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TILT STABILITY IN NONLINEAR PROGRAMMING UNDER MANGASARIAN–FROMOVITZ CONSTRAINT QUALIFICATION

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The paper concerns the study of tilt stability of local minimizers in standard problems of nonlinear programming. This notion plays an important role in both theoretical and numerical aspects of optimization and has drawn a lot of attention in optimization theory and its applications, especially in recent years. Under the classical Mangasarian–Fromovitz Constraint Qualification, we establish relationships between tilt stability and some other stability notions in constrained optimization. Involving further the well-known Constant Rank Constraint Qualification, we derive new necessary and sufficient conditions for tilt-stable local minimizers.

Keywords: variational analysis, second-order theory, generalized differentiation, tilt stability

Classification: 49J52, 90C30, 90C31

1. INTRODUCTION

It has been well recognized in modern variational analysis that appropriate stability concepts play a fundamental role in almost all areas of optimization theory and its applications; see, e. g., the books [2, 5, 7, 15, 22] and the references therein. Among the most important and attractive stability notions are those related to *Lipschitzian stability* that provide not only qualitative but also quantitative amount of information needed, in particular, for the justification of numerical algorithms.

In this paper we focus on the study of *tilt stability* of local minimizers introduced by Poliquin and Rockafellar [28] in the general extended-real-valued framework of unconstrained optimization. This remarkable type of Lipschitzian stability, primarily motivated by supporting computational work, prevents the disproportional change of a local minimizer or threatening its uniqueness under tilt perturbations of the objective; see Section 2 below for the precise definition. The main result of [28] provides a very impressive characterization of tilt-stable local minimizers for a broad class of extended-real-valued objectives in terms of positive-definiteness of the *second-order subdifferential/generalized*

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Hessian in the sense of Mordukhovich [21]. Furthermore, by developing new second-order subdifferential calculus rules and computing the aforementioned constructions in the settings of interest (cf. also [9, 22, 23, 24]), Mordukhovich and Rockafellar [25] have recently obtained efficient characterizations of tilt-stable minimizers for some special classes of problems in constrained optimization entirely in terms of their initial data. Among them are usual C^2 problems of nonlinear programming (NLP) with equality and inequality constraints, where tilt stability is fully characterized by [25, Theorem 5.2] via the classical *Strong Second-Order Sufficient Condition* (SSOSC), provided the validity of the *Linear Independence Constraint Qualification* (LICQ). As a consequence of this result and previously known developments in nonlinear programming, tilt stability of local minimizers for NLP has proved to be equivalent [25, Corollary 5.3] to Robinson's *strong regularity* [31] of the associated Karush–Kuhn–Tucker (KKT) system – again under the validity of LICQ.

In the other lines of developments, Lewis and Zhang [18, Theorem 6.3 and Proposition 7.2] and Drusvyatskiy and Lewis [6, Theorem 3.3][1] have recently characterized tilt stability of local minimizers for general classes of extended-real-valued functions in terms of strong metric regularity of the limiting subdifferential mappings as well as via a certain uniform quadratic growth condition imposed on the objective. The latter characterization is similar to that in Bonnans and Shapiro [2, Theorem 5.36] obtained for a particular class of problems; cf. also Aragón Artacho and Geoffroy [1, Corollary 3.9 and Theorem 3.10] for the case of convex objective functions.

This paper concerns the study of tilt-stable local minimizers for problems of nonlinear programming with C^2 data. Although our major results hold in the general NLP setting with both inequality and equality constraints of the C^2 type, we primarily focus on nonlinear programs with only inequality constraints, which reflect the essence of our new developments. Among the main motivations for our study was the desire to relax the rather restrictive LICQ requirement in the aforementioned characterization of tilt stability established in [25]. In this way we first obtain, imposing only MFCQ, a characterization of tilt-stable local minimizers via strong metric regularity of a certain set-valued mapping associated with the first-order stationarity conditions for the NLP under consideration with no explicit Lagrange multipliers, i. e., not in terms of the corresponding Karush–Kuhn–Tucker (KKT) system. Then imposing in addition the Constant Rank Constraint Qualification (CRCQ) allows us to derive necessary and sufficient conditions for tilt-stable minimizers formulated entirely via the initial program data. The results obtained in this paper significantly extend, in particular, those derived in [25] for C^2 nonlinear program satisfying LICQ at the reference optimal solutions.

It is worth mentioning that, although we consider NLP only with twice continuously differentiable data and the corresponding stationarity conditions do not include anything but classical normals to the simplest convex cones, the main variational machinery of this paper is based on *nonconvex* tools of generalized differentiation briefly discussed in the next section together with some other preliminaries.

The rest of the paper is organized as follows. Section 2 contains some basic definitions and preliminaries from variational analysis widely used in the paper. We also give here

[1] We got familiar with the latter preprint when this paper was basically completed. The main results of [6, 18] are complement to those presented here.

an implementation of the aforementioned generalized Hessian characterization of tilt stability from [28] for the case of NLP models, which is the starting point of our study. Section 3 is devoted to deriving our main results on necessary and sufficient conditions for tilt stability of local minimizers in NLP under various constraint qualifications in terms of the problem data. Section 4 presents two examples illustrating the results obtained and relationships between them. Concluding remarks are given in Section 5, where we also discuss some topics of our future research in this direction.

2. BASIC CONSTRUCTIONS AND PRELIMINARIES FROM VARIATIONAL ANALYSIS AND TILT STABILITY

The main object under consideration in this paper is the following nonlinear program:

$$\begin{array}{ll} \text{minimize} & f_0(x) \ \text{ with } \ x \in \mathbb{R}^n \\ \text{subject to} & q_i(x) \le 0 \ \text{ for } \ i = 1, \ldots, s, \end{array} \tag{2.1}$$

where all the functions $f_0, q_i \colon \mathbb{R}^n \to \mathbb{R}$ are assumed to be twice continuously differentiable (C^2) around the points in question. Denote the set of *feasible solutions* to (2.1) by

$$\Gamma := \big\{ x \in \mathbb{R}^n \big|\ q_i(x) \le 0, \quad i = 1, \ldots, s \big\}. \tag{2.2}$$

The original NLP problem (2.1) can be obviously rewritten in the unconstrained form:

$$\text{minimize} \ \ f(x) := f_0(x) + \delta_\Gamma(x), \quad x \in \mathbb{R}^n, \tag{2.3}$$

where $\delta_\Gamma(x)$ is the indicator function of the set Γ, which is equal to 0 for $x \in \Gamma$ and to ∞ for $x \notin \Gamma$. The fundamental stability notion studied below is formulated for arbitrary extended-real-valued functions $f \colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ as follows [28].

