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Existence of Solutions for a Coupled Hadamard Fractional System of Integral Equations in Local Generalized Morrey Spaces

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Abstract: This paper introduces a new measure of non-compactness within a bounded domain of $\mathbb{R}^{\mathbb{N}}$ in the generalized Morrey space. This measure is used to establish the existence of solutions for a coupled Hadamard fractional system of integral equations in generalized Morrey spaces. To illustrate the application of the main result, an example is presented.

Keywords: non-compactness measure; hadamard fractional system of integral equations; non-compactness; boundedness

MSC: 26A33; 34A08; 45G15; 47H08

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1. Introduction

Fractional calculus (FC) extends the traditional concepts of integer-order calculus by introducing fractional-order derivatives and integrals [1,2]. This generalization allows for a more flexible and comprehensive mathematical framework, which is particularly valuable for modeling complex systems that exhibit non-locality and memory effects phenomena that standard calculus struggles to address effectively. FC has emerged as a critical area of research due to its unique ability to capture and describe processes where the history of the system or its spatial interactions play a crucial role. Over the past few years, fractional calculus has gained substantial attention, with a rapidly growing body of research exploring its wide range of applications across various scientific and engineering fields [3–8]. The increasing interest in FC is largely driven by its success in modeling physical systems that require more than local and immediate effects for accurate representation. For instance, FC has proven to be highly effective in describing diffusion processes, where the rate of diffusion depends on past states, and in control theory, where systems with memory or delayed responses are better understood through fractional models. Similarly, heat conduction and electromagnetics benefit from FC's ability to model systems that involve long-range interactions and cumulative effects over time [9–11].

Results regarding compactness in the spaces $L^p(\mathbb{R}^d)$, with $1 \leq p < \infty$, and $C(K)$, meaning the space of continuous functions on a compact metric space K with values in $\mathbb{R}$, are essential for establishing the existence of solutions to functional integral, integral, and differential equations.

Kuratowski [12] proposed the first definition of the measure of non-compactness. According to his definition, letting Ω be any bounded subset of a metric space, if it is possible to cover Ω with a finite number of balls, then the smallest diameter among these balls is referred to as the Kuratowski's measure of non-compactness. Saha et al. [13] applied the measure within the Hölder space, specifically when studying nonlinear functional integral equations with changed arguments. Mehravaran et al. [14] introduced a set of measures of non-compactness in the locally Sobolev space and then used it to study the existence of solutions for a class of Volterra integro-differential nonlinear equations.

Mehravaran et al. [15] proposed a measure of non-compactness for regulated functions in a specific space, and demonstrated a Darbo-type fixed point theorem. Metwali and Vishnu [16] defined a compactness criterion in Lebesgue spaces and constructed a related measure of non-compactness. They applied it using a modified Darbo-type fixed point theorem to prove that products of n-Hammerstein integral equations have monotonic integrable solutions. Tamimi et al. [17] developed a measure of non-compactness in the generalized Morrey space and used it to address the existence of solutions to systems of nonlinear integral equations. Zhu and Han [18] used the measure of non-compactness, the convex-power condensing operator, the solution operator, and the Banach contraction principle to investigate nonlinear time fractional-order (FO) partial integro-differential equations. Zhu and Han [19] used the Schauder fixed point theorem to explore the approximate controllability of mixed-type non-autonomous FO differential equations. Zhu et al. [20] adopted the measure of non-compactness, the β-resolvent family, and fixed point theorems to study the existence and uniqueness of mild solutions for FO non-autonomous evolution equations. Banaei and Mursaleen [21] established Darbo's fixed point theorems using measures of non-compactness and weakly JS-contractive conditions in Banach spaces. They applied these results to a system of differential equations. Aghajani and Haghighi [22] employed measures of non-compactness and the Darbo fixed point theorem to establish criteria for the existence of solutions to systems of nonlinear equations in Banach spaces. Bokayev et al. [23] provided sufficient conditions for the pre-compactness of sets in the generalized Morrey spaces M^p_w. Arab et al. [24] introduced compact sets in C^k and C^k_0 spaces and developed new measures of non-compactness. They applied them to prove the existence of solutions for functional integro-differential equations.

We can generalize many important theorems from L^p to functions in generalized Morrey spaces. The benefit of working in Morrey spaces, as compared to L^p, is that we can study functions without compact support. This space controls the local integrability of functions, and it makes Morrey spaces more flexible for certain applications. Tamimi et al. [17] developed the first measure of noncompactness in the generalized Morrey spaces; this measure was defined on $\mathbb{R}^{\mathbb{N}}$, and it is a powerful tool for studying the integral equation on unbounded domains.

