μ which solves the corresponding martingale problem.[19]

We follow this procedure to perform our simulations. That is, for an arbitrary $T > 0$, we decompose the time interval $[0, T]$ into small parts, and approximate the nonlinear process by the corresponding compound Poisson process with the drift. First, with our parameter values and no feedback loop ($c = 0$), the stability analysis predicts damped oscillation as a probability distribution μ_t converges to μ^* because eigenvalues are a conjugate pair with Re $\lambda_\pm = \mathit{trace} < 0$. In fact, this behavior is confirmed by our simulation shown in Figure 4.

Next, we investigate how the stability of the system depends on the parameter c and whether the loss of the stability occurs as the stability analysis above predicts. We set $\sigma^{*2} = 1.50$ obtained from the simulation in Figure 4. The results are shown in Figures from 5 to 10. As our stability analysis predicts, when c is small, the distribution is (locally) stable and only small variations around the equilibrium point m^* and σ^{*2} are observed because of the finite size effect of N. In this situation, the stabilization effect of randomness overwhelms the depression effect of uncertainty on investment. The randomness disturbs and impedes the generation of collective behaviors, and, therefore, the unique stochastic macro-equilibrium is achieved. The system is essentially the same as the one with $c = 0$.

However, this stability is lost when c is larger than some critical point. The stabilization effect

[19] Here, we consider the weak limit in the sense of the distributions on the Skorokhod space of cadlag paths.

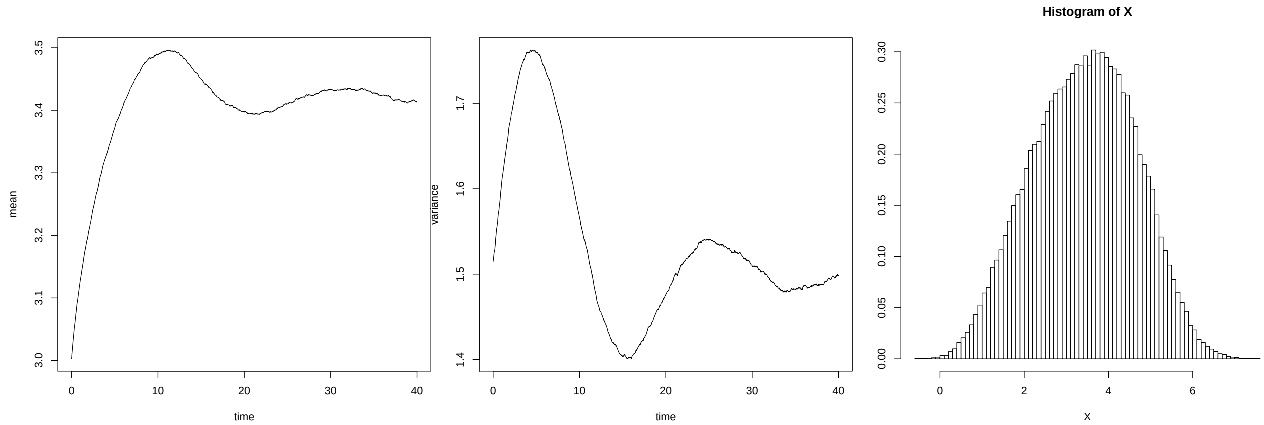

Figure 4: Simulations: $c = 0, N = 100000, a = 5, \delta = 0.2, \sigma_j^2 = 3/4$. The initial distribution is Gaussian distribution with $mean = 3.0, variance = 1.50$. Excluding transient period, the mean and variance of X_t under the stationary distribution μ^* is $mean = 3.42(0.00292), variance = 1.50(0.00606)$ (s.d.).

caused by randomness is no longer able to inhibit the effect of uncertainty. Non-negligible fraction of firms compared to the economy as a whole simultaneously "wait and see" in the face of the increase in uncertainty σ_t^2, resulting in the decrease in m_t and, equivalently, aggregate investment (Figure 9 and 10).