Definition 2.1. (tilt stability) A point $\bar{x} \in \mathbb{R}^n$ is said to give a TILT-STABLE local minimum of the function $f \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ if the value $f(\bar{x})$ is finite and there exists a number $\varrho > 0$ such that the mapping

$$M \colon p \mapsto \operatorname*{argmin}_{\|x - \bar{x}\| \le \varrho} \big[f(x) - f(\bar{x}) - \langle p, x - \bar{x} \rangle \big]$$

is single-valued and Lipschitzian on some neighborhood of $p = 0$ with $M(0) = \{\bar{x}\}$.

Applying Definition 2.1 to the function f in (2.3), we clearly specify that $\bar{x} \in \Gamma$. Denoting further $q(x) := (q_1(x), \ldots, q_s(x))$, recall that the classical *Mangasarian–Fromovitz Constraint Qualification* (*MFCQ*) at $\bar{x} \in \Gamma$ for problem (2.1) amounts to the implication

$$\left. \begin{array}{l} \big(\nabla q(\bar{x})\big)^T \lambda = 0 \\ \lambda \in N_{\mathbb{R}^s_-}\big(q(\bar{x})\big) \end{array} \right\} \Longrightarrow \lambda = 0, \tag{2.4}$$

where "T" indicates the matrix transposition, and where $N_{\mathbb{R}^s_-}$ stands for the standard normal cone of convex analysis. It is easy to verify that the validity of MFCQ (2.4) ensures that the feasible set Γ in (2.2) is *fully amenable* at $\bar{x}$ in the sense of [33, Definition 10.23], and hence the indicator function δ_Γ is *prox-regular* and *subdifferentially* *continuous* on $\mathcal{O}\cap\Gamma$, where $\mathcal{O}$ is a neighborhood of $\bar{x}$; see [33, Proposition 13.32]. The latter two properties are the major assumptions needed for the characterization of tilt-stable local minimizers of a general function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ in [28, Theorem 1.3]. This allows us to employ the aforementioned characterization to our NLP setting (2.3) under the standing MFCQ (2.4); see Theorem 2.2 and its proof below for more details.

To proceed with adopting the main result of [28] in the framework of our problem (2.3), we need some generalized differential constructions of variational analysis used throughout the whole paper. Recall first the (Painlevé–Kuratowski) *outer limit* of a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ as $x \to \bar{x}$ defined by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \big\{v\in\mathbb{R}^m\big|\ \exists\, x_k\to\bar{x},\ v_k\to v\ \text{ with }\ v_k\in F(x_k)\ \text{ as }\ k\in\mathbb{N}\big\}, \tag{2.5}$$

where $\mathbb{N} := \{1,2,\ldots\}$. Given further a nonempty set $A \in \mathbb{R}^n$, the (Bouligand–Severi) *contingent cone* to A at $\bar{x}\in A$ is

$$T_A(\bar{x}) := \operatorname*{Lim\,sup}_{t\downarrow 0}\frac{A-\bar{x}}{t} = \big\{d\in\mathbb{R}^m\big|\ \exists\, t_k\downarrow 0,\ d_k\to d\ \text{ with }\ \bar{x}+t_kd_k\in A,\ k\in\mathbb{N}\big\}. \tag{2.6}$$

The (Fréchet) *regular normal cone* to A at $\bar{x}\in A$ is given equivalently by

$$\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^0 = \Big\{v\in\mathbb{R}^n\Big|\ \limsup_{x\xrightarrow{A}\bar{x}}\frac{\langle v,x-\bar{x}\rangle}{\|x-\bar{x}\|}\le 0\Big\}, \tag{2.7}$$

where T^0 stands for the negative polar of the cone T, and the symbol $x \xrightarrow{A} \bar{x}$ signifies that $x\to\bar{x}$ with $x\in A$. The (Mordukhovich) *limiting normal cone* to A at $\bar{x}\in A$ can be equivalently defined via the outer limit (2.5) by

$$N_A(\bar{x}) := \operatorname*{Lim\,sup}_{x\xrightarrow{A}\bar{x}}\widehat{N}_A(x) = \operatorname*{Lim\,sup}_{x\to\bar{x}}\big\{\operatorname{cone}\big[x-\Pi_A(x)\big]\big\} \tag{2.8}$$

in the case of locally closed sets A in the second representation, where Π_A stands for the Euclidean projector onto the set A, and where the symbol "cone" indicates the (nonconvex) conic hull of the set in question. Invoking our basic normal cone construction (2.8), we define the (first-order) *subdifferential* $\partial f(\bar{x})$ of f at $\bar{x}$ for an extended-real-valued function $f\colon\mathbb{R}^n\to\overline{\mathbb{R}}$, finite at $\bar{x}$, and the *coderivative* $D^*\Phi(\bar{x},\bar{y})$ of a set-valued mapping $\Phi\colon\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ at $(\bar{x},\bar{y})\in\operatorname{gph}\Phi := \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m|\ y\in\Phi(x)\}$ by, respectively,

$$\partial f(\bar{x}) := \big\{v\in\mathbb{R}^n\big|\ (v,-1)\in N_{\operatorname{epi} f}\big(\bar{x},f(\bar{x})\big)\big\}, \tag{2.9}$$

$$D^*\Phi(\bar{x},\bar{y})(u) := \big\{v\in\mathbb{R}^n\big|\ (v,-u)\in N_{\operatorname{gph}\Phi}(\bar{x},\bar{y})\big\},\quad u\in\mathbb{R}^m, \tag{2.10}$$

with $\operatorname{epi} f := \{(x,\mu)\in\mathbb{R}^n\times\mathbb{R}|\ \mu\ge f(x)\}$ in (2.9). We always omit $\bar{y}=\Phi(\bar{x})$ in the coderivative notation when Φ is single-valued. Note that $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$ if f is smooth near $\bar{x}$ while the coderivative (2.10) reduces to the adjoint (transposed) Jacobian operator

