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Application of the Concept of Entropy to Equilibrium in Macroeconomics[*]

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Abstract

Entropic consideration plays the essential role in understanding equilibrium in the macroeconomy comprising heterogeneous agents. Curiously, it has been long ignored in economics. In fact, the idea of entropy is completely opposite to that of mainstream macroeconomics which is founded on representative agent models. Macroeconomics is meant to analyze the macroeconomy which consists of 10 million households and one million firms. Despite a large number of micro agents, the standard model focuses on the behavior of the representative micro agent. We show that entropy serves as a key concept in economics as well as in physics by taking the distribution of labor productivity in Japan as an illustrative example. Negative temperature, which is abnormal in nature, turns out to be normal to explain the distribution of workers on the low-to-medium productivity side. Also, the empirical result on the high productivity side indicates limited capacities of leading firms for workers. We construct a statistical physics model that is valid in the whole range of productivity.

Keywords: Heterogeneous agents, Microeconomic foundations, Unemployment, Productivity, Entropy, Keynes' principle of effective demand
JEL classification: D39, E10, J64

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I. Introduction

In this paper, drawing on our previous research, we explain that the concept of entropy can be usefully applied to macroeconomics. The macroeconomy consists of 10^7 heterogeneous households and 10^6 firms. Despite of this simple fact, entropy which plays the essential role in statistical physics has been long ignored in macroeconomics. It is physicists who first noticed the potential usefulness of the concept and applied it to socioeconomic systems (Montroll, 1981, 1987). Following their lead, we proposed a new concept of stochastic equilibrium in macroeconomics (Yoshikawa, 2003; Aoki and Yoshikawa, 2007; Iyetomi, 2012; Yoshikawa, 2014; Aoyama et al., 2014). Especially, we focused on distribution of workers across different levels of labor productivity. It is parallel to distribution of particles across different energy levels in physics.

Although entropy serves as a key concept in economics as well as in physics, there is crucial difference between the two disciplines. In macroeconomics, we can usefully identify the level of labor productivity with that of energy. In physics, a free particle tends to achieve lower energy whereas worker always tries to find a job with higher productivity. Thus, reasonable measure of temperature necessarily becomes negative, not positive, in economics. Our data of one million Japanese firms shows how labor productivity is distributed across firms and workers. A statistical theory based on the entropy maximum principle, reinforced with a concept of negative temperature, works well to reproduce increasing distribution of workers on low-to-medium productivity side; more than 90% of workers belong to this regime. The significant difference in the economic temperatures was detected between manufacturing and non-manufacturing sectors, indicating imbalance in effective demand across two sectors; in contrast, they are almost in equilibrium with respect to exchanges of workers.

Decreasing distribution of workers in a power-law form on high productivity side requires another explanation. We made an additional assumption that a sector or firm with high productivity can accommodate only a limited number of workers. The macroeconomy consists of many firms with different levels of productivity. Differences in productivity arise from different capital stocks, levels of technology and/or demand conditions facing firms. We call a group of firms with the same level of productivity a *cluster*. Workers randomly move from a cluster to another for various reasons at various times. Despite of these random changes, the distribution of labor productivity as a whole remains stable because those incessant random movements balance with each other. This balancing must be achieved for each cluster, and is called detail-balance. We worked out a general treatment of this detail-balance using particle-correlation theory à la Costantini and Garibaldi (1989). In doing so, we made an assumption that the number of workers who belong to clusters with high productivity is constrained. We thus derived a general formula for the productivity distribution to explain the two distinctive observational facts in a unified way, using the detail-balance condition necessary for equilibration in the Ehrenfest-Brillouin model.

In this paper we give an integrated account of what have been achieved by us in the previous works (Yoshikawa, 2003; Aoki and Yoshikawa, 2007; Iyetomi, 2012; Yoshikawa, 2014; Aoyama et al., 2014), together with new additional contributions. We use another dataset of financial statements of firms in Japan extracted from a worldwide database. This allows us not only to confirm the empirical results previously obtained but also to study diversity of labor productivity from a global point of view. Theoretically, we rederive the general formula for the productivity distribution within equilibrium statistical physics not resorting to the Ehrenfest-Brillouin model. This amounts to development of a grand canonical formulation for *distinguishable* particles of the same kind, while identical particles are *indistinguishable* in quantum theory. The new formulation encompasses the Maxwell-Boltzmann, the Fermi-Dirac, and the Bose-Einstein distributions, three typical distributions in statistical physics, on an equal footing.

The following section explains the motivation of the method of statistical physics and the concept of stochastic macro-equilibrium. Section III obtains distribution of labor productivity from an extensive financial database of Japanese firms. Section IV then constructs a theoretical model to explain it using a grand canonical formulation. Section V is devoted to validation of the model through comparison with the empirical results, accompanied by derivation of equilibrium conditions for multi-sector systems. The final section offers brief concluding remarks.

II. Stochastic Macro-equilibrium — The Basic Idea

We consider distribution of workers across different levels of labor productivity. Workers are always interested in better job opportunities, and occasionally change their jobs. While workers search for suitable jobs, firms also search for suitable workers. Firm's job offer is, of course, conditional on its economic performance. The present analysis focuses on the firm's labor productivity. The firm's labor productivity increases thanks to capital accumulation and technical progress or innovations. However, those job sites with high productivity remain only *potential* unless firms face high enough demand for their products; firms may not post job vacancy signs or even discharge the existing workers when demand is low.

The motivation for the method of statistical physics is as follows. Though we assume that firms with higher productivity make more attractive job offers to workers, we do not know how attractive they are to which workers. Whenever possible, workers move to firms with higher productivity, but we never know particular reasons for such moves. For workers to move to firms with higher productivity, it is necessary that those firms must decide to fill the vacant job sites, and post enough number of vacancy signs and/or make enough hiring efforts. They post such vacancy signs and make hiring efforts only when they face an increase of demand for their products, and decide to raise the level of production. It also goes without saying that high productivity firms keep their existing workers only when they face high enough demand.