Given that global stability of μ^* (Doob's theorem) when $c = 0$ and its instability when $c > c^*$, one might expect that there exists the set of probability measures in $\mathcal{M}(\mathbb{R})$ such that an initial distribution chosen in this subset converges to μ^* (the basin of attraction) when $0 < c < c^*$. Although it is highly unrealistic to seek this subset exactly, this conjecture is supported by the following simulations and numerical solutions of equations (8) and (9). We set Gaussian distribution with the mean $m^* + am_+$ and variance $\sigma^{*2} + a\sigma_+^2$ as the initial distribution, where a is a parameter representing the degree of aggregate shocks. Then, we study whether the resulting distribution converges to μ^* or diverges. Namely, when (temporary) aggregate shocks in the direction of (m_+, σ_+^2) hit the system, we investigate to what degree the stabilization effect caused by randomness stand the aggregate shocks. The result is shown in Figure 11.[20] The upper region above $\circ$ ($\times$) represents a unstable region where the stabilization effect is defeated by the aggregate shocks, estimated from simulations (equations (8), (9)). The stability region shrinks as c approaches to c^*. In other words,

[20]As $c \to c^*$ from the left, our simulations suggest that fluctuations around μ^* and σ^{*2} due to the error of the finite size N rapidly increase, and finally explode. Although this is an inevitable consequence of the emergence of collective behavior, in principle, it is impossible to determines the exact value of c^* by simulation methods.

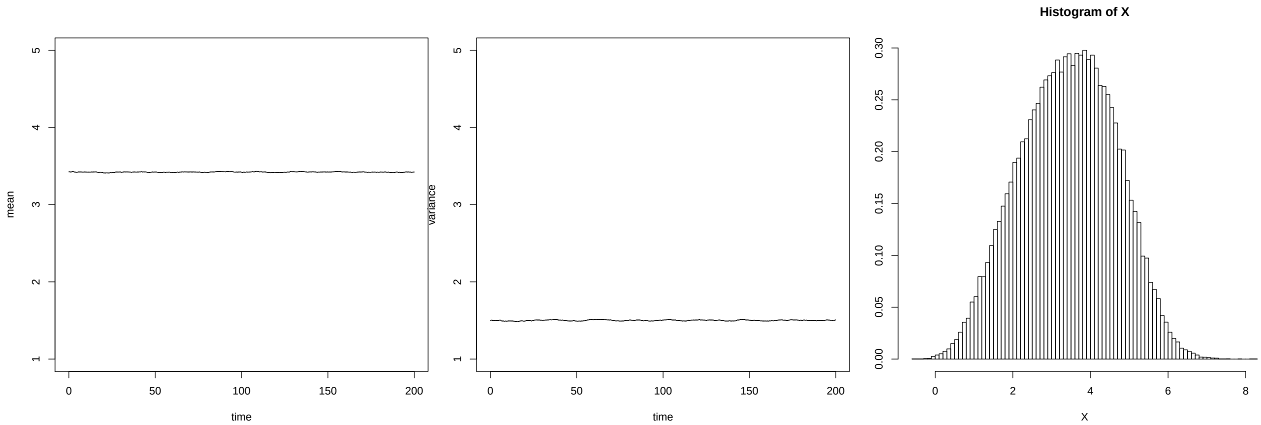

Figure 5: Simulations: $c = 0.2$ and other parameters are the same as in Figure 4. The initial distribution is Gaussian distribution with $mean = 3.42, variance = 1.8$. Excluding transient period, the mean and variance of X_t under the stationary distribution μ^* is $mean = 3.42(0.00381), variance = 1.50(0.00624)$ (s.d.).