$$D^*\Phi(\bar{x})(u) = \big\{\nabla\Phi(\bar{x})^T u\big\}\ \text{ for all }\ u\in I\!\!R^m$$

for smooth single-valued mappings. However, even in the simplest cases of nonconvex functions f and nonsmooth mappings Φ, the subdifferential (2.9) and the coderivative (2.10) *cannot* be dual/adjoint to any derivative-like constructions in primal spaces due to the intrinsic *nonconvexity* of the generated limiting normal cone (2.8); see, e. g., the case of $f = -|x|$ and $\Phi(x) = |x|$ at $\bar{x} = 0 \in \mathbb{R}$, respectively. Nevertheless, the normal cone (2.8) and the corresponding subdifferential and coderivative constructions (2.9) and (2.10) are robust and enjoy *full calculi* in the general frameworks due to the *extremal/variational principles* and related techniques of variational analysis. We refer the reader to the books [22, 33] and the bibliographies therein for comprehensive developments.

The main result of [28] characterizes tilt-stability of local minimizers of $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ from Definition 2.1 via the *positive-definiteness* of the *second-order subdifferential* (or *generalized Hessian*) of f at $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$ defined in [21] by

$$\partial^2 f(\bar{x}, \bar{v})(w) := (D^* \partial f)(\bar{x}, \bar{v})(w), \quad w \in \mathbb{R}^n, \tag{2.11}$$

which reduces to the classical (symmetric) Hessian

$$\partial^2 f(\bar{x})(w) = \big\{\nabla^2 f(\bar{x}) w\big\}, \quad w \in \mathbb{R}^n, \tag{2.12}$$

when $f \in C^2$ around $\bar{x}$; see [22, Proposition 1.119]. If f in (2.11) is the indicator function δ_Γ of the set Γ, then we have

$$\partial^2 \delta(\bar{x}, \bar{v})(w) = D^* N_\Gamma(\bar{x}, \bar{v})(w), \quad w \in \mathbb{R}^n, \tag{2.13}$$

whenever $\bar{x} \in \Gamma$ and $\bar{v} \in N_\Gamma(\bar{x})$. It is more convenient for us to use in this paper the "coderivative-of-the-normal-cone" expression (2.13) for the generalized Hessian of the set indicator function instead of the second-order subdifferential notation (2.11).

Now we are ready to present an implementation of the tilt-stability characterization from [28, Theorem 1.3] for the case of nonlinear program (2.1) with C^2 data.

Theorem 2.2. (tilt stability in NLP via generalized Hessians) Let $\bar{x} \in \Gamma$ be a feasible solution to (2.1) such that MFCQ (2.4) holds at $\bar{x}$ and that

$$0 \in \nabla f_0(\bar{x}) + N_\Gamma(\bar{x}). \tag{2.14}$$

Then $\bar{x}$ is a tilt-stable local minimizer of (2.1) if and only if

$$\langle w, \nabla^2 f_0(\bar{x}) w\rangle > -\langle z, w\rangle \ \text{ whenever } \ z \in D^* N_\Gamma\big(\bar{x}, -\nabla f_0(\bar{x})\big)(w),\ w \neq 0. \tag{2.15}$$

Proof. As mentioned above, the validity of MFCQ ensures that the indicator function of the feasible set Γ is prox-regular and subdifferentially continuous at $\bar{x} \in \Gamma$. It is well known that these properties are preserved when a C^2 function is added. Thus applying [28, Theorem 1.3] to the function f defined in (2.3) tells us that the point $\bar{x}$ having $0 \in \partial f(\bar{x})$ is a tilt-stable local minimizer for (2.3) if and only if

$$\langle z, w\rangle > 0 \ \text{ whenever } \ z \in \partial^2 f(\bar{x}, 0)(w),\ w \neq 0. \tag{2.16}$$

Since f_0 is smooth in (2.3), it follows from the elementary first-order subdifferential sum rule (see, e. g., [22, Proposition 1.107]) that the stationarity condition $0 \in \partial f(\bar{x})$ is equivalent to (2.14). Employing now in (2.3) the second-order subdifferential sum rule from [22, Proposition 1.121] with taking relationships (2.12)–(2.14) into account, we get

$$\partial^2(f_0 + \delta_\Gamma)(\bar{x}, 0)(w) = \nabla^2 f_0(\bar{x})w + D^* N_\Gamma\big(\bar{x}, -\nabla f_0(\bar{x})\big)(w), \quad w \in \mathbb{R}^n,$$

which justifies the equivalence between the second-order conditions (2.16) and (2.15) and thus completes the proof of the theorem. □

Note that characterization (2.15) of tilt-stable minimizers is given not in terms of the initial data of the NLP (2.1) under consideration while involving the generalized Hessian of the indicator function to the feasible set Γ. This is just our *starting point* to derive verifiable necessary and sufficient conditions for tilt stability in Section 3. Meantime we observe that due to the robustness of MFCQ there is a neighborhood $\mathcal{O}$ of $\bar{x}$ such that the first-order equality chain rule

$$N_\Gamma(x) = \big(\nabla q(x)\big)^T N_{\mathbb{R}^s_-}\big(q(x)\big) \tag{2.17}$$

holds for all $x \in \mathcal{O}$; see, e. g., [33]. Thus the stationarity condition

$$0 \in \nabla f_0(x) + N_\Gamma(x)$$

is locally equivalent to the *generalized equation* (GE)

$$0 \in \nabla f_0(x) + \big(\nabla q(x)\big)^T N_{\mathbb{R}^s_-}\big(q(x)\big). \tag{2.18}$$

By the straightforward description of the normal cone

$$\lambda \in N_{\mathbb{R}^s_-}\big(q(x)\big) \Longleftrightarrow q(x) \le 0,\ \lambda \ge 0,\ \langle \lambda, q(x)\rangle = 0,$$

the introduced GE (2.18) amounts to the standard *KKT system*

$$\begin{aligned} &0 = \nabla_x \mathcal{L}(x, \lambda),\\ &q(x) \le 0,\ \lambda \ge 0,\ \langle \lambda, q(x)\rangle = 0, \end{aligned} \tag{2.19}$$

where $\lambda \in \mathbb{R}^s$ is the corresponding *Lagrange multiplier*, and where

$$\mathcal{L}(x, \lambda) := f_0(x) + \langle \lambda, q(x)\rangle, \quad x \in \mathbb{R}^n,$$

is the *Lagrangian* associated with program (2.1).