The question we ask is what the distribution of employed workers is across firms whose productivities differ. Because microeconomic shocks to both workers and firms are so complex and unspecifiable, optimization exercises based on representative agent assumptions do not help us much. In particular, we never know how the aggregate demand is distributed across firms. Besides, among other things, the job arrival rate, the job separation rate, and the probability distribution of wages (or more generally measure of the desirability of the job) differ across workers and firms. This recognition is precisely the starting point of the fundamental method of statistical physics. Foley (1994), in his seminal application of this approach to general equilibrium theory, called the idea "statistical equilibrium theory of markets". Following the lead of Foley (1994), Yoshikawa (2003) applied the concept to macroeconomics. At first, one might think that allowing too large a dispersion of individual characteristics leaves so many degrees of freedom that almost anything can happen. However, it turns out that the methods of statistical physics provide us not only with qualitative results but also with quantitative predictions.

In the present model, the fundamental constraint on the economy as a whole is aggregate demand D. Accordingly, to each firm facing the downward-sloping kinked individual demand curve, the level of demand for its product is the fundamental constraint. The problem is how the aggregate demand D is allocated to these monopolistically competitive firms. Our model gives a solution to this problem. The method is standard in statistical physics. The basic idea behind the analysis can be explained with the help of the simplest case. We focus on productivity dispersion here.

Suppose that n_k workers belong to firms whose productivity is c_k ($k = 1,\ 2,\ \cdots,\ K$), where there are K levels of productivity in the economy arranged in the ascending order: $c_1 < c_2 < \cdots < c_K$. The total number N of workers and the total output Y in the economy are then given by

$$N = \sum_{k=1}^{K} n_k \ , \tag{1}$$

and

$$Y = \sum_{k=1}^{K} c_k n_k \ , \tag{2}$$

respectively. A vector $n = (n_1, n_2, \cdots, n_K)$ shows a particular allocation of workers across firms with different productivities. The combinatorial number W_n of obtaining this allocation, n is equal to that of throwing N balls to K different boxes. Because the number of all the possible ways to allocate N different balls to K different boxes is K^N, the probability that a particular allocation $n = (n_1, n_2, \cdots, n_K)$ is obtained is

$$P_n = \frac{W_N(n)}{K^N} = \frac{1}{K^N} \frac{N!}{\prod_{k=1}^{K} n_k!} \tag{3}$$

It is the fundamental postulate of statistical physics that the state or the allocation $n = (n_1, n_2, \cdots, n_K)$ which maximizes the probability P_n or (3) *under macro constraints* is to be realized[1]. The idea is similar to maximum likelihood in statistics/econometrics. Maximizing P_n is equivalent to maximizing $\ln P_n$. Applying the Stirling formula for large number we find that the maximization of $\ln P_n$ is equivalent to that of S

$$S = \ln W_N(n) = -N \sum_{k=1}^{K} p_k \ln p_k \quad (p_k = \frac{n_k}{N}) \ , \tag{4}$$

where S is the Shannon entropy, and captures the combinatorial aspect of the problem. Though the combinatorial consideration summarized in the entropy plays a decisive role for the final outcome that is not the whole story, of course. The qualification "under macro-constraints" is crucial.

The first macro-constraint concerns the labor endowment, (1). The second macro-constraint concerns the effective demand. We assume that given aggregate demand D is balanced by the total output Y, (2):

$$D = Y = \sum_{k=1}^{K} c_k n_k \ . \tag{5}$$

In our analysis, we explicitly analyze the allocation of labor $(n_1, n_2, \cdots, n_K)$. The allocation of labor basically corresponds to the allocation of the aggregate demand to monopolistically competitive firms.

To maximize entropy S *under two macro-constraints* (1) and (2), set up the following Lagrangean form L:

$$L = -N \sum_{k=1}^{K} \left(\frac{n_k}{N}\right) \ln \left(\frac{n_k}{N}\right) - \alpha N - \beta D \ , \tag{6}$$

with two Lagrangean multipliers, α and β. Maximization of this Lagrangean form with respect to n_k leads us to the first-order variational condition:

$$\delta L = \delta S - \alpha \delta N - \beta \delta D = 0 \ , \tag{7}$$

with the following variations,

[1]To be precise, it is to be realized in the sense of expected value. In physics, variance is normally so small relative to expected value that we practically always observe the expected value.

$$\delta S = \sum_{k=1}^{K} \delta n_k \ln\left(\frac{n_k}{N}\right) \ , \tag{8}$$

$$\delta N = \sum_{k=1}^{K} \delta n_k \ , \tag{9}$$

$$\delta D = \sum_{k=1}^{K} c_k \delta n_k \ . \tag{10}$$

Equation (7) coupled with (8), (9), and (10) determines n_k as

$$\ln\left(\frac{n_k}{N}\right) = -\alpha - \beta c_k \quad (k = 1,\ 2,\ \cdots,\ K) \ . \tag{11}$$

Because n_k/N sums up to one, we obtain

$$\frac{n_k}{N} = e^{-\alpha-\beta c_k} = \frac{e^{-\beta c_k}}{\sum\limits_{k=1}^{K} e^{-\beta c_k}} \ . \tag{12}$$

Thus, the number of workers working at firms with productivity c_k is exponentially distributed. It is known as the Maxwell-Boltzmann distribution in physics.

Here arises a crucial difference between economics and physics. In physics, c_k corresponds to the level of energy. Whenever possible, particles tend to move toward the *lowest* energy level. To the contrary, in economics, workers always strive for better jobs offered by firms with higher productivity c_k. As a result of optimization under unobservable respective constraints, workers move to better jobs. In fact, if allowed all the workers would move up to the job sites with the highest productivity, c_K. This situation corresponds to the textbook Pareto optimal Walrasian equilibrium with no frictions and uncertainty. However, this state is actually impossible unless the level of aggregate demand D is so high as equal to the maximum level $D_{\max} = c_K N$. When D is lower than $D_{\max}$, the story is quite different. Some workers — a majority of workers, in fact must work at job sites with productivity lower than c_K.

How are workers distributed over job sites with different productivity? Obviously, it depends on the level of aggregate demand. When D reaches its lowest level, $D_{\min}$, workers are distributed evenly across all the sectors with different levels of productivity, $c_1, c_2, \cdots, c_K$. Here, $D_{\min}$ is defined as $D_{\min} = N(c_1 + c_2 + \cdots + c_k)/K$. It is easy to see that the lower the level of D is, the greater the combinatorial number of distribution $(n_1, n_2, \cdots, n_K)$ which satisfies aggregate demand constraint (2) becomes.