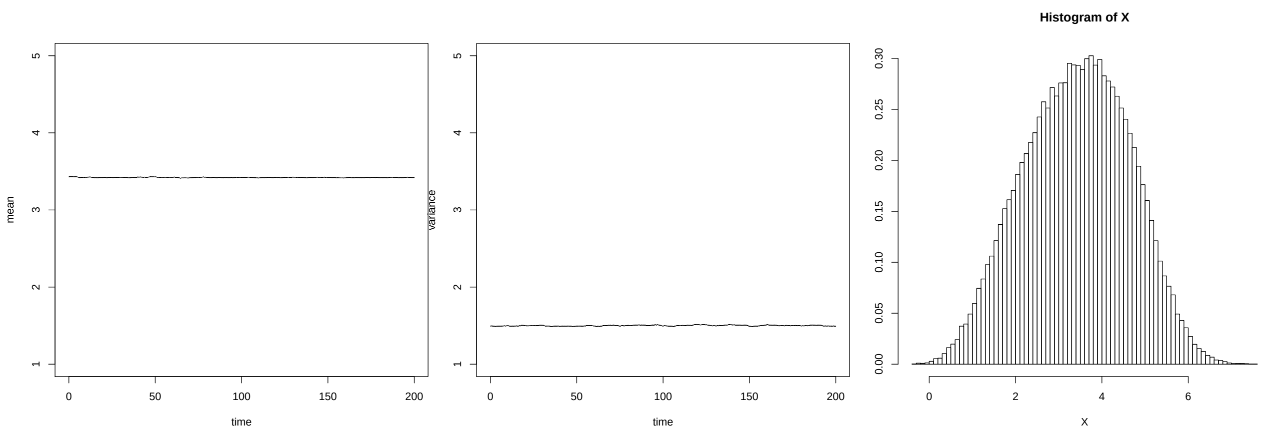

Figure 6: Simulations: $c = 0.6$ and other parameters are the same as in Figure 4. The initial distribution is the same as in Figure 5. Excluding transient period, the mean and variance of X_t under the stationary distribution μ^* is $mean = 3.42(0.00350), variance = 1.50(0.00601)$ (s.d.).

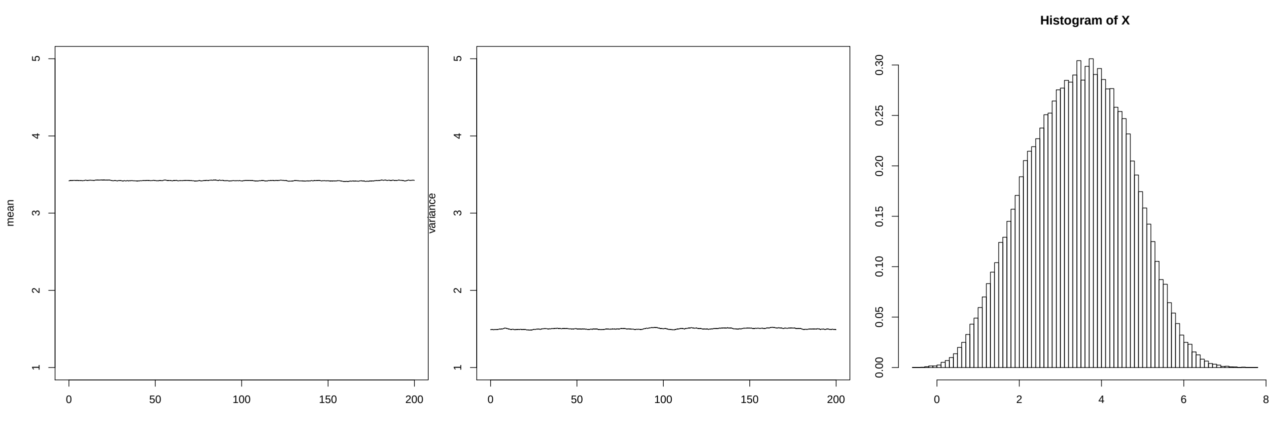

Figure 7: Simulations: $c = 1.0$ and other parameters are the same as in Figure 4. The initial distribution is the same as in Figure 5. Excluding transient period, the mean and variance of X_t under the stationary distribution μ^* is $mean = 3.42(0.00399), variance = 1.50(0.00745)$ (s.d.).