Consider further the *active index set* given by

$$I(\bar{x}) := \big\{ i \in 1, \ldots, s \big|\ q_i(\bar{x}) = 0 \big\}$$

and recall that the *Constant Rank Constraint Qualification* (*CRCQ*) holds at $\bar{x}$ if there is a neighborhood $\mathcal{W}$ of $\bar{x}$ such that the system of gradient vectors $\{\nabla q_i(x) |\ i \in J\}$ has constant rank in $\mathcal{W}$ for any index set $J \subset I(\bar{x})$. Note that CRCQ at $\bar{x}$ neither implies nor is implied by MFCQ at $\bar{x}$. Being introduced a long time ago [13], CRCQ and its variants have recently gained a lot of attention in nonlinear optimization; see, e. g., [19, 20].

We conclude this section with recalling an abstract version of *Robinson's strong regularity* [31] formulated in [3] and used in what follows.

Definition 2.3. (strong metric regularity) A set-valued mapping $\Phi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called strongly metrically regular at $(\bar{x}, \bar{y}) \in \operatorname{gph} \Phi$ if its inverse mapping Φ^{-1} has a Lipschitzian single-valued localization around $(\bar{x}, \bar{y})$, i. e., there are neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ and a single-valued Lipschitz continuous mapping $\varphi\colon \mathcal{V} \to \mathcal{U}$ such that we have $\varphi(\bar{y}) = \bar{x}$ and

$$\Phi^{-1}(y) \cap \mathcal{U} = \big\{\varphi(y)\big\} \text{ for all } y \in \mathcal{V}.$$

3. NECESSARY AND SUFFICIENT CONDITIONS FOR TILT-STABLE MINIMIZERS IN NLP

We start this section with characterizing tilt-stable local minimizers of (2.1) via the strong metric regularity of a set-valued mapping associated with the generalized equation (2.18) under the validity of MFCQ. Denote

$$\Psi(x) := \nabla f_0(x) + \big(\nabla q(x)\big)^T N_{\mathbb{R}^s_-}\big(q(x)\big), \quad x \in \mathbb{R}^n, \tag{3.1}$$

which is the right-hand side of the GE in (2.18). We first present the following lemma of its own interest that is sorted out from [4, Lemma 2.5] while the underlying result goes back to [14, Theorem 1] and [32, Theorem 4.3]. Recall that a local minimizer $\bar{x}$ of f is *isolated* if there are no other local minimizers of f in some neighborhood of $\bar{x}$.

Lemma 3.1. (isolated local minimizers under MFCQ) Let $\bar{x}$ be an isolated local minimizer of (2.1) under the validity of MFCQ at $\bar{x}$. Then the mapping $X\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, which assigns to each $p \in \mathbb{R}^n$ the set of optimal solutions to the perturbed program

$$\begin{array}{ll} \text{minimize} & f_0(x) - \langle p, x - \bar{x}\rangle \\ \text{subject to} & q(x) \le 0, \end{array}$$

enjoys the following property: For every neighborhood $\mathcal{M}$ of $\bar{x}$ there is a neighborhood $\mathcal{N}$ of $0_{\mathbb{R}_n}$ with

$$X(p) \cap \mathcal{M} \neq \emptyset \text{ whenever } p \in \mathcal{N}.$$

Now we are ready to derive the aforementioned characterization of tilt stability in (2.1) via the strong metric regularity of mapping (3.1), with no explicit involvement of Lagrange multipliers.

Theorem 3.2. (tilt stability versus strong metric regularity of GE under MFCQ) Let $\bar{x}$ be a local minimizer in (2.1) satisfying MFCQ. Then $\bar{x}$ is a tilt-stable local minimizer of (2.1) if and only if the mapping Ψ in (3.1) is strongly metrically regular at $(\bar{x}, 0)$.

Proof. We first justify the *sufficiency* of strong metric regularity of Ψ at $(\bar{x}, 0)$ for $\bar{x}$ to be a tilt-stable local minimizer in (2.1). To proceed in this way, observe that

$$M(p) \cap W = X(p) \cap W \text{ whenever } W \subset \operatorname{int} \mathbb{B}(\bar{x}; \rho), \quad p \in \mathbb{R}^n, \tag{3.2}$$

where the mappings M and X are taken from Definition 2.1 and Lemma 3.1, respectively. Choose $\rho > 0$ so small that $\mathbb{B}(\bar{x}; \rho) \subset \mathcal{O}$ with the neighborhood $\mathcal{O}$ of $\bar{x}$ coming from

(2.17) due to robustness of MFCQ. It follows by the construction of Ψ that each solution $x \in X(p) \cap \mathbb{B}(\bar{x};\gamma)$ with $\gamma \in (0,\rho)$ satisfies the relation $p \in \Psi(x)$. Furthermore, the strong metric regularity of Ψ at $(\bar{x},0)$ ensures the existence of neighborhoods $\mathcal{U}$ of 0 and $\mathcal{V}$ of $\bar{x}$ and a Lipschitzian single-valued mapping $\sigma\colon \mathcal{U} \to \mathcal{V}$ such that $\sigma(0) = \bar{x}$ and

$$\Psi^{-1}(p) \cap \mathcal{V} = \big\{\sigma(p)\big\} \ \text{ for all } \ p \in \mathcal{U}.$$

Shrinking these neighborhoods if necessary so that $\mathcal{V} \subset \mathbb{B}(\bar{x};\gamma)$, we get

$$X(p) \cap \mathcal{V} \subset \Psi^{-1}(p) \cap \mathcal{V} = \big\{\sigma(p)\big\} \ \text{ for all } \ p \in \mathcal{U}.$$

This gives us $X(0) \cap \mathcal{V} = \{\bar{x}\}$, and hence $\bar{x}$ is an isolated local minimizer of (2.1). We can thus apply Lemma 3.1 and, by taking (3.2) into account, conclude that there is a neighborhood $\widetilde{\mathcal{U}} \subset \mathcal{U}$ of $\bar{x}$ such that

$$M(p) \cap \mathcal{V} = X(p) \cap \mathcal{V} = \big\{\sigma(p)\big\} \ \text{ for all } \ p \in \widetilde{\mathcal{U}}.$$

This shows that $\bar{x}$ is a tilt-stable minimizer in (2.1).