As explained above, the combinatorial number W_n of a particular allocation $n = (n_1, n_2, \cdots, n_K)$ is basically equivalent to the Shannon entropy, S defined by (4). The entropy S increases when D decreases. For example, in the extreme case where D is equal to the maximum level $D_{\max}$, all the workers work at job sites with the highest productivity. In this case, the entropy S becomes zero, its lowest level because $n_K/N = 1$ and $n_k/N = 0$ $(k \neq K)$. In the other extreme where aggregate demand is equal to the minimum level $D_{\min}$, we have $n_k = N/K$, and the entropy S defined by (4) reached its maximum level, $\ln K$. The relation between the entropy S and the level of aggregate demand D, therefore, is schematically drawn in Figure 1.

At this stage, we can recall that the Lagrangean multiplier β in (6) for aggregate demand constraint is equal to

$$\beta = \frac{\partial L}{\partial D} = \frac{\partial S}{\partial D} \ , \tag{13}$$

where β is the slope of the tangent of the curve as shown in Figure 1, and, therefore, is negative.

In physics, β is normally positive. This difference arises because workers strive for job sites with higher productivity, not the other way round (Iyetomi, 2012). In physics, β is equal to the inverse of temperature, or more precisely, temperature is defined as the inverse of $\partial S/\partial D$ when S is the entropy and D energy. Thus, negative β means the negative temperature. It may sound odd, but the notion of negative temperature is perfectly legitimate in such systems as the one in the present analysis; see Section 73 of Landau and Lifshitz (Landau and Lifshitz, 1980) and Appendix E of Kittel and Kroemer (Kittel and Kroemer, 1980). With negative β, the exponential distribution (11) is upward-sloping. However, unless the aggregate demand is equal to (or greater than) the maximum level, $D_{\max}$, workers' efforts to reach job sites with the highest productivity c_K must be frustrated because firms with the highest productivity do not employ a large number of workers and are less aggressive in recruitment, and accordingly it becomes harder for workers to find such jobs. As a consequence, workers are distributed over all the job-sites with different levels of productivity.

The maximization of entropy under the aggregate demand constraint (6), in fact, balances two forces. On one hand, whenever possible, workers move to better jobs identified with job sites with higher productivity. It is the outcome of successful job matching resulting from the worker's search and the firm's recruitment. When the level of aggregate demand is high, this force dominates. However, when D is lower than $D_{\max}$, there are in general a number of different allocations $(n_1, n_2, \cdots, n_K)$ which are consistent with D.

As we argued above, micro shocks facing both workers and firms are truly unspecifiable. We simply do not know which firms with what productivity face how much demand constraint and need to employ how many workers with what qualifications. We do not know which workers are seeking what kind of jobs with how much productivity, either. Here comes the maximization of entropy. It gives us the distribution $(n_1, n_2, \cdots, n_K)$ which corresponds to the maximum combinatorial number consistent with given D.

The standard economic theory, search theory in particular, emphasizes non-trivial job matching in labor market with frictions and uncertainty. Our analysis shows that the matching of high productivity jobs is ultimately conditioned by the level of aggregate demand. That is, uncertainty and frictions emphasized by the standard search theory are *not* exogenously given, but depend crucially on aggregate demand. In a booming gold-rush town, one does not waste a minute to find a good job! The opposite holds in a depressed city.

It is essential to understand that the present approach does *not* regard economic agents' behaviors as random. Certainly, firms and workers maximize their profits and utilities. The present analysis, in fact, presumes that workers always strive for better jobs characterized by higher productivity. Randomness underneath the entropy maximization comes from the fact that both the objective functions of and constraints facing a large number of economic agents are constantly subject to *unspecifiable* micro shocks. We must recall that the number of households is of order 10^7, and the number of firms, 10^6. Therefore, there is nothing for outside observers, namely economists analyzing the macroeconomy but to regard a particular allocation *under macro-constraints* as equiprobable. Then it is most likely that the allocation of the aggregate demand and workers which maximizes the probability P_n or (3) *under macroconstraints* is realized.

This method has been time and again successful in natural sciences when we analyze object comprising many micro elements. Economists might be still skeptical of the validity of the method in economics saying that inorganic atoms and molecules comprising gas are essentially different from optimizing economic agents. Every student of economics knows that behavior of dynamically optimizing economic agent such as the Ramsey consumer is described by the Euler equation for a problem of calculus of variation. On the surface, such a sophisticated economic behavior must look remote from "mechanical" movements of an inorganic particle which only satisfy the law of motion. However, every student of physics knows that the Newtonian law of motion is actually nothing but the Euler equation for a certain variational problem; particles minimize the energy or the Hamiltonian! It is called the principle of least action: see Chapter 19 of Feynman (1964). Therefore, behavior of dynamically optimizing economic agent and motions of inorganic particle are on a par to the extent that they both satisfy the Euler equation for respective variational problem. The method of statistical physics can be usefully applied not because motions of micro units are "mechanical," but because object or system under investigation comprises many micro units individual movements of which we are unable to know.

The above analysis shows that the distribution of workers at firms with different productivities depends crucially on the level of aggregate demand. Though the simple model is useful to explain the basic idea, it is too simple to apply to the empirically observed distribution of labor productivity.

III. Empirical Distribution of Productivity

In this study we take advantage of the Orbis database (Bureau van Dijk Electronic Publishing, Brussels, Belgium), possessing information on over 120 million firms across the globe. We focus on Japanese firms with non-empty entries in annual operating revenue Y and the number n of employees, and define the labor productivity c of firms as follows:

$$c := \frac{Y}{n} \ . \tag{14}$$

We end up with 31,512 manufacturing firms and 304,006 non-manufacturing firms in 2012. The data source is different from that of our previous studies (Souma et al., 2009; Iyetomi, 2012; Aoyama et al., 2014), which were based on the dataset constructed by unifying two domestic databases, the Nikkei Economic Electric Database (NEEDS) (Nikkei Digital Media, Inc., Tokyo, Japan) for large firms and the Credit Risk Database (CRD) (CRD Association, Tokyo, Japan) for small to medium-sized firms. The Orbis database used here may enable us to extend our study to the labor productivity in other countries and do international comparisons of its dynamics.