In this paper, we introduce the new noncompactness measure defined on a bounded domain of $\mathbb{R}^{\mathbb{N}}$. This measure can be applied to investigate the solution for integral equations on every bounded domain of the generalized Morrey spaces. By using the measure of noncompactness within generalized Morrey spaces, we investigate the existence of solutions for the Hadamard FO system of integral equations

$$\begin{cases} u(\xi) = f_1\Big(\xi, u(\xi), v(\xi)), \int\limits_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_1(\eta, u(\eta), v(\eta))}{\eta} d\eta\Big), \\ v(\xi) = f_2\Big(\xi, u(\xi), v(\xi)), \int\limits_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_2(\eta, u(\eta), v(\eta))}{\eta} d\eta\Big), \end{cases} \tag{1}$$

where $\xi \in [0, T]$, $\eta \in \mathbb{R}$, $\alpha \in (0, 1)$, and $f_i : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ for $i = \{1, 2\}$.

Definition 1 ([25])**.** *Given the function $f : [a, \infty) \to \mathbb{R}$, with $a > 0$, its left-sided Hadamard FO integral of order $\alpha > 0$ is*

$$H^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds,$$

assuming that the integral exists, $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$, and $\ln(\cdot)$ denotes the natural logarithm function.

Definition 2 ([25])**.** *For $f \in C^n[a,\infty)$ and $a, \alpha > 0$, the α-order left-sided Hadamard FO derivative is*

$$H^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt} \right)^n \int_a^t \left(\ln \frac{t}{s} \right)^{n-\alpha-1} \frac{f(s)}{s} \, ds,$$

where $n-1 < \alpha \le n$ and $n = \lceil \alpha \rceil + 1$, with $\lceil \cdot \rceil$ denoting the Gaussian function.

Hadamard first introduced the concept of Hadamard fractional-order (FO) calculus in his work [26]. This form of fractional calculus differs significantly from the more commonly known Riemann–Liouville (R-L) fractional calculus, particularly in terms of the kernel functions that define the fractional derivatives. In a recent study, Ould et al. [27] explored the uniqueness of mild solutions, as well as Ulam–Hyers and Ulam–Hyers-Rassias stability, for abstract fractional differential equations of Sobolev type with nonlocal boundary conditions. Their work employed the Hadamard derivative to achieve these results. Additionally, Maazouz et al. [28] tackled the issue of existence and uniqueness of solutions for boundary value problems related to nonlinear fractional-order pantograph equations. Their approach involved the use of a variable-order fractional derivative of Hadamard type. Furthering the exploration of Hadamard derivatives, Benkerrouche et al. [29] analyzed impulsive boundary value problems (BVP) for differential equations with variable fractional order that include the Caputo–Hadamard fractional derivative. Moreover, Ntouyas et al. [30] initiated research into fractional boundary value problems involving a combination of sequential Riemann–Liouville and Hadamard–Caputo fractional derivatives, with iterated fractional integral boundary conditions supplementing the differential equations. Indeed, the Hadamard and the R-L FO calculus use the kernel functions $k_H(t,s) = (\log \frac{t}{s})^{\alpha-1}$ and $k_{R-L}(t,s) = (t-s)^{\alpha-1}$, respectively. For any $\mu > 0$, $k_H(\mu t, \mu s) = k_H(t,s)$, while $k_{R-L}(\mu t, \mu s) = \mu^{\alpha-1} k_{R-L}(t,s) \neq k_R(t,s)$, highlighting another difference. For more information on the Hadamard FO calculus, please see references [25,30–33].

In the following, we propose a novel measure of non-compactness within a bounded domain of $\mathbb{R}^N$ in the context of generalized Morrey spaces. This newly defined measure is employed to analyze the existence of solutions for a coupled Hadamard FO system of integral equations in those spaces. Additionally, we provide an example that illustrates the applicability of the attained results and highlights potential avenues for further research and applications in a variety of fields.

The rest of this research paper is organized as follows. Section 2 introduces some basic definitions and results. Section 3 establishes a new measure of non-compactness within the framework of generalized Morrey spaces on bounded domains. Section 4 provides an example to confirm the effectiveness and applicability of this measure. Finally, Section 5 summarizes the key findings.

2. Preliminaries

This section provides some fundamental definitions and results that are necessary for the work presented in this paper.