To justify the *converse* implication in the theorem, recall that the tilt stability of a local minimizer $\bar{x}$ for a lower semicontinuous, prox-regular, and subdifferentially continuous function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ yields the strong metric regularity of the subgradient mapping ∂f at $(\bar{x},0)$ by [18, Proposition 7.2] and [6, Theorem 3.3]. In the case of our function f from (2.3) the aforementioned assumptions on f are satisfied due to the imposed MFCQ at $\bar{x}$. Furthermore, applying in (2.3) the first-order subdifferential sum rule from [22, Proposition 1.107] with the smooth function f_0 gives us

$$\partial f(x) = \nabla f_0(x) + \partial\delta_\Gamma(x) = \Psi(x) \ \text{ for all } \ x \ \text{ near } \ \bar{x},$$

which ensures therefore the strong metric regularity of the mapping Ψ at $(\bar{x},0)$ and thus completes the proof of the theorem. □

Note that the strong metric regularity characterization of tilt-stable local minimizers in Theorem 3.2 is not expressed via the initial data of the nonlinear program (2.1) under consideration. To derive verifiable necessary and sufficient condition for tilt stability entirely in terms of the program data, we have to involve additional assumptions and arguments. The next result obtained in this direction provides *sufficient conditions* for tilt stability in (2.1) by adding the *constant rank* qualification to our standing MFCQ (2.4) imposed in Theorem 3.2 and by invoking *Lagrange multipliers*. We are based here on Theorem 2.2 and calculus results of variational analysis.

Theorem 3.3. (sufficient coderivative condition for tilt stability under MFCQ and CRCQ) Let $\bar{x} \in \Gamma$ be a feasible solution to (2.1) satisfying the first-order stationarity condition (2.14). Assume further that both MFCQ and CRCQ hold at $\bar{x}$ and that for all Lagrange multipliers $\lambda \in \mathbb{R}^s_+$ satisfying the KKT relationships

$$\nabla_x \mathcal{L}(\bar{x},\lambda) = 0 \ \text{ and } \ \langle q(\bar{x}),\lambda\rangle = 0 \tag{3.3}$$

and all vectors $w \in \mathbb{R}^n \setminus \{0\}$ we have the sufficient coderivative condition

$$\langle w, \nabla^2_{xx}\mathcal{L}(\bar{x},\lambda)w\rangle > -\langle z, \nabla q(\bar{x})w\rangle \ \text{ whenever } \ z \in D^* N_{\mathbb{R}^s_-}\big(q(\bar{x}),\lambda\big)\big(\nabla q(\bar{x})w\big). \tag{3.4}$$

Then $\bar{x}$ is a tilt-stable local minimizer in the nonlinear program (2.1).

P r o o f . It follows from Theorem 2.2 above that, under the assumed first-order stationarity condition (2.14) and MFCQ (2.4) at the point $\bar{x} \in \Gamma$, its local optimality and tilt stability in (2.1) follow from the fulfillment of the second-order condition (2.15). To check the validity of condition (2.15) under the additional assumptions of this theorem, we use the second-order calculus formula

$$D^* N_\Gamma(\bar{x}, \bar{v})(u) \subset \bigcup_{\substack{\lambda \in N_{\mathbb{R}^s_-}(q(\bar{x})) \\ \bar{v} = (\nabla q(\bar{x}))^T \lambda}} \Big\{ \Big(\sum_{i=1}^{s} \lambda_i \nabla^2 q_i(\bar{x})\Big) u + \big(\nabla q(\bar{x})\big)^T D^* N_{\mathbb{R}^s_-}\big(q(\bar{x}), \lambda\big)\big(\nabla q(\bar{x}) u\big)\Big\}, \tag{3.5}$$

where $\bar{v} = -\nabla f_0(\bar{x})$, and where u is an arbitrary vector from $\mathbb{R}^n$. While this formula was originated in [23, Theorem 3.4] and then developed in [24, Theorem 3.1], its most recent version used here was proved in [9, Theorem 3.3] under the *calmness* assumption on the set-valued mappings $M_J : \mathbb{R}^{|J|} \rightrightarrows \mathbb{R}^n$ for $J \subset I(\bar{x})$ defined by

$$M_J(\vartheta) := \big\{ x \in \mathbb{R}^n \big|\ q_i(x) = \vartheta_i,\ i \in J \big\}$$

at $(0, \bar{x})$, meaning that for each $J \subset I(\bar{x})$ there exist neighborhoods $\mathcal{N}_J$ of $0_{\mathbb{R}^{|J|}}$ and $\mathcal{M}_J$ of $\bar{x}$ as well as a real number $L \geq 0$ such that

$$M_J(\vartheta) \cap \mathcal{M}_J \subset M_J(0) + L\|\vartheta\| \ \text{ for all } \ \vartheta \in \mathcal{N}_J.$$

It has been recently shown in [20, Theorem 1] that the calmness of a perturbed system of equalities and inequalities is implied by CRCQ at the respective point. Since the imposed CRCQ assumption at $\bar{x}$ is valid (by the definition) also for all subsystems generating the maps $M_J, J \subset I(\bar{x})$, we may conclude that the maps M_J indeed do possess the required calmness property. To complete the proof of the theorem, it remains to insert the inclusion (3.5) into condition (2.15). □

Next we proceed with deriving *necessary conditions* for tilt-stable local minimizers of (2.1) involving Lagrange multipliers under both MFCQ and CRCQ. In what follows Λ stands for the set of all Lagrange multipliers $\lambda \in \mathbb{R}^s$ satisfying the relationships in (3.3). Denote by

$$\mathcal{K} := T_\Gamma(\bar{x}) \cap \big\{\nabla f_0(\bar{x})\big\}^\perp \tag{3.6}$$

the *critical cone* to the feasible set Γ at $\bar{x} \in \Gamma$ with respect to $\nabla f_0(\bar{x})$, where $T_\Gamma(\bar{x})$ is the contingent cone to Γ at $\bar{x}$ defined in (2.6).