Figure 2 shows the probability density function (PDF) of firms and workers with respect to $\log c$, empirically determined from the Orbis database in 2012. Here c is measured in units of 10^3 USD per person. The fact that the major peak of the latter is shifted to right compared to that of the former indicates that the average number $\bar{n}$ of workers per firm increases in this region. In fact, Figure 3 shows the functional dependence of $\bar{n}$ on the labor productivity c of firms. We observe that as the productivity rises, it first goes up to about $n \simeq 100$ and then decreases. Iyetomi (2012) explained the upward-sloping distribution in the low productivity region by introducing the negative temperature theory. The downward-sloping part in the high productivity region is close to linear (denoted by the dotted line) in this double-log plot. This indicates that it obeys the power law:

$$\bar{n} \propto c^{-\gamma} \ . \tag{15}$$

The new empirical observations ascertain our previous conclusion (Iyetomi, 2012; Aoyama et al., 2014) that the broad shape of distribution of productivity among firms is quite robust and universal. The number of workers exponentially increases as c increases up to a certain level of productivity, and then it decreases with the power law form (15) in the high productivity region. The latter behavior of $\bar{n}$ may be somewhat counter-intuitive, because firms that have achieved higher productivity through innovations and high-quality management would continue to grow larger and larger leading to monotonically increasing $\bar{n}$ with c.

In the previous section we demonstrated that the entropy maximization under macro constraints leads to an exponential distribution. This distribution with negative β can explain the broad pattern of the left-hand side of the distribution shown in Figure 3, namely an upward-sloping exponential distribution (Iyetomi, 2012).

However, we cannot reproduce the downward-sloping power distribution for high productivity firms. To explain it, we need to make an additional assumption that the number of potentially available high-productivity jobs is limited and it decreases as the level of productivity rises (Aoyama et al., 2014).

Potential jobs f_j are created by firms by accumulating capital and/or introducing new technologies, particularly new products. On the other hand, they are destroyed by firms' losing demand for their products permanently. Schumpeterian innovations by way of creative destruction raise the levels of some potential jobs, but at the same time lower the levels of others. In this way, the number of potential jobs with a particular level of productivity keeps changing. Note, however, that they remain only *potential* because firms do not necessarily attempt to fill all the job sites with workers. To fill them, firms either keep the existing workers on the job or post job vacancy signs and make enough hiring efforts, but they are economic decisions and depend crucially on the economic conditions facing firms. The number of potential job sites, therefore, is not exactly equal to, but rather imposes a ceiling on the sum of the number of filled job sites, or employment and the unfilled jobs.

Under reasonable assumptions, distribution of potential job sites with high productivity becomes downward-sloping power law. Adapting the model of Marsili and Zhang (1998), we can derive a power-law distribution such as the one for the tail of the empirically observed distribution of labor productivity; see Yoshikawa (2014) for details. However, the determination of employment by firms with various levels of productivity is another matter. To fill potential job sites with workers is the firm's economic decision. The most important constraining factor is the level of demand facing the firm in the product market. To fill potential job sites, the firm must either keep the existing workers on the job, or make enough hiring efforts including posting vacancy signs toward successful job matching. Such actions of the firms and job search of workers are purposeful. However, micro shocks affecting firms and workers are just unspecifiable. Then, how are workers actually employed at firms with various levels of productivity? This is the problem we considered in the previous section. In what follows, we will consider it in a more general framework.

The number of workers working at the firms with productivity c_k, namely n_k is

$$n_k \in \{0,\ 1,\ \cdots,\ g_k\} \quad (k = 1,\ 2, \cdots K)\ . \tag{16}$$

Here, g_k is the number of potential jobs with productivity c_k, and puts a ceiling on n_k. We assume that in the low productivity region, g_k is so large that n_k is virtually unconstrained by g_k. In contrast, in the high productivity region, g_k constrains n_k and it actually diminishes in a power form as we have analyzed above. When the number of potential jobs with high productivity is limited, behavior of economic agents necessarily becomes correlated; If good jobs are taken by some workers, it becomes more difficult for others to find such jobs. The present analysis precisely does it by introducing ceilings on n_j.

IV. Grand Canonical Formulation

One way to analyze the equilibrium distribution of labor productivity is to maximize entropy just as we did in Section II. Alternatively, Garibaldi and Scalas (2010) suggested that we could usefully apply the Ehrenfest-Brillouin model, a Markov chain to analyze the problem. In our previous paper (Aoyama et al., 2014), following their suggestion, we developed such a stochastic model to incorporate the assumption that a sector can accommodate a limited number of workers depending on its productivity. Here we derive the identical model within a framework of statistical physics taking advantage of the idea of grand canonical ensemble.

In the Ehrenfest-Brillouin model, we first considered the minimal binary processes with time reversal symmetry as depicted in Figure 4. We then took the probability flux $N(i, j; k, \ell)$ for transition from (i, j) to (k, ℓ) in the following form:

$$N(i, j; k, \ell) \propto n_i n_j L(c_k, n_k) L(c_\ell, n_\ell) \ . \tag{17}$$

Here the factor $L(c, n)$ limits the number n of workers in a firm of productivity c at $n = g(c)$, so that it is characterized by

$$1 \geq L(c, n) \geq 0 \quad \text{for} \quad n \leq g(c) \ , \tag{18}$$

$$L(c, n) = 0 \quad \text{for} \quad n > g(c) \ . \tag{19}$$

To proceed further, we adopted such a simple linear model for $L(c, n)$ as shown in Figure 5:

$$L(c, n) = \begin{cases} \dfrac{g(c) - n}{g(c)} = 1 - \dfrac{n}{g(c)} & \text{for } n \leq g(c) \ , \\ 0 & \text{for } n \geq g(c) \ . \end{cases} \tag{20}$$

Finally detailed balancing between the transition from (i, j) to (k, ℓ) and its reverse enabled us to derive the equilibrium distribution $\bar{n}(c)$ given by

$$\bar{n}(c) = \frac{g(c)}{g(c)/q(c) + 1} = \left[\frac{1}{q(c)} + \frac{1}{g(c)} \right]^{-1} \ , \tag{21}$$

with

$$q(c) = e^{\beta(\mu - c)} \ . \tag{22}$$

As noted by Garibaldi and Scalas (2010), the form (21) reduces to the Maxwell-Boltzmann (MB), the Fermi-Dirac (FD), and the Bose-Einstein (BE) distributions at $g^{-1} = 0, 1, \text{and } -1$, respectively.