Definition 3 ([21])**.** *The mapping $\mu : \Lambda_\Theta \longrightarrow [0, +\infty]$ is a measure of non-compactness in Θ if:*

A1. $\ker \mu := \{U \in \Lambda_\Theta : \mu(U) = 0\}$ *is Definition 3. nonempty and* $\ker \mu \subset D_\Theta$*,*
A2. $U \subseteq V \Longrightarrow \mu(U) \le \mu(V)$*,*
A3. $\mu(\overline{U}) = \mu(U) = \mu(\mathrm{Conv}\, U)$*,*
A4. $\mu(cU + (1-c)V) \le c\mu(U) + (1-c)\mu(V)$*, for each* $c \in [0,1]$*,*
A5. If $\{U_n\}$ *is a sequence of closed sets from* Λ_Θ*, such that* $U_{n+1} \subseteq U_n$ *and* $\lim_{n\to\infty} \mu(U_n) = 0$*, then the set* $U_\infty = \bigcap_{n=1}^{\infty} U_n \neq \emptyset$*, where* D_Θ *is the set of all precompact sets in* Λ_Θ*, and* Λ_Θ *is the set of all nonempty, bounded subsets of* Θ*.*

Theorem 1 ([34]). *Consider a nonempty, bounded, convex, and closed subset F within a Banach space* Ω*, with a continuous mapping* $E : \Omega \longrightarrow \Omega$*, and let* μ *be a measure of non-compactness on* Ω*. If, for any nonempty subset U of* Ω*, a constant* $c \in [0, 1)$ *exists so that*

$$\mu(EU) \leq c\mu(U),$$

then E possesses at least one fixed point in Ω*.*

Theorem 2 ([22]). *Consider a nonempty, bounded, convex, and closed subset E of a Banach space V, and let* μ *be an arbitrary measure of non-compactness on V. Suppose that, for* $n \in \mathbb{N}$ *and* $i = 1, 2, \ldots, n$*, continuous operators* $\mathcal{H}_i : E^n \to E$ *exist, and for any subsets* $U_1, \ldots, U_n$ *of E, a constant* $\lambda \in [0, 1)$ *exists so that*

$$\mu(\mathcal{H}_i(U_1, \ldots, U_n)) \leq \lambda \max\{\mu(U_1), \ldots, \mu(U_n)\}. \tag{2}$$

Then, for every i, there exist elements $\gamma_1^*, \ldots, \gamma_n^* \in E$*, so that* $\mathcal{H}_i(\gamma_1^*, \ldots, \gamma_n^*) = \gamma_i^*$*.*

Definition 4 ([23]). *Let* $\Omega_{p\infty}^N$ *be the set of all non-negative, measurable and not equivalent to zero functions* $\vartheta : (0, +\infty) \longrightarrow (0, +\infty)$*, which for some* $\xi > 0$

$$||\vartheta(.)r^{\frac{N}{p}}||_{L^\infty(0,\xi)} < \infty,$$

and for all $\xi > 0$

$$||\vartheta(.)||_{L^\infty(\xi,+\infty)} < \infty,$$

where $1 \leq p \leq \infty$*.*

Definition 5 ([23]). *The local generalized Morrey space* $M_p^\vartheta \equiv M_p^\vartheta(\Omega)$ *on* $\Omega = [a_1, b_1] \times ... \times [a_n \times b_n] \subseteq \mathbb{R}^N$ *is the finite quasi-norm*

$$||f||_p^\vartheta \equiv \sup_{\substack{u\in\Omega, \\ 0<r<diam(\Omega)}} ||\vartheta(.)||f||_{L^p(B(u,.))}||_{L^\infty(0,r)}, \tag{3}$$

where $\vartheta \in \Omega_{p\infty}^N$*.*

Theorem 3. *The space* $M_p^\varpi(\mathbb{R}^N)$ *is a non-trivial normed vector space if and only if* $\varpi \in \Omega_{p\infty}^N$*.*

Example 1. *Let us consider*

$$f(x) = \begin{cases} \sqrt{\pi}, & x \in [2\nu, 2\nu + 1], \\ -\sqrt{\pi}, & x \in (2\nu - 1, 2\nu), \end{cases}$$

where $\nu \in \mathbb{Z}$*. If* $\vartheta(r) = r^{-\frac{1}{2}}$*, we have*

$$\sup_{x\in\mathbb{R}} ||\varpi(.)||f||_{L^2(B(x,0))}||_{L^\infty(0,+\infty)} = \sup_{x\in\mathbb{R},r>0} |r^{-\frac{1}{2}}(\int_{x-r}^{x+r} f^2(s)ds)^{\frac{1}{2}}| = \sqrt{2\pi}.$$