Theorem 3.4. (necessary coderivative condition for tilt stability under MFCQ and CRCQ) Let $\bar{x}$ be a tilt-stable local minimizer of the nonlinear program (2.1), where both MFCQ and CRCQ hold. Then the necessary coderivative condition

$$\langle w, \nabla^2_{xx}\mathcal{L}(\bar{x}, \lambda) w\rangle + \langle z, w\rangle > 0 \ \text{ whenever } z \in D^* N_{\mathcal{K}}(0,0)(w) \tag{3.7}$$

is satisfied for any $\lambda \in \Lambda$ and any $w \in \mathbb{R}^n \setminus \{0\}$.

P r o o f. It has been recently shown in [11, Theorem 5] that, under the validity of both MFCQ and CRCQ, the contingent cone (2.6) to the graph of the normal cone mapping to the feasible set (2.2) is represented by

$$T_{\operatorname{gph} N_\Gamma}(\bar{x},\bar{v}) = T_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{x},\bar{v}) = \left\{ (a,b) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, b \in \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{x}) \right) a + N_{\mathcal{K}}(a) \right\}, \tag{3.8}$$

where λ is an arbitrary multiplier satisfying the KKT conditions (3.3) with $\bar{v} = -\nabla f_0(\bar{x})$. Furthermore, it follows from [33, Proposition 6.27] and the form of the set Γ in (2.2) that

$$N_{\operatorname{gph} N_\Gamma}(\bar{x},\bar{v}) \supset N_{T_{\operatorname{gph} N_\Gamma}(\bar{x},\bar{v})}(0,0) = \left\{ \begin{bmatrix} u - \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{x}) \right) y \\ y \end{bmatrix} \,\middle|\, u \in D^* N_{\mathcal{K}}(0,0)(-y) \right\}, \tag{3.9}$$

where deriving the equality benefits from [33, Exercise 6.7]. Putting $w := -y$ in (3.9) gives us the implication

$$z \in \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{x}) \right) w + D^* N_{\mathcal{K}}(0,0)(w) \Longrightarrow z \in D^* N_\Gamma(\bar{x},\bar{v})(w).$$

Substituting this implication into the generalized Hessian characterization (2.15) of tilt-stable local minimizers in Theorem 2.2, we arrive at the coderivative condition (3.7), which is therefore necessary for the tilt stability of $\bar{x}$. □

Observe the similarity (Hessian of the Lagrangian) and the difference (coderivatives of the different normal cone mappings) in sufficient (3.4) and necessary (3.7) second-order conditions for tilt-stable minimizers obtained under the same constraint qualifications MFCQ and CRCQ. It can also be seen that both conditions (3.4) and (3.7) containing coderivatives are not fully in terms of the initial problem data of (2.1). However, to express them entirely via the problem data, we can employ the available calculations of the coderivatives of the normal mappings involved (i. e., the second-order subdifferentials/generalized Hessians of the corresponding indicator functions); see, e. g., [3, 8, 9, 25, 26] and the references therein.

Alternatively, we can proceed with elaborating the characterization of tilt stability from Theorem 3.2 by involving verifiable conditions for the *strong metric regularity* of the GE mapping (3.1) at $(\bar{x},0)$. It follows from [29, Theorem 2] that this mapping Ψ is strongly metrically regular at $(\bar{x},0)$ if in addition to MFCQ and CRCQ the classical *Strong Second-Order Sufficient Condition* (SSOSC) holds at $\bar{x}$: for all $\lambda \in \Lambda$ we have

$$\langle w, \nabla^2_{xx}\mathcal{L}(\bar{x},\lambda)w\rangle > 0 \;\text{ whenever} \tag{3.10}$$

$$\langle \nabla q_i(\bar{x}), w\rangle = 0 \;\text{ for all }\; i \in I_+(\bar{x},\lambda) := \left\{ j \in \{1,\ldots,s\} \,\middle|\, \lambda_j > 0 \right\} \;\text{ and }\; w \neq 0.$$

The next result of its own interest shows that the coderivative condition (3.4) of Theorem 3.3 is in fact *equivalent* to the SSOSC defined in (3.10), and hence the latter condition ensures the tilt stability of a local minimizer $\bar{x}$ in (2.1) under the validity of MFCQ and CRCQ at this point due to either Theorem 3.2 or Theorem 3.3.

Theorem 3.5. (tilt stability from SSOSC under MFCQ and CRCQ) Take any vector $\lambda \in \Lambda$. Then the coderivative condition (3.4) in Theorem 3.3 holds if and only if SSOSC (3.10) is fulfilled. Thus $\bar{x}$ is a tilt-stable local minimizer in (2.1) provided that all the three conditions MFCQ, CRCQ, and SSOSC are satisfied at $\bar{x}$.

Proof. Assume without loss of generality that $I(\bar{x}) = \{1, \ldots, s\}$ and denote by

$$I_0(\bar{x}, \lambda) := I(\bar{x}) \setminus I_+(\bar{x}, \lambda)$$

the index set of *weakly active* inequality constraints. For simplicity we omit in what follows the arguments $(\bar{x}, \lambda)$ in the index notation I_+ and I_0. It is proved in [26, Lemma 2.2] that

$$\big\{w \in \mathbb{R}^n \big|\, D^* N_{\mathbb{R}^s_-}\big(q(\bar{x}), \lambda\big)\big(\nabla q(\bar{x})w\big) \neq \emptyset\big\} = \ker \nabla q_{I_+}(\bar{x}). \tag{3.11}$$

By virtue of SSOSC (3.10) and (3.11) one has that

$$\langle w, \nabla^2_{xx} \mathcal{L}(\bar{x}, \lambda) w\rangle > 0$$

for all $w \neq 0$ such that $D^* N_{\mathbb{R}^s_-}(q(\bar{x}), \lambda)(\nabla q(\bar{x})w) \neq \emptyset$. From the monotonicity result in [28, Theorem 2.1] one has that for any vector $w \in \mathbb{R}^n$ the real number

$$\vartheta := \big\langle \nabla q(\bar{x})w, D^* N_{\mathbb{R}^s_-}\big(q(\bar{x}), \lambda\big)\big(\nabla q(\bar{x})w\big)\big\rangle$$

is nonnegative. Hence, the implication SSOSC$\Longrightarrow$(3.4) holds true. To justify the converse implication (3.4)$\Longrightarrow$SSOSC, observe that

$$\vartheta = \left\{ \langle \nabla q_{I_0}(\bar{x})w, z\rangle \;\middle|\; \begin{array}{lll} z_i \geq 0 & \text{when} & \langle \nabla q_i(\bar{x}), w\rangle > 0 \\ z_i \in \mathbb{R} & \text{when} & \langle \nabla q_i(\bar{x}), w\rangle = 0, \qquad i \in I_0 \\ z_i = 0 & \text{when} & \langle \nabla q_i(\bar{x}), w\rangle < 0 \end{array} \right\} (\geq 0). \tag{3.12}$$