Now we begin with the grand canonical modelling for $\bar{n}(c)$ by going back to the situation that sectors or firms can accommodate as many workers as they wish. In this case, the partition function Z_N, corresponding to the Lagrangian (6), is written down as

$$Z_N = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \cdots \sum_{n_K=0}^{N} W_N(n) \exp \left[-\beta \sum_{k=1}^{K} c_k n_k \right] \ , \tag{23}$$

where the multiple summation with respect to the number of workers at each productivity level is carried out subject to the constraint (1). Adoption of a grand canonical formulation make us free from such a troublesome restriction. We thus think about the situation in which the system under study is connected to a worker reservoir with "chemical potential" μ_0. The grand partition function $\varXi$ is then defined as

$$\varXi = \sum_{N=0}^{\infty} e^{\beta\mu_0 N} \binom{N_\infty}{N} Z_N \ , \tag{24}$$

where N_∞ is the total number of workers in the extended system and $\binom{N_\infty}{N}$ denotes the binomial coefficient. Note that the combinatorial factor in (24) stems from the statistical assumption that workers are *distinguishable*. Since one can assume $N \ll N_\infty$, the combinatorial factor is well approximated by

$$\binom{N_\infty}{N} \simeq \frac{N_\infty^N}{N!} \ . \tag{25}$$

With this approximation, we can calculate (24) as

$$\varXi = \prod_{k=1}^{K} \varXi(c_k) \ , \tag{26}$$

where we set

$$\varXi(c) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{\beta(\mu-c)n} = \exp\left[e^{\beta(\mu-c)}\right] \ , \tag{27}$$

and

$$e^{\beta\mu} = N_\infty e^{\beta\mu_0}. \tag{28}$$

The equilibrium distribution $\bar{n}(c)$ is then obtained from $\varXi(c)$ according to

$$\bar{n}(c) = q(c)\frac{\partial \ln \varXi(c)}{\partial q(c)} \ , \tag{29}$$

where $q(c)$ is defined by (22). Certainly substitution of (27) in this formula gives the Maxwell-Boltzmann distribution.

Next we impose the restriction on the distribution of productivity by ceiling the number of job positions at each productivity level c:

$$\varXi(c) = \sum_{n=0}^{\infty} f(n; g(c)) \frac{1}{n!} q(c)^n \ , \tag{30}$$

where we introduced the ceiling function $f(n; g(c))$ which is characterized by

$$1 \geq f(n; g(c)) \geq 0 \quad \text{for } n \leq g(c) \ , \tag{31}$$

$$f(n; g(c)) = 0 \quad \text{for } n > g(c) \ . \tag{32}$$

These two conditions, (31) and (32), imposed on $f(n; g(c))$ correspond to (18) and (19) on $L(c, n)$, respectively. The equation (30) is compared with (23) in Yoshikawa (2014), where workers are treated as being *indistinguishable*.

In order to reproduce (21), in fact, we only have to choose $f(k; g(c))$ in the following form:

$$f(n; g(c)) = \binom{g(c)}{n} \frac{n!}{g(c)^n} = \frac{g(c)!}{(g(c) - n)! g(c)^n} \ , \tag{33}$$

supplemented with (32). Then, the grand partition function (30) is explicitly calculated as

$$\Xi(c) = \sum_{k=0}^{g(c)} \binom{g(c)}{k} \left(\frac{q(c)}{g(c)} \right)^k = \left(1 + \frac{q(c)}{g(c)} \right)^{g(c)} \ . \tag{34}$$

We can derive the desired result (21) from this grand partition function through (29). The formula (34) has been already obtained heuristically by Aoyama et al. (2014).

It should be remarked that (33) is well defined even at $g = -1$:

$$f(n; g = -1) = \binom{-1}{n} \frac{n!}{(-1)^n} = n! \ , \tag{35}$$

with

$$\binom{-1}{n} = \frac{(-1)(-2) \cdots (-n)}{n!} = (-1)^n \ . \tag{36}$$

This leads to the BE distribution as expected. We thus see that the ceiling function (33) is capable of representing the three typical distributions, MB, FD, and BE, in a unified way. This is within a framework of the grand canonical formulation for *distinguishable* particles. On the other hand, the grand canonical formulation for *indistinguishable* particles as in the case of quantum physics can accommodates only the FD and the BE distributions; the MB distribution is obtained by taking the classical limit of $\bar{n} \ll 1$ in either of the two quantum distributions. These relationships are diagramed in Figure 6.

In passing we note that the Gaussian approximation to $f(n; g(c))$ in (33),

$$f(n; g(c)) \simeq \exp \left[-\frac{n^2}{2g(c)} \right] \ , \tag{37}$$

is valid for $g(c) \gg 1$. Although we assumed $n \ll g(c)$ besides to derive (37), the additional condition is automatically satisfied by the resulting Gaussian form; its relevant range of n is $n \lesssim \sqrt{2g(c)} \ll g(c)$ for $g(c) \gg 1$. Figure 7 compares the results calculated in the exact functional form for $g = 10$ with the corresponding results in the Gaussian form. We see the approximation works well even for a not so large value of g.

V. Fitting of the Model to the Japanese Data: Manufacturing and Non-manufacturing Sectors

First, we note that when there is no limit to the number of the workers, *i.e.*, $g \to \infty$, (21) boils down to the Maxwell-Boltzmann distribution,

$$\bar{n}_{\rm MB}(c) = e^{-\beta(c-\mu)}. \tag{38}$$

When we apply (38) to low-to-intermediate range of c where $\bar{n}$ is an exponentially increasing function of c as observed in Figure 3, we must have

$$\beta < 0 \ . \tag{39}$$

The negative β is tantamount to the negative temperature (Landau and Lifshitz, 1980; Kittel and Kroemer, 1980). The current model thus accommodates the Boltzmann statistics model with negative temperature advanced by Iyetomi (2012) as a special case.