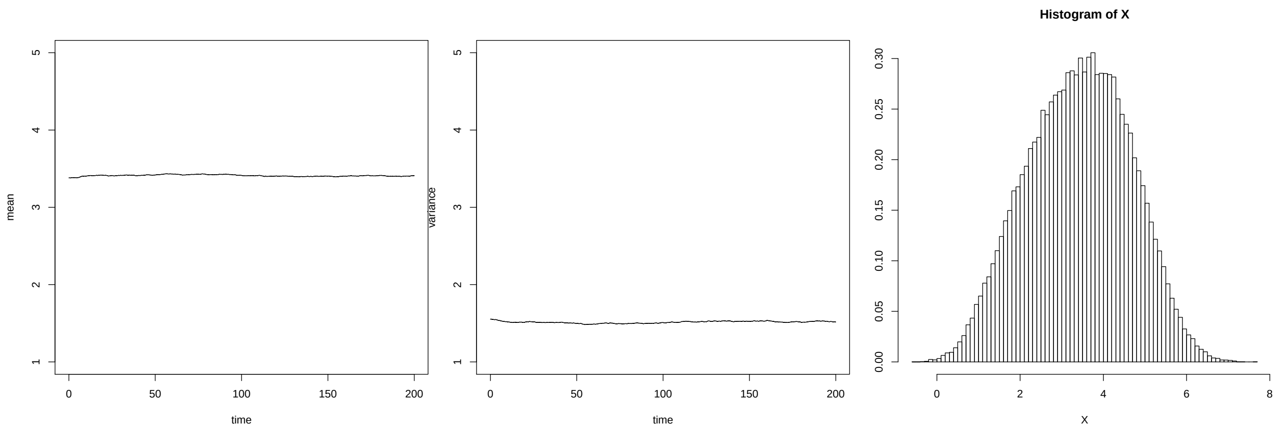

Figure 8: Simulations: $c = 1.7$ and other parameters are the same as in Figure 4. The initial distribution is the same as in Figure 5. Excluding transient period, the mean and variance of X_t under the stationary distribution μ^* is $mean = 3.41(0.0108), variance = 1.51(0.0133)$ (s.d.).

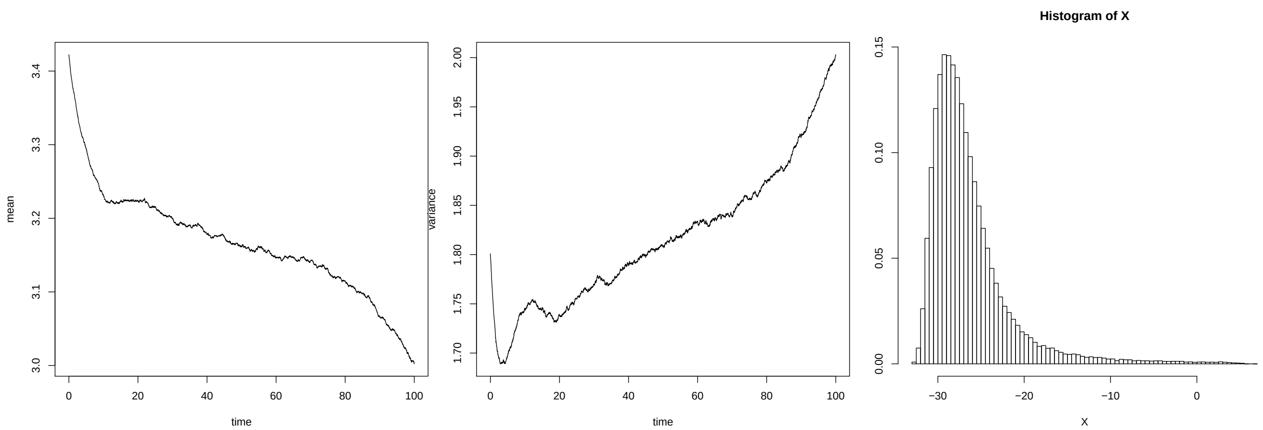

Figure 9: Simulations: $c = 1.8$ and other parameters are the same as in Figure 4. The initial distribution is the same as in Figure 5.

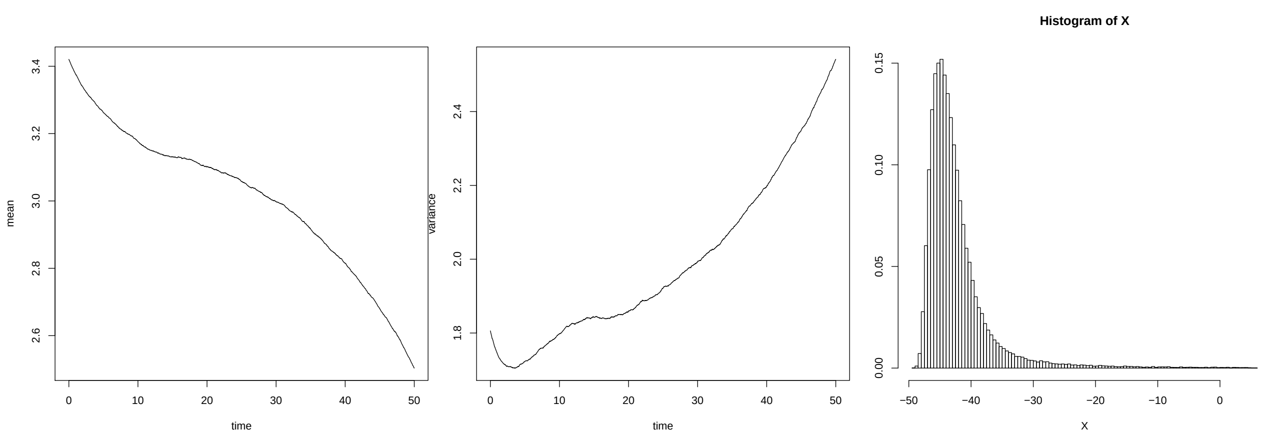

Figure 10: Simulations: $c = 1.9$ and other parameters are the same as in Figure 4. The initial distribution is the same as in Figure 5.