Example 2. *We consider*

$$g(x) = \begin{cases} \text{sech}(x), & x \in [2\nu, 2\nu + 1], \\ 0.5.\tanh(x), & x \in (2\nu + 1, 2\nu + 2), \end{cases}$$

where $v \in \nvDash\mathbb{Z}$*, which belongs to* $M_2^{\vartheta}(\mathbb{R}$ *If* $\vartheta(r) = r^{-\frac{1}{2}}$*, and we have*

$$\sup_{x\in\mathbb{R}} ||\varpi(.)||f||_{L^2(B(x,0))}||_{L^\infty(0,+\infty)} = \sup_{x\in\mathbb{R},r>0} |r^{-\frac{1}{2}}(\int_{x-r}^{x+r} f^2(s)ds)^{\frac{1}{2}}| = \sqrt{2\pi}.$$

This shows that discontinuous functions can belong to the generalized Morrey space. Figure 1 presents a plot of this function.

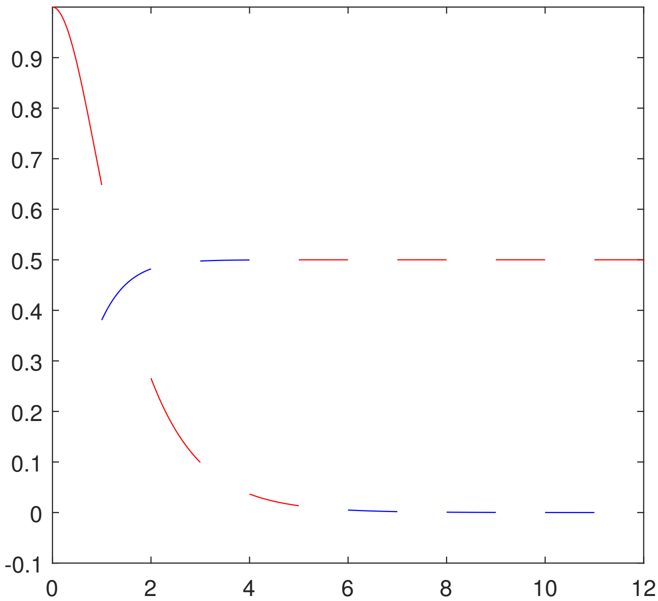

Figure 1. Morrey spaces.

Theorem 4 ([23])**.** *Suppose that* $\vartheta \in \Omega^N_{p\infty}$ *and* $1 \le p \le \infty$*. If, for a subset Q of* $M_p^{\vartheta}(\mathbb{R}^N)$*,*

D1. $\sup_{g\in Q} ||g||_p^{\vartheta} < \infty$,

D2. $\lim_{T\to\infty} \sup_{g\in Q} ||g\chi_{B^c(0,T)}||_p^{\vartheta} = 0$,

D3. $\lim_{|d|\to 0} \sup_{g\in Q} ||T_d g - g||_p^{\vartheta} = 0$,

where $T_d g(u) := g(u+d)$ *for all* $u, d \in \mathbb{R}^N$*, then the subset Q is precompact in* $M_p^{\vartheta}(\mathbb{R}^N)$*.*

Lemma 1 ([24])**.** *Assume that E is a metric space and that, for each* $\epsilon > 0$*, there exist a constant* $\gamma > 0$*, a metric space F and a mapping* $\psi : E \longrightarrow F$*, such that* $\psi[E]$ *is totally bounded. Moreover, for* $u, v \in E$*, if* $d(u,v) < \varepsilon$*, then* $d(\psi(u), \psi(v)) < \gamma$*. Under these conditions, E is totally bounded.*

3. Measure of Non-Compactness on Bounded Domains

This section establishes a novel measure of non-compactness within the framework of generalized Morrey spaces on bounded domains. This new measure builds upon existing concepts of non-compactness, offering a more refined tool for analyzing the compactness properties of sets in these spaces.