Secondly, we recall the observation that the power law (15) holds for $\bar{n}$ in the high productivity side. We can use this empirical fact to determine the functional form for $g(c)$. Equation (21) implies that when temperature is negative, $\bar{n}$ approaches $g(c)$ in the limit $c \to \infty$. These arguments lead us to adopt the following ansatz for $g(c)$,

$$g(c) = Ac^{-\gamma}. \tag{40}$$

Given the present model, explaining the empirically observed distribution of productivity is equivalent to determining four parameters, β, μ, A, and γ in (21) and (40). We estimate these four parameters by the χ^2 fit to the empirical results. Figure 8 demonstrates the results of the best fit for three datasets of firms, namely, those for all the sectors, the manufacturing sector, and the non-manufacturing sector[2]. The fitted parameters are listed in Table 1, together with the crossover productivity $c_{\rm p}$ separating low-to-medium and high productivity regimes, where $c_{\rm p}$ is defined according to

$$\frac{\partial \bar{n}(c)}{\partial c} = \bar{n}^2 \left(\frac{1}{g(c)} \frac{\partial \ln g(c)}{\partial c} + \frac{1}{q(c)} \frac{\partial \ln q(c)}{\partial c} \right) = 0 \ . \tag{41}$$

The present model is quite successful in unifying the two opposing functional behaviors of the average number of workers with low-to-medium and high productivities.

In the above, we treated the economy as a whole and also regarded it as consisting of two sectors. Economic systems are generally inhomogeneous, since they can be decomposed into various components such as industrial sectors, regional sectors and business groups. Here we derive necessary conditions for such economic subsystems to be in equilibrium to each other. We are required to modify the standard

[2] The extraction of the empirical distributions, especially for the manufacturing sector, may be hampered considerably by unavailability of material costs in the data.

derivation (Landau and Lifshitz, 1980) of those conditions in physics, because we have to take into account distinguishability of workers.

Let us suppose that an economic system consists of two subsystems A and B with N_{A} and N_{B} workers and demands of D_{A} and D_{B}, respectively. The total number $W_{\mathrm{A+B}}$ of microscopic states of the whole system is calculated as

$$W_{\mathrm{A+B}}\left(N_{\mathrm{A}}, N_{\mathrm{B}}; D_{\mathrm{A}}, D_{\mathrm{B}}\right) = \frac{N!}{N_{\mathrm{A}}!N_{\mathrm{B}}!} W_{\mathrm{A}}\left(N_{\mathrm{A}}, D_{\mathrm{A}}\right) W_{\mathrm{B}}\left(N_{\mathrm{B}}, D_{\mathrm{B}}\right) \ , \tag{42}$$

where we assume that the total number of workers, $N = N_{\mathrm{A}} + N_{\mathrm{B}}$, and the total demand, $D = D_{\mathrm{A}} + D_{\mathrm{B}}$, are conserved during contact of the two subsystems. The prefactor in the right-hand side of (42) counts how many ways to distribute *distinguishable* workers between the two subsystems. This counting is unnecessary in the case of statistical physics because identical particles are indistinguishable in nature. From (42) we obtain the entropy of the whole system as

$$S_{\mathrm{A+B}} \simeq S_{\mathrm{A}} + S_{\mathrm{B}} + N \ln N - N_{\mathrm{A}} \ln N_{\mathrm{A}} - N_{\mathrm{B}} \ln N_{\mathrm{B}} \ . \tag{43}$$

It should be remarked that the last three terms arising from the extra counting factor destroy the additivity of the entropy; the entropy of the whole system is not separable into components of the subsystems.

The entropy maximum principle determines the most probable distribution of workers and demands between the two subsystems. Since both N and D are assumed to be constant, the variational condition is explicitly written down with (7) as

$$\begin{aligned} \delta S_{\mathrm{A+B}} &= \delta S_{\mathrm{A}} + \delta S_{\mathrm{B}} - \delta N_{\mathrm{A}} \ln N_{\mathrm{A}} - \delta N_{\mathrm{B}} \ln N_{\mathrm{B}} \\ &= \left(\alpha_{\mathrm{A}} - \alpha_{\mathrm{B}} - \ln N_{\mathrm{A}} + \ln N_{\mathrm{B}}\right) \delta N_{\mathrm{A}} + \left(\beta_{\mathrm{A}} - \beta_{\mathrm{B}}\right) \delta D_{\mathrm{A}} \\ &= 0 \ . \end{aligned} \tag{44}$$

where δN_{A} and δD_{A} denote the infinitesimal variations of N_{A} and D_{A} which are independent of each other. We thus obtain the following equilibrium conditions:

$$\beta_{\mathrm{A}} = \beta_{\mathrm{B}} \ , \tag{45}$$

$$N_{\mathrm{A}} e^{-\alpha_{\mathrm{A}}} = N_{\mathrm{B}} e^{-\alpha_{\mathrm{B}}} \ . \tag{46}$$

The first condition (45) shares the same form as in physics. It is required for the system to be in equilibrium against exchange of demands. The second condition (46) guarantees no macroscopic flow of workers between the two subsystems. The equation, having the number of workers on each side, is unfamiliar to physicists; distinguishability of workers brings the extra factor to the normal form. Demand flows from subsystem A to subsystem B if $\beta_{\mathrm{A}} < \beta_{\mathrm{B}}$ and in the reversed direction if $\beta_{\mathrm{A}} > \beta_{\mathrm{B}}$. Also workers flow from A to B if $N_{\mathrm{A}} e^{-\alpha_{\mathrm{A}}} > N_{\mathrm{B}} e^{-\alpha_{\mathrm{B}}}$ and vice versa in the opposite case.