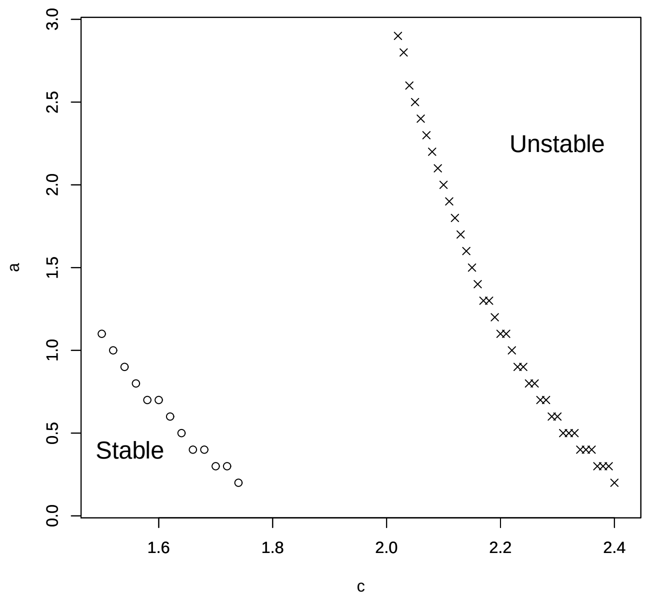

Figure 11: Boundary between stable and unstable regions. $\times$ and $\circ$ denote the estimates numerically obtained by equation (8), (9) and simulations, respectively. Parameters used in the simulations are the same Figure 4. For more details, see the explanation in the main text.

as firms are more sensitive to the environment, the stability is undermined. The stability region ends up disappearing when $c > c^*$, and, therefore, a tiny aggregate shock is sufficient to cause collective behavior, that is, a drop in aggregate investment.[21] From the simulations in Figure 11, the critical point c^* lies around 1.8, which is relatively close to the value obtained by our stability analysis in Section 5 (the critical point is calculated to be 2.41). We can conclude that Gaussian approximation well describes the evolution of the probability distribution qualitatively and quantitatively.

7 Conclusion

In this paper, we have shown that heterogeneity across the behavior of firms (or randomness) contributes to the stability of the macroeconomy. Namely, ceaseless and stochastic activity at the micro-level creates the stabilizing force in the entire economy. Because of this stabilization effect,

[21] Furthermore, because the number of firms $N < \infty$ in a real economy, the collective behavior can be induced by the finite size effect, that is, the fluctuation due to $N < \infty$ without any aggregate shocks.

the distribution describing the macroeconomy converges to the (stochastic macro) equilibrium point even in the absence of market adjustment mechanisms (i.e., the adjustment of interest rate). This stabilization effect supports the concept of stochastic macro-equilibrium by Tobin (1972), that is, while individual units incessantly move, the aggregate outcome shows stability.

In addition, we have focused on the effect of uncertainty. Investment activity always involves uncertainty over the macroeconomic environment. Facing high uncertainty, firms prefer to "wait and see" how things turn out rather than undertaking an irreversible action. As long as the sensitivity to uncertainty is smaller than a critical value, the stochastic macro-equilibrium is achieved because the stabilization effect caused by randomness overwhelms the destabilizing effect by uncertainty. However, the destabilizing force gradually becomes bigger as individual investors become more sensitive to uncertainty. Finally, the destabilizing force defeats the stabilization effect caused by randomness if the sensitivity to uncertainty exceeds the critical value. As a result, under the condition that firms are highly sensitive to uncertainty, they collectively "wait and see" instead of investing, and this collective inactive behavior breaks down the macro-stability and results in a decrease in the aggregate investment. The collective inaction of investment activities can lead the economy into recession without any substantial aggregate shocks.

Heterogeneity and feedback–loops are salient features in macroeconomy. They can generate collective behavior observed only at the macroscopic level, completely different from microeconomic behavior. As well as theoretical investigations like this paper, it is also meaningful to show the importance of interactions by empirical data. For example, Lux (2012) estimates the parameters of a stochastic model based on social interactions. These are promising subjects for future research.

A Appendix

A.1 Feller Processes

Here, we discuss the validity of the assumptions of Feller processes (the Feller property and strong continuity) from an economic standpoint.