Theorem 5. *Assume that* Ω *is a compact subset of* $\mathbb{R}^N$*. If* $F \subset M_p^{\vartheta}(\Omega)$ *is bounded and equicontinuous with norm of local generalized Morrey space, then F is totally bounded in* $M_p^{\vartheta}(\Omega)$*.*

Proof. By using the equicontinuity of F and compactness of Ω for all $\varepsilon > 0$, we can obtain a finite set of $u_1, ..., u_m \in \Omega$, such that $|u - u_i| < \varepsilon$, with neighborhood $U_1, ..., U_m$ covering Ω, so that $||f(u) - f(u_i)||_p^{\vartheta} < \varepsilon$. Define $\varphi : F \longrightarrow \mathbb{R}^m$, such that $\varphi(f) := (f(u_1), ..., f(u_m))$

by boundedness of F. The image $\varphi[F]$ is totally bounded on $\mathbb{R}^m$. Additionally, if $f, g \in F$ satisfy $||\varphi(f) - \varphi(g)||_p^\vartheta < \varepsilon$, then by Lemma 1, F is totally bounded. □

Theorem 6. *Let $\mathcal{H}$ be a bounded subset of $M_p^\vartheta(\Omega)$, with $1 \le p \le \infty$, and*

$$\varpi(g, \varepsilon) := \sup\Big\{||T_d g - g||_p^\vartheta : |d| < \epsilon\Big\}, \tag{4}$$

$$\varpi(\mathcal{H}, \varepsilon) := \sup\{\varpi(g, \varepsilon) : g \in \mathcal{H}\}, \tag{5}$$

$$\varpi_0(\mathcal{H}) := \lim_{\epsilon \to 0} \varpi(\mathcal{H}, \varepsilon), \tag{6}$$

where $T_d g(\xi) := g(\xi + d)$ for all $\xi, d \in \Omega$. Therefore, $\varpi_0 : \Omega_{M_p^\vartheta(\Omega)} \longrightarrow \mathbb{R}^+$ is a measure of non-compactness on $M_p^\vartheta(\Omega)$.

Proof. Let $\mathcal{H} \in \Lambda_\Omega$, such that $\varpi_0(\mathcal{H}) = 0$. According to Equations (4)–(6), it holds that $\lim\limits_{|d| \to 0} ||T_d g - g||_p^\vartheta = 0$ uniformly for all $g \in \mathcal{H}$. Therefore, by Theorem 5, we conclude that $\mathcal{H}$ is precompact. The condition $(A2)$ is straightforward, and thus $\varphi_0(\mathcal{H}) \le \varpi_0(\bar{\mathcal{H}})$. To prove the reverse inequality, assume that $g \in \bar{\mathcal{H}}$ and that there exists a sequence $\{g_n\}_{n=1}^\infty \subset \mathcal{H}$ such that $g_n \to g$ in $M_p^\vartheta(\Omega)$. For $\epsilon > 0$, a $k \in \mathbb{N}$ exists such that, for all $m > k$, $||g - g_m||_p^\vartheta < \dfrac{\epsilon}{2}$. For each $d \in \mathbb{R}^N$ with $|d| < \varepsilon$

$$\begin{aligned}
&||T_d g - g||_p^\vartheta \le ||T_d g - T_d g_m||_p^\vartheta + ||T_d g_m - g_m||_p^\vartheta + ||g_m - g||_p^\vartheta \\
&= ||g_m - g||_p^\vartheta + ||T_d g_m - g_m||_p^\vartheta + ||g - g_m||_p^\vartheta \\
&\le \varepsilon + \varpi(g_m, \varepsilon).
\end{aligned}$$

Therefore, $\varpi(\bar{\mathcal{H}}, \varepsilon) \le \varepsilon + \varpi(\mathcal{H}, \varepsilon)$. Hence,

$$\varpi_0(\bar{\mathcal{H}}) \le \varpi_0(\mathcal{H}). \tag{7}$$

This completes the proof of (A3). Because the norm has a triangular property, one can check (A4) quickly. In the following, the correctness of (A5) is verified. Suppose that $\{g_n\}$ is a sequence formed of closed sets in $\Lambda_{M_p^\vartheta}(\Omega)$, so that $g_{n+1} \subseteq g_n$ and $\lim_{n\to\infty} \varpi_0(\mathcal{H}_n) = 0$. For any $n \in \mathbb{N}$, select $g_n \in \mathcal{H}_n$. Now, it needs to be shown that g_n converges to some g_0. Suppose that one defines $H = \{g_n\}$. Then, it is necessary to prove that H is compact. Hence, it must be proven that H is equicontinuous and bounded. In the following, it is known that $\lim\limits_{n\to\infty} \varpi_0(g_n) = 0$, so for any $\epsilon > 0$, there exists a $L > 0$ such that, for $m > L$, $\varpi_0(g_m) < \epsilon$. Then, $\|T_d g_m - g_m\|_p < \epsilon$. This implies equicontinuity of $\{g_m\}_{m=L}^\infty$. By applying Lemma 1, it can be concluded that $\{g_m\}_{m=L}^\infty$ is compact, and thus, $\{g_i\}_{i=1}^L$ is also compact. □

4. Applications within Generalized Morrey Spaces

This section illustrates the practical usefulness of the newly introduced measure of non-compactness. An example is carefully chosen to show how the new measure can be effectively applied to solve complex problems within generalized Morrey spaces.