Table 1 shows that the temperature of the non-manufacturing sector is significantly lower than that of the manufacturing sector. This fact implies that there is a much wider demand gap in the non-manufacturing sector. The system as a whole is thus far away from equilibrium in exchanges of product demand between the two sectors. In contrast, β times μ takes almost the same value for the two sectors, indicating that the subsystems seem to be balanced against flow of workers. Note that comparison of (12) and (38) rewrites the equilibrium condition (46) as

$$\beta_{\mathrm{A}}\mu_{\mathrm{A}} = \beta_{\mathrm{B}}\mu_{\mathrm{B}} \ . \tag{47}$$

These empirical findings on equilibration of the Japanese economic system with respect to exchanges of demand and workers have been established by Iyetomi (2012) and Aoyama et al. (2014) on the basis of the alternative dataset for the years spanning from 2000 through 2009.

VI. Concluding Remarks

The concept of stochastic macro-equilibrium is motivated by the presence of all kinds of unspecifiable micro shocks. At first, one might think that allowing all kinds of unspecifiable micro shocks leaves so many degrees of freedom that almost anything can happen. However, the methods of statistical physics — the maximization of entropy under macro-constraints — actually provide us with the quantitative prediction about the equilibrium distribution of productivity.

It is extremely important to recognize that the present approach does not regard behaviors of workers and firms as random. They certainly maximize their objective functions perhaps dynamically in their respective stochastic environments. The maximization of entropy under the aggregate demand constraint (6), in fact, balances two forces. On one hand, whenever possible, workers are assumed to move to better jobs which are identified with job sites with higher productivity. Firms make efforts for hiring good workers under demand constraint in the goods market. It is the outcome of successful job matching resulting from the worker's search and the firm's recruitment. When the level of aggregate demand is high, this force dominates because demand for labor of high productivity firms is high. However, as the aggregate demand gets lower the number of possible allocations consistent with the level of aggregate demand increases. More workers are forced to be satisfied with or look for low productivity jobs. Randomness which plays a crucial role in our analysis basically comes from the fact that demand constraints in the product market facing firms with different productivity, and optimizing behaviors of workers and firms under such constraints are so complex and unspecifiable that those of us who analyze the macroeconomy must take micro behaviors as random. The method is straight-forward, and does not require any arbitrary assumptions on the behavior of economic agents.

Though the concept of entropy is little known in economics, it is useful for understanding the macroeconomy which comprises many heterogeneous agents. Based on the methods of statistical physics, we quantitatively show how labor is mobilized when the aggregate demand rises. The level of aggregate demand is the ultimate factor conditioning the outcome of random matching of workers and firms. By so doing, it changes not only unemployment but also the distribution of productivity, and as a consequence, the level of aggregate output. This is the market mechanism beneath Keynes' principle of effective demand (Keynes, 1936); See Yoshikawa (2014) for details.

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	β	μ	A	γ	c_{p}	$\beta\mu$
All	-4.32	-0.48	2.57	0.90	1.32	2.09
Manucacturing	-8.68	-0.26	4.75	0.97	0.73	2.27
Non-manufacturing	-2.13	-1.07	0.68	0.76	2.15	2.28

Table 1. Estimated parameters in (21) and the position of the peak c_{p} determined by (41) for the results given in Figure 8. The units are 10^{-3} for β, 10^3 for μ, 10^5 for A, and 10^3 for c_{p}.

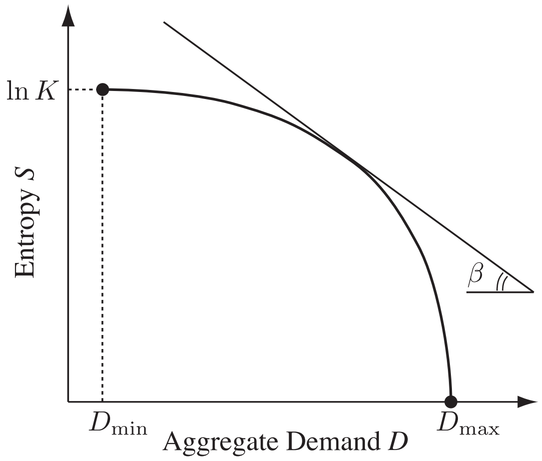

Figure 1. Relationship between entropy S and aggregate demand D, where β is a Lagrangean multiplier in (6) in the text.

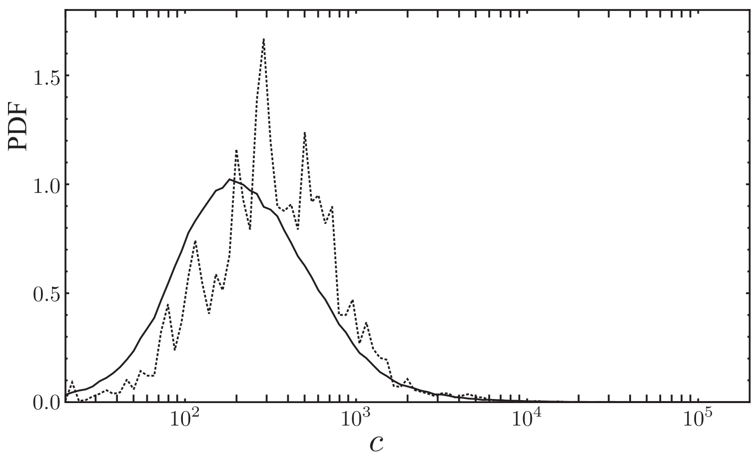

Figure 2. Probability Density Function (PDF) of firms' $\log c$ (solid line) and workers' $\log c$ (dotted line) in 2012. The labor productivity c is measured in units of 10^3 USD/person.