First, we consider the strong continuity. It can be shown that the strong continuity is equivalent to pointwise convergence (e.g., Lemma 1.4 in Böttcher et al. (2013)):

$$\lim_{t\to 0} T_t u(x) = u(x) \quad \forall u \in C_\infty(\mathbb{R}),\ x \in \mathbb{R}. \tag{16}$$

Because $T_t u(x)$ is the expectation value after time t with an initial point x, (16) simply means that the process X_t stay at its initial value *in probability* if t is small. Note that it does not exclude the possibility that the process X_t jumps at around 0. The property (16) means that this probability converges to 0 as $t \to 0$. In this sense, this property is closely related to the *stochastic continuity* discussed in the context of Lévy processes. Because there is no reason to assume that X_t jumps exactly at 0 (or some pre-determined points) with positive probability, the strong continuity is a plausible assumption in an economic sense.

Second, to discuss the Feller property, we introduce an additional assumption:

Assumption 5 *For all $t > 0$ and any increasing sequence of bounded sets $B_n \in \mathcal{B}(\mathbb{R})$ with $\cup_{n\geq 1} B_n = \mathbb{R}$, the transition function $p_t(x, B_n)$ satisfies the following property;*

$$\lim_{|x|\to\infty} p_t(x, B_n) = 0, \quad \forall n \geq 1.$$

This is a reasonable assumption in an economic sense. For example, suppose that there exist increasing adjustment costs $\psi(\Delta X_t)$ such that $\psi'(\Delta X_t) > 0, \psi''(\Delta X_t) > 0$, or financial (or physical) constraints, and, therefore, capital adjustment to some target level cannot be done at one time when X_t is extremely small. In such a situation, Assumption 5 can be justified.

Then, the following lemma implies the Feller property.

Lemma 6 *Assumption 5 implies the Feller property.*

Proof. Let $u \in C_\infty(\mathbb{R})$. For an arbitrary ϵ, there exists N such that $|u| < \epsilon$ on $\mathbb{R} \setminus B_n$ for all $n \geq N$. Therefore,

$$\begin{aligned} |T_t u(x)| &\leq \int_{B_n} |u(y)| p_t(x, dy) + \int_{\mathbb{R}\setminus B_n} |u(y)| p_t(x, dy) \\ &\leq ||u||_\infty p_t(x, B_n) + \epsilon \end{aligned}$$

where $||\cdot||_\infty$ denotes the sup norm on $C_\infty(\mathbb{R})$.

Because ϵ is arbitrary, Assumption 5 implies that $T_t u \in C_\infty(\mathbb{R})$. ■

A.2 Equation (6)

Here, we discuss the conditions under which equation (6) holds. To begin with, we introduce the following metric on the set of probability measures $\mathcal{M}(\mathbb{R})$:

Definition 7 *The pth Wasserstein-Kantorovich metrics between two probability measures ρ_1 and ρ_2 with finite pth moment is defined as*

$$W_p(\rho_1, \rho_2) = \Big(\inf_{\rho} \int |y_1 - y_2|^p \rho(dy_1 dy_2)\Big)^{1/p} \tag{17}$$

where inf is taken over the collection of all probability measures ρ on $\mathbb{R}^2$ with marginals ρ_1 and ρ_2, respectively (i.e., the coupling of ρ_1 and ρ_2).

In order to compare the Lévy measures, the Wasserstein-Kantorovich metrics are extended to finite and unbounded measures in Kolokoltsov (2011). However, because the Lévy measure, $\lambda\nu$, is finite in our case, we can use another (more tractable) generalization of the Wasserstein-Kantorovich metrics. Indeed, a look on the proof of his theorem (Theorems 1.2 and 1.3) shows that the metrics are only used to estimate (and bound) the distance of the Lévy measures and, therefore, we can employ the following generalization of the Wasserstein-Kantorovich metrics:

Definition 8 *The distance between two finite measures ρ_1 and ρ_2 ($\rho_1(\mathbb{R}) = n > 0$ and $\rho_2(\mathbb{R}) = m > 0$, respectively) with finite pth moment is defined as*

$$W_p^*(\rho_1, \rho_2) = |n - m| + W_p(\frac{\rho_1}{n}, \frac{\rho_2}{m}). \tag{18}$$

The axioms of metric are satisfied.