We examine the system (1) within the framework of generalized Morrey spaces and demonstrate that if system (1) satisfies the conditions

A. Function $J_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous and function $f_i : \mathbb{R}^4 \longrightarrow \mathbb{R}$ is measurable. Therefore, positive constants λ_1, λ_2 and λ_3 exist, such that $0 < \lambda_1 + \lambda_2 < 1$, and

$$\begin{aligned}
&|f_i(\xi_1, u_1, v_1, z_1) - f_i(\xi_2, u_2, v_2, z_2)| \le |J_i(\xi_1) - J_i(\xi_2)| + \lambda_1 |u_1(\xi) - u_2(\xi)| + \\
&\lambda_2 |v_1(\xi) - v_2(\xi)| + \lambda_3 |z_1(\xi) - z_2(\xi)|,
\end{aligned} \tag{8}$$

where $i \in \{1, 2\}$ and $\lambda_3 \in \mathbb{R}_+$.

B. The function $b_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is bounded and continuous, with a positive constant M such that $|b_i(\xi, u, v)| \leq M$.

C. $f_{i,0}(\xi) := f_i(\xi, 0, 0, 0)$ belongs to $M_p^{\vartheta}(\Omega) \cap L^p(\Omega)$.

Then, the system (1) has at least one solution.

Theorem 7. *Under conditions A–C, the system (1) has at least one solution in $M_p^{\vartheta}(\Omega)$.*

Proof. We start with

$$||u||_p^{\vartheta} \equiv \sup_{\substack{u\in\Omega,\\ 0<r<\text{diam}(\Omega)}} ||\varpi_{(.)}||u||_{L^p(B(u,.))}||_{L^{\infty}(0,r)} < r_0,$$

$$||v||_p^{\vartheta} \equiv \sup_{\substack{u\in\Omega,\\ 0<r<\text{diam}(\Omega)}} ||\varpi_{(.)}||v||_{L^p(B(v,.))}||_{L^{\infty}(0,r)} < r_0,$$

and

$$F_i(u(\xi), v(\xi)) = f_i\Big(\xi, u(\xi), v(\xi), \int_1^{\xi} (\ln \frac{\xi}{\eta})^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta\Big). \tag{9}$$

According to the Cauchy–Schwartz inequality, we obtain

$$|F_i(u,v)(\xi)| \leq |f_i(\xi, u(\xi), v(\xi)), \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta) - f_i(\xi,0,0,0) + f_i(\xi,0,0,0)| \tag{10}$$

$$\leq \lambda_1 |u(\xi)| + \lambda_2 |v(\xi)| + \lambda_3 M (T-1)^{\frac{1}{2}} \frac{1}{2\alpha - 1} (\ln(T))^{2\alpha-1}. \tag{11}$$

By the Lebesgue-dominated convergence theorem, we have

$$||F_i(u,v)(\xi)||_p^{\vartheta} \leq \lambda_1 ||u(\xi)||_p^{\vartheta} + \lambda_2 ||v(\xi)||_p^{\vartheta} + \lambda_3 M (T-1)^{\frac{1}{2}} \frac{1}{2\alpha-1} (\ln(T))^{2\alpha-1}. \tag{12}$$

We choose r_0 such that

$$\frac{\lambda_3 M (T-1)^{\frac{1}{2}} (\ln(T))^{2\alpha-1}}{(2\alpha-1)(1-\lambda_1-\lambda_2)} < r_0, \tag{13}$$

Therefore, we prove that F_i is a self-map. We show that $u_n \to u$ and $v_n \to v$ in B_{r_0} with norm $M_p^{\vartheta}(\Omega)$. So, we show that $F_i(u_n, v_n) \to F_i(u, v)$ for $i \in 1, 2$. By using A and C, we get

$$\begin{aligned}
&|f_i(\xi, u_n(\xi), v_n(\xi)), \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_i(\eta, u_n(\eta), v_n(\eta))}{\eta} d\eta) \\
&\quad - f_i(\xi, u(\xi), v(\xi)), \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta)| \\
&\leq \lambda_1 |u_n(\xi) - u(\xi)| + \lambda_2 |v_n(\xi) - v(\xi)| \\
&\quad + \lambda_3 \left| \int_1^{\xi} \left(\ln\left(\frac{\xi}{\eta}\right)\right)^{\alpha-1} \frac{b_i(\eta, u_n(\eta), v_n(\eta)) - b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta \right|.
\end{aligned} \tag{14}$$