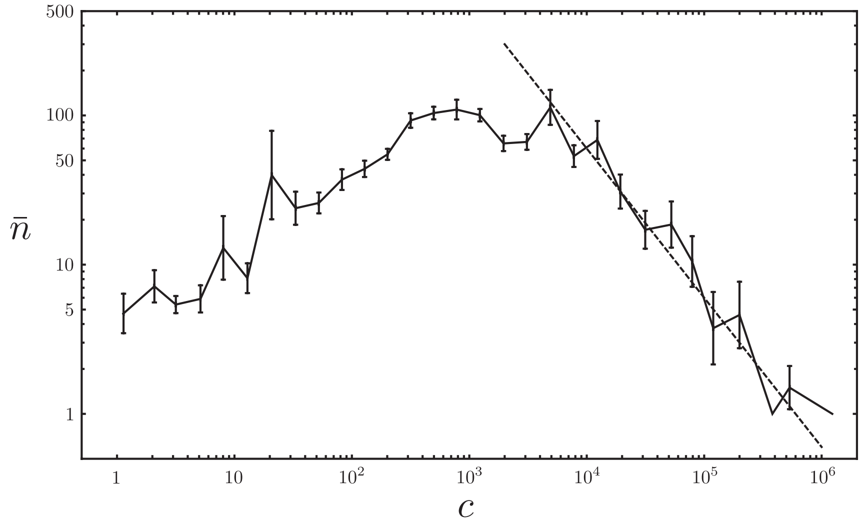

Figure 3. Dependence of the average number $\bar{n}$ of workers of individual firms on the labor productivity c (dots with their error bars connected by thick lines) in 2012. The dotted straight line has the gradient -1, that is, $\bar{n} \propto 1/c$.

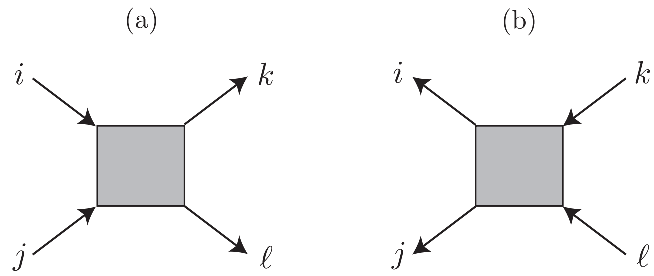

Figure 4. Elementary binary processes where (a) two workers at firms i and j simultaneously move to firms k and ℓ and (b) two workers move in the time-reversal way with the same probability as the forward process.

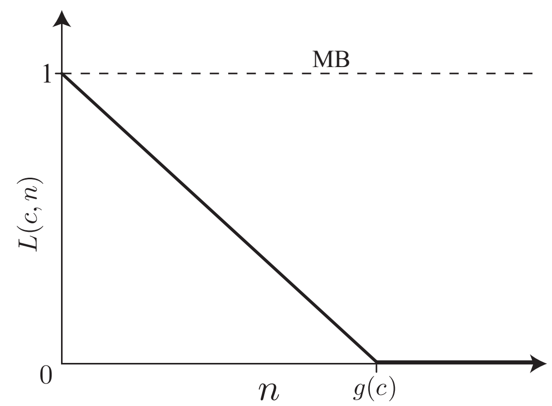

Figure 5. A model of worker limitation $L(n, c)$ in (20); $L(n, c) = 1$ corresponds to the Maxwell-Boltzmann (MB) distribution.

For **indistinguishable** particles,

$$\left.\begin{array}{l} g = 1\text{: Fermi-Dirac dist.} \\ g = \infty\text{: Bose-Einstein dist.} \end{array}\right\} \xrightarrow[n \ll 1]{} \text{Maxwell-Boltzmann dist.}$$

For **distinguishable** particles,

$g = 1\,(1/g = 1)$: Fermi-Dirac dist.

$g = \infty\,(1/g = 0)$: Maxwell-Boltzmann dist.

$g = -1\,(1/g = -1)$: Bose-Einstein dist.

Figure 6. Relationship of the grand partition function (30) supplemented by (33) for distinguishable particles to the Maxwell-Boltzmann, the Fermi-Dirac, and the Bose-Einstein distributions, which is compared with the case of the grand canonical formulation for indistinguishable particles.

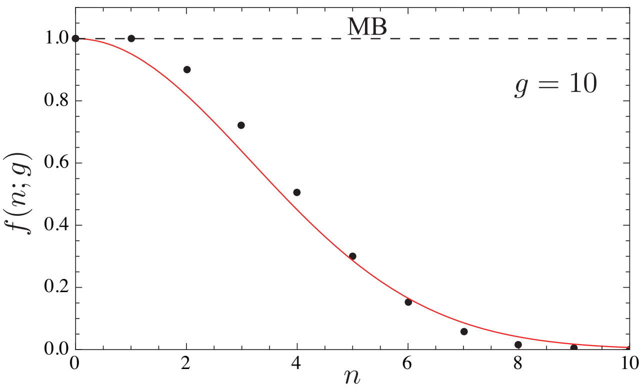

Figure 7. Ceiling function $f(n;g)$, (33), calculated in the exact functional form (dots) and the approximated Gaussian form (solid curve) at $g = 10$. Setting $f(n;g) = 1$ results in the Maxwell-Boltzmann (MB) distribution.

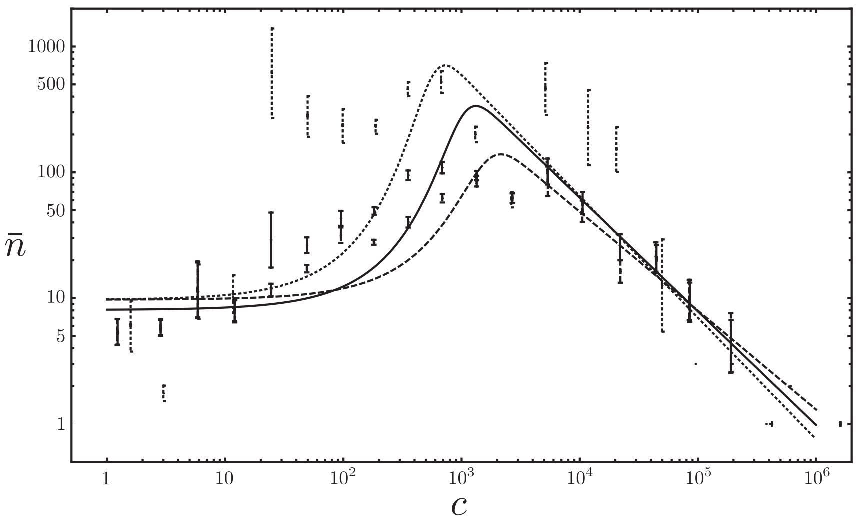

Figure 8. The best fits to the empirical data in 2012 as demonstrated in Figure 3; the solid curve is for all sectors, the dotted curve for the manufacturing sector, and the dashed curve for the non-manufacturing sector.