Furthermore, for every $w \in \ker \nabla q_{I_+}(\bar{x})$ there is a vector $z \in D^* N_{\mathbb{R}^s_-}(q(\bar{x}), \lambda)(\nabla q(\bar{x})w)$ such that $\vartheta = 0$. Thus condition (3.4) yields that

$$\langle w, \nabla^2_{xx} \mathcal{L}(\bar{x}, \lambda) w\rangle > 0 \;\text{ for all }\; w \in \ker \nabla q_{I_+}(\bar{x}) \setminus \{0\},$$

justifying (3.4)$\Longrightarrow$SSOSC and so the equivalence between these two conditions.

The sufficiency of SSOSC for the tilt stability of $\bar{x}$ in (2.1) follows now from Theorem 3.3 under the validity of MFCQ and CRCQ at $\bar{x}$. On the other hand, it is also a consequence of Theorem 3.2 due the aforementioned result of [29, Theorem 2] on the strong metric regularity at $(\bar{x}, 0)$ of the mapping Ψ from (3.1). $\square$

The last theorem of this section concerns the relationship between tilt stability of local minimizers for the nonlinear program and a special variant of the *calmness* property[2] of the *canonically perturbed KKT system*

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \Phi(x, \lambda), \tag{3.13}$$

[2]For the definition of the "standard" calmness property, cf., e. g., [5, Chapter 3H].

where the mapping $\Phi\colon \mathbb{R}^n \times \mathbb{R}^s \rightrightarrows \mathbb{R}^n \times \mathbb{R}^s$ is defined by

$$\Phi(x,\lambda) := \begin{bmatrix} \nabla_x \mathcal{L}(x,\lambda) \\ -q(x) \end{bmatrix} + N_{\mathbb{R}^n \times \mathbb{R}^s_+}(x,\lambda). \tag{3.14}$$

Note that, in contrast to Ψ from (3.1), the defined mapping Φ in (3.14) depends on Lagrange multipliers. It is easy to see that

$$p \in \Psi(x) \Longleftrightarrow \exists \lambda \text{ such that } \begin{pmatrix} p \\ 0 \end{pmatrix} \in \Phi(x,\lambda).$$

Now we are ready to show that the tilt stability of local minimizers in (2.1) implies, under the validity of both MFCQ and CRCQ, a calmness-type property of the canonically perturbed KKT system (3.13).

Theorem 3.6. (calmness property of tilt-stable minimizers) Let $\bar{x}$ be a tilt-stable local minimizer in (2.1), where both MFCQ and CRCQ hold, and let $\bar{\lambda} \in \Lambda$. Then there is a neighborhood $\mathcal{V}$ of $(\bar{x},\bar{\lambda})$, a neighborhood $\mathcal{U}$ of $(0_{\mathbb{R}^n} \times 0_{\mathbb{R}^s})$ and a number $\ell \geq 0$ such that we have the estimate

$$\operatorname{dist}\big((x,\lambda),\{\bar{x}\} \times \Lambda\big) \leq \ell \|(a,b)\| \text{ for all } (x,\lambda,a,b) \in \operatorname{gph}\Phi,\ (x,\lambda) \in \mathcal{V} \text{ and } (a,b) \in \mathcal{U}. \tag{3.15}$$

P r o o f. Pick any multiplier $\bar{\lambda} \in \Lambda$ and by Theorem 3.4 get the necessary coderivative condition (3.7) for the tilt-stable local minimizer $\bar{x}$. Taking into account that the regular normal cone (2.7) is always included in the limiting one, we conclude from (3.7) that

$$\langle w, \nabla^2_{xx}\mathcal{L}(\bar{x},\bar{\lambda})w\rangle > 0 \tag{3.16}$$

for all $w \neq 0$ such that $\langle z, w\rangle = 0$ and

$$(z,-w) \in \widehat{N}_{\operatorname{gph} N_{\mathcal{K}}}(0,0), \tag{3.17}$$

where $\mathcal{K}$ is the critical cone (3.6). Since $\mathcal{K}$ is a convex polyhedral cone with vertex at 0 and since the critical cone to it at 0 with respect to 0 again gives $\mathcal{K}$, the Reduction Lemma from [3] tells us that

$$T_{\operatorname{gph} N_{\mathcal{K}}}(0,0) = \operatorname{gph} N_{\mathcal{K}}.$$

Following further the proof of [3, Theorem 2], we arrive at the representation

$$\widehat{N}_{\operatorname{gph} N_{\mathcal{K}}}(0,0) = \mathcal{K}^0 \times \mathcal{K}.$$

From and (3.16) and (3.17) we now conclude, by choosing $z = 0 \in \mathcal{K}^0$ and $w \in \mathcal{K}$, that

$$\langle w, \nabla^2_{xx}\mathcal{L}(\bar{x},\bar{\lambda})w\rangle > 0 \text{ when } w \in \mathcal{K} \setminus \{0\}. \tag{3.18}$$

To conclude finally that the calmness property (3.15) holds and thus complete the proof of the theorem, it remains to apply [12, Corollary 1] observing that the second-order condition (3.18) implies the noncriticality of the multiplier $\bar{\lambda}$, cf. [12, Remark 1]. □

To summarize the results obtained in this paper on tilt stability of local minimizers and its relationships with some other stability properties in nonlinear programming, let us present the diagram depicted in Figure 1.

In this figure we can see various characterizations of tilt stability as well as sufficient conditions and necessary conditions for this notion. Observe that none of these relationships requires the uniqueness of the corresponding Lagrange multipliers. Such a uniqueness is ensured by the Linear Independence Constraint Qualification (LICQ) under which we obtain the following immediate consequence of the obtained results. This corollary was first derived in the recent paper [25] by a different way. Recall that LICQ holds at $\bar{x}$ for the nonlinear program (2.1) if the gradient vectors $\nabla q_i(\bar{x})$ of all the constraint functions active at $\bar{x}$ are linearly independent.