By the Lebesgue-dominated convergence theorem, we have

$$
\begin{aligned}
&||f_i(\xi, u_n(\xi), v_n(\xi)), \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_i(\eta, u_n(\eta), v_n(\eta))}{\eta} d\eta) \\
&\quad - f_i(\xi, u(\xi), v(\xi)), \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta)||_p^{\vartheta} \\
&\leq \lambda_1 ||u_n(\xi) - u(\xi)||_p^{\vartheta} + \lambda_2 ||v_n(\xi) - v(\xi)||_p^{\vartheta} \\
&\quad + \lambda_3 \left|\left| \int_1^{\xi} \left(\ln\left(\frac{\xi}{\eta}\right)\right)^{\alpha-1} \frac{b_i(\eta, u_n(\eta), v_n(\eta)) - b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta \right|\right|_p^{\vartheta}.
\end{aligned}
\tag{15}
$$

Then $||F_i(u_n, v_n) - F_i(u, v)||_p^{\vartheta} \to 0$ as $n \to \infty$, confirming that F_i is continuous in the local generalized Morrey space.

Now, we show that, for the nonempty subsets $X_1, X_2 \in B_{r_0}$, we have

$$
\varpi(F_i(X_1, X_2)) \leq C_1 \max \varpi(X_1, X_2), \tag{16}
$$

with $C_1 := \max\{\lambda_1, \lambda_2\}$. We choose $u \in X_1$ and $v \in X_2$. For an arbitrary $\varepsilon > 0$ and $\xi, d \in \Omega$ with $|d| < \varepsilon$, we have:

$$
\begin{aligned}
&|F_i(u, v)(\xi + d) - F_i(u, v)(\xi)| = \\
&|f_i(\xi + d, u(\xi + d), v(\xi + d), \int_1^{\xi+d} (\ln(\frac{\xi + d}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta) \\
&- f_i(\xi, u(\xi), v(\xi), \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta)| \\
&< |J_i(\xi + d) - J_i(d)| + \lambda_1 |u(\xi + d) - u(\xi)| + \lambda_2 |v(\xi + d) - v(\xi)| \\
&+ \lambda_3 | \int_1^{\xi+d} (\ln(\frac{\xi + d}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta - \ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta|.
\end{aligned}
\tag{17}
$$

By the Lebesgue-dominated convergence theorem, we have

$$
\begin{aligned}
&||F_i(u, v)(\xi + d) - F_i(u, v)(\xi)||_p^{\vartheta} = \\
&< ||J_i(\xi + d) - J_i(d)||_p^{\vartheta} + \lambda_1 ||u(\xi + d) - u(\xi)||_p^{\vartheta} + \lambda_2 ||v(\xi + d) - v(\xi)||_p^{\vartheta} \\
&+ \lambda_3 || \int_1^{\xi+d} (\ln(\frac{\xi + d}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta - \ln(\frac{\xi}{\eta}))^{\alpha-1} \frac{b_i(\eta, u(\eta), v(\eta))}{\eta} d\eta||_p^{\vartheta}.
\end{aligned}
\tag{18}
$$

Applying the Riesz–Fischer and the Lebesgue-dominated convergence theorems, the proof is completed. □

Remark 1. *In a similar manner, the investigation of both finite and infinite systems of fractional integral equations within the context of generalized Morrey spaces constitutes an important area of research. For an in-depth analysis of this subject, readers are encouraged to consult the thorough examination found in reference [17].*

Example 3. *Here, we present an example that illustrates the effectiveness and applicability of Theorem 7. For every positive constant T such that $\xi \in [0, T]$, we consider a system of fractional differential equations:*

$$
\begin{cases}
u(\xi) = e^{-\xi} + \dfrac{u(\xi)}{e^{\xi} + 3} + \dfrac{v(\xi)}{\xi^2 + 3} + \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\frac{-1}{6}} \dfrac{\arctan(u(\eta) + v(\eta))}{\eta} d\eta, \\
v(\xi) = \dfrac{1}{1 + \cosh(\xi)} + \dfrac{u(\xi)}{20 + \cos(\xi)} + \dfrac{v(\xi)}{\pi + \cos^2(\xi)} + \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{\frac{-1}{6}} \dfrac{\sin(u(\eta) + v(\eta))}{\eta} d\eta.
\end{cases}
\tag{19}
$$