Theorem 1.3 in Kolokoltsov (2011) can be rewritten as follows:

Theorem 9 *Suppose that*

$$B_\mu f(x) = b(x)\frac{df(x)}{dx} + \int_{\mathbb{R}} (f(x+y) - f(x))\lambda(x, \sigma_\mu^2)\nu(x, dy), \;\; \nu(x, \cdot) \in \mathcal{M}(\mathbb{R}). \tag{19}$$

and

$$|b(x)-b(y)|+W_1^*(\lambda(x,\sigma_\mu^2)\nu(x;.),\lambda(y,\sigma_\eta^2)\nu(y;.))\leq\kappa(|x-y|+W_1^*(\mu,\eta)) \quad (20)$$

holds with a constant κ*. Then, there exists a process* $X_\mu(t)$ *solving*

$$X(t)=X+\int_0^t dY_s(X(s),\mathcal{L}(X(s))),\ \mathcal{L}(X)=\mu, \quad (21)$$

where the solution is defined as the limit of the Euler type approximation (15). Moreover, the processes

$$M(t)=f(X_\mu(t))-f(X_\mu(t))-\int_0^t\big(B_{\mathcal{L}(X_\mu(s))}f(X_\mu(s))\big)ds \quad (22)$$

are martingales for any $f\in C^1(\mathbb{R})$*, and the distributions* $\mu_t=\mathcal{L}(X_\mu(t))$ *satisfy equation (6).*

Therefore, to prove that equation (6) holds, we need to check $W_1^*(\lambda(x,\sigma_\mu^2)\nu(x;.),\lambda(y,\sigma_\eta^2)\nu(y;.))\leq\kappa(|x-y|+W_1^*(\mu,\eta))$ for some κ.[22]

We need additional assumptions:

Assumption 10 *The intensity function* $\lambda>0$ *is Lipschitz continuous with some constant* c_1*, and* $|\sigma_\mu^2-\sigma_\eta^2|\leq c_2\sup_{f\in Lip}|(f,\mu)-(f,\eta)|$ *for* μ,η*, the laws of the processes generated by (15). Here, Lip is the collection of continuous function* f *such that* $|f(x)-f(y)|\leq|x-y|$ *for* $\forall x,y\in\mathbb{R}$*.*

The assumption about λ is technical. The assumption $|\sigma_\mu^2-\sigma_\eta^2|\leq c_2\sup_{f\in Lip}|(f,\mu)-(f,\eta)|$ means that the difference in variance is mainly determined by the "central region" of the measures. For example, if the measures, μ and η, have a compact support, it can be shown that the assumption is satisfied.[23] Here, we assume that Assumption 10 is satisfied. Then, we obtain the desired lemma:

Lemma 11 *Let Assumption 10 be satisfied. Then,*

$$W_1^*(\lambda(x,\sigma_\mu^2)\nu(x;.),\lambda(y,\sigma_\eta^2)\nu(y;.))\leq\kappa(|x-y|+W_1^*(\mu,\eta)) \quad (23)$$

[22] In our model, $b(x)=-\delta$.

[23] However, in general, we cannot assume that the laws of processes generated by (15) have a compact support because they are Lévy processes.

for some $\kappa > 0$.

Proof. By the triangle inequality,

$$W_1^*(\lambda(x,\sigma_\mu^2)\nu(x;.),\lambda(y,\sigma_\eta^2)\nu(y;.)) \leq W_1^*(\lambda(x,\sigma_\mu^2)\nu(x;.),\lambda(y,\sigma_\eta^2)\nu(x;.)) + W_1^*(\lambda(y,\sigma_\eta^2)\nu(x;.),\lambda(y,\sigma_\eta^2)\nu(y;.)).$$

The second term on the right hand side is the distance between the measures with the same total measure, and, therefore, is equal to $W_1(\nu(x;.),\nu(y;.))$ (for each x and y, ν is a probability measure). Because of the construction of ν, the following estimate holds:

$$W_1(\nu(x;.),\nu(y;.)) = \inf_\rho \int |z_1 - z_2|\rho(dz_1 dz_2) \leq \int |z - (z + (x-y))|\nu(dz) \leq |x-y|. \tag{24}$$

For the first term, by triangle inequality,

$$\begin{aligned} W_1^*(\lambda(x,\sigma_\mu^2)\nu(x;.),\lambda(y,\sigma_\eta^2)\nu(x;.)) &\leq |\lambda(x,\sigma_\mu^2) - \lambda(y,\sigma_\eta^2)| & (25)\\ &\leq |\lambda(x,\sigma_\mu^2) - \lambda(y,\sigma_\mu^2)| + |\lambda(y,\sigma_\mu^2) - \lambda(y,\sigma_\eta^2)|. & (26)\end{aligned}$$