According to system (1) *and Equation* (9), *we obtain*

$$\begin{cases} f_1(\xi,u,v,z) = e^{-\xi} + \dfrac{u}{e^{\xi}+3} + \dfrac{v}{\xi^2+3} + z, \\ f_2(\xi,u,v,z) = \dfrac{1}{1+\cosh(\xi)} + \dfrac{u}{20+\cos(\xi)} + \dfrac{v}{\pi+\cos^2(\xi)} + z. \end{cases}$$

We set $\vartheta = r^{\frac{-1}{2}}$, *and* $\alpha = \dfrac{1}{2}$. *Since* $f_1(\xi,0,0,0) = e^{-\xi}$ *and* $f_2(\xi,0,0,0) = \dfrac{1}{1+\cosh(\xi)}$, *then* $||J_1||_p^{\vartheta} \leq \sqrt{2}$, *and* $||J_2||_p^{\vartheta} \leq \dfrac{1}{\sqrt{2}}$. *Furthermore,*

$$\lambda_1 = \max\left\{ \sup_{0\leq\xi\leq T}\left\{\frac{1}{e^{\xi}+3}\right\}, \sup_{0\leq\xi\leq T}\left\{\frac{1}{\cos(\xi)+20}\right\}\right\} = \frac{1}{3},$$
$$\lambda_2 = \max\left\{ \sup_{0\leq\xi\leq T}\left\{\frac{1}{\xi^2+3}\right\}, \sup_{0\leq\xi\leq T}\left\{\frac{1}{\cos^2(\xi)+\pi}\right\}\right\} = \frac{1}{3},$$
$$\lambda_3 = 1.$$

Hence, conditions $A - C$ *are satisfied. Then, for each* $p \geq 1$, *the system* (19) *has a solution in* M_p^{ϑ}.

Example 4. *We demonstrate that the fractional integral system* (20) *possesses a solution within the space* M_p^{ϑ}:

$$\begin{cases} u(\xi) = \arctan(\xi^2+1) + \dfrac{\cos(v(\xi))}{|\sinh(\xi)|+15} + \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{-\frac{1}{6}} \dfrac{\cos(v(\eta))}{\eta} d\eta, \\ v(\xi) = \tanh(\xi) + \dfrac{\arctan(u(\xi)+v(\xi))}{\xi^5+20} + \int_1^{\xi} (\ln(\frac{\xi}{\eta}))^{-\frac{1}{6}} \dfrac{\mathrm{sech}(u(\eta))}{\eta} d\eta. \end{cases} \tag{20}$$

According to system (1) *and Equation* (9), *we obtain*

$$\begin{cases} f_1(\xi,u,v,z) = \arctan(\xi^2+1) + \dfrac{\cos(v)}{|\sinh(\xi)|+15} + u, \\ f_2(\xi,u,v,z) = \tanh(\xi) + \dfrac{\arctan(u+v)}{\xi^5+20} + u. \end{cases}$$

We set $\varpi(r) = r^{\frac{-1}{3}}$, *and* $\alpha = \dfrac{1}{3}$. *Since* $f_1(\xi,0,0,0) = \arctan(\xi^2+1)$ *and* $f_2(\xi,0,0,0) = \tanh(\xi)$, *then* $||J_1||_p^{\vartheta} \leq \frac{\sqrt[3]{2}}{8}\pi^2$, *and* $||J_2||_p^{\vartheta} \leq \sqrt[3]{2}$. *Additionally,*

$$\lambda_1 = \max\left\{0, \sup_{0\leq\xi\leq T}\left\{\frac{1}{\xi^2+20}\right\}\right\} = \frac{1}{20},$$
$$\lambda_2 = \max\left\{ \sup_{0\leq\xi\leq T}\left\{\frac{1}{|\sinh(\xi)|+15}\right\}, \sup_{0\leq\xi\leq T}\left\{\frac{1}{\xi^2+20}\right\}\right\} = \frac{1}{15},$$
$$\lambda_3 = 1.$$

Hence, conditions A–C are satisfied. Then, for each $p \geq 1$, *the system* (20) *has a solution in* M_p^{ϑ}.

5. Concluding Remarks

This paper introduced a novel measure of non-compactness within a bounded domain in $\mathbb{R}^N$ for generalized Morrey spaces. This measure was utilized to demonstrate the existence of solutions for a coupled Hadamard fractional-order system of integral equations in these spaces. Examples were provided to illustrate the application of the main result.

The paper also suggested future research directions, including the study of infinite systems of integral equations in Morrey spaces and extending these results to other domains, such as Morrey spaces on $\mathbb{R}^2$.

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