Because, by the Monge-Kantorovich theorem, $W_1(\mu,\eta)$ for $\mu,\eta \in \mathcal{M}(\mathbb{R})$ can be written as

$$W_1(\mu,\eta) = \sup_{f\in Lip} |(f,\mu) - (f,\eta)|,$$

Using this theorem and Assumption 10, we obtain $|\sigma_\mu^2 - \sigma_\eta^2| \leq c_2 W_1(\mu,\eta)$.

Thus, combining these estimates and Lipschitz continuity of λ, there exists a constant κ such that $W_1^*(\lambda(x,\sigma_\mu^2)\nu(x;.),\lambda(y,\sigma_\eta^2)\nu(x;.)) \leq \kappa(|x-y| + W_1^*(\mu,\eta))$. Therefore, our claim follows. ■

A.3 Stability Analysis

In this section, we investigate the properties of the following system of differential equations of the first moment (mean) m and the second central moment $\sigma^2 \geq 0$ obtained in Section 5:[24]

$$\begin{aligned} \frac{dm}{dt} &= (a - m + \sigma^2)e^{-m-\sigma^2/2-c(\sigma^2-\sigma^{2*})}, & (27)\\ \frac{d\sigma^2}{dt} &= [(a-m)^2 - \sigma^2(\sigma^2+1) + \sigma_j^2]e^{-m-\sigma^2/2-c(\sigma^2-\sigma^{2*})}, & (28)\end{aligned}$$

[24]To simplify the notation, the subscript t representing time is omitted below.

An equilibrium point of the system of (27) and (28), $(m^*, \sigma^{2^*}) \in \mathbb{R} \times \mathbb{R}_+$, is defined as a solution of the following system of simultaneous equations:

$$e^{-m^*+\sigma^{2^*}/2}(a - m^* + \sigma^{2^*}) = \delta, \tag{29}$$

$$(a - m^*)^2 + \sigma_j^2 = \sigma^{2^*}(\sigma^{2^*} + 1). \tag{30}$$

One can easily find that $dm/dt = d\sigma^2/dt = 0$ if and only if $(m, \sigma^2) = (m^*, \sigma^{2^*})$. Concerning the existence and uniqueness of an equilibrium (m^*, σ^{2^*}), the following proposition holds.

Theorem 12 *There uniquely exists an equilibrium point* $(m^*, \sigma^{2^*}) \in \mathbb{R} \times \mathbb{R}_+$.

Proof. Eq. (30) can be solved for $\sigma^2 \geq 0$ as

$$\sigma^2 = \frac{\sqrt{1 + 4[(a - \sigma^2)^2 + \sigma_j^2]} - 1}{2} > 0. \tag{31}$$

Let $z = a - m$. Then, substituting (31) in (29), we have

$$g(z)h(z) = \delta e^a, \tag{32}$$

where $g(z)$ and $h(z)$ are defined as follows:

$$\begin{aligned} g(z) &= z + \frac{\sqrt{1 + 4(z^2 + \sigma_j^2)} - 1}{2}, \\ h(z) &= \exp\Big(z + \frac{\sqrt{1 + 4(z^2 + \sigma_j^2)} - 1}{4}\Big) > 0. \end{aligned}$$

Since the right hand side of (32) is positive, for some z to satisfy (32), we must have

$$g(z) > 0,$$

or

$$z > \frac{3}{8} - \frac{\sigma_j^2}{2} \equiv \underline{z}. \tag{33}$$

Both g and h are positive and strictly increasing in z within the range of (33) and that we have

$g(\underline{z})h(\underline{z}) = 0$ and $g(\infty)h(\infty) = \infty$. Hence, there uniquely exists a z^* that meets (32). Letting $m^* = a - z^*$ and $\sigma^{2*} = [-1 + \sqrt{1 + 4(z^{*2} + \sigma_j^2)}]/2$, we can find that $(m^*, \sigma^{2*}) \in \mathbb{R} \times \mathbb{R}_+$ is a unique equilibrium point. ■

Corollary 13 *Assume that the following condition is satisfied:*

$$\sigma_j^2 \leq \frac{3}{4}, \tag{34}$$

Then, we have $m^* < a$.

Proof. It is obvious from (33) in Proof of Theorem 12. ■

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