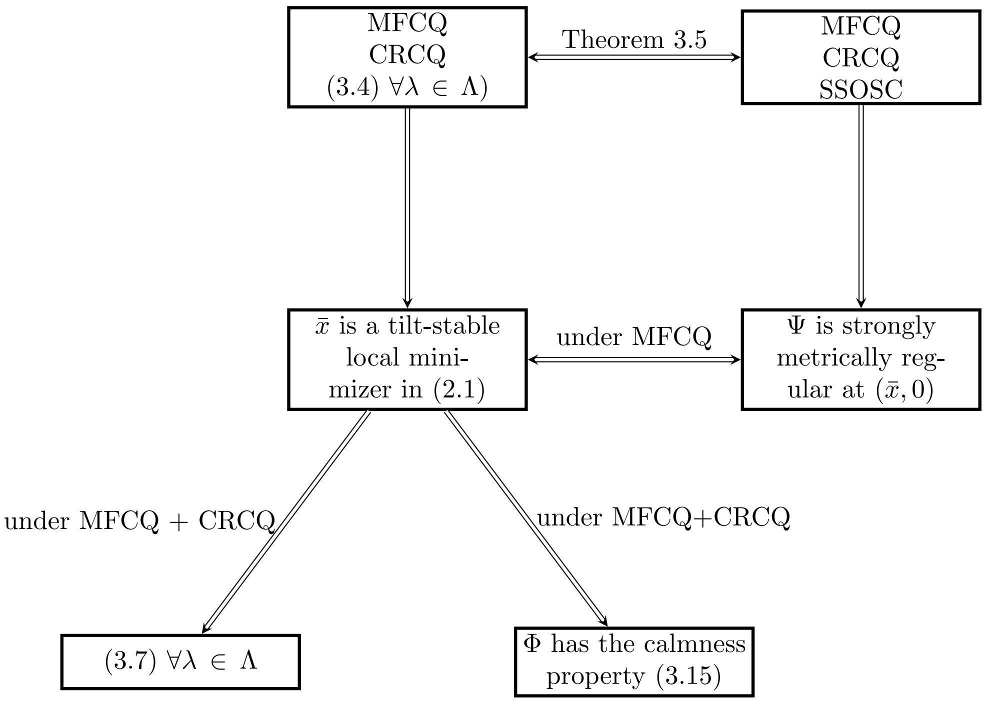

Fig. 1. Relationships of tilt stability with some conditions and stability properties.

Corollary 3.7. (characterizations of tilt stability under LICQ) Let $\bar{x} \in \Gamma$ be a feasible solution to the nonlinear program (2.1), and let LICQ hold at $\bar{x}$. Then $\bar{x}$ is a tilt-stable local minimizer of (2.1) if and only if SSOSC (3.10) is satisfied at $\bar{x}$. Furthermore, in this setting the tilt stability of $\bar{x}$ is equivalent to Robinson's strong regularity of Ψ at $(\bar{x}, 0)$, which amounts to the strong metric regularity of the associated KKT mapping Φ (3.14) at $(\bar{x}, \bar{\lambda}, 0)$, where $\bar{\lambda}$ is the corresponding unique Lagrange multiplier.

P r o o f . It is easy to see that the set of KKT Lagrange multipliers Λ associated with $\bar{x}$ is a singleton $\Lambda = \{\bar{\lambda}\}$ and that we have the representation

$$\begin{aligned} D^*N_\Gamma(\bar{x}, -\nabla f(\bar{x}))(u) &= \left(\sum_{i=1}^{s} \bar{\lambda}_i \nabla^2 q_i(\bar{x})\right) u + \big(\nabla q(\bar{x})\big)^T D^* N_{\mathbb{R}^s_-}\big(q(\bar{x}), \bar{\lambda}\big)\big(\nabla q(\bar{x}) u\big) \\ &= \left(\sum_{i=1}^{s} \bar{\lambda}_i \nabla^2 q_i(\bar{x})\right) u + D^* N_{\mathcal{K}}(0,0)(u) \ \text{ for all } \ u \in \mathbb{R}^n; \quad (3.19) \end{aligned}$$

cf. [10, Proposition 2]. Combining now our results from Theorems 3.3–3.5 with that of [3, Theorem 6], we justify all the statements of this corollary. □

4. ILLUSTRATIVE EXAMPLES

In this section we present two academic examples that illustrate some of the results on tilt stability derived in Section 3 without the validity of LICQ.

Example 4.1. (violation of tilt stability under MFCQ and CRCQ) Consider the following three-dimensional nonlinear program:

$$\begin{array}{ll} \text{minimize} & \langle a, x\rangle + \frac{1}{2}(x_3)^2 + (x_1)^3 \\ \text{subject to} & \\ x \in \Gamma := \left\{ x \in \mathbb{R}^3 \right. & \left| \begin{array}{l} -\frac{1}{2}x_1^2 + x_1 - x_3 \le 0, \\ -\frac{1}{2}x_1^2 - x_1 - x_3 \le 0, \\ -\frac{1}{2}x_2^2 + x_2 - x_3 \le 0, \\ -\frac{1}{2}x_2^2 - x_2 - x_3 \le 0, \\ -\frac{1}{4}(x_1^2 + x_2^2) + \frac{1}{2}(x_1 + x_2) - x_3 \le 0 \end{array} \right\} \end{array} \quad (4.1)$$

with $a = (-0.3, -0.7, 1)$. It is not hard to check that $\bar{x} = 0$ is a solution to program (4.1) and both MFCQ and CRCQ are fulfilled at $\bar{x}$. To test whether $\bar{x}$ is a *tilt-stable* local minimizer in (4.1), let us apply the second-order necessary condition (3.7) obtained in Theorem 3.4. To verify the underlying condition, we choose, e. g., $\bar{\lambda} = (0.3, 0, 0.7, 0, 0) \in \Lambda$ and calculate the Hessian of the Lagrangian

$$\nabla^2_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} -0.3 & 0 & 0 \\ 0 & -0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as well as the critical cone (3.6) and its polar given by

$$\mathcal{K} = \mathbb{R}_+(1,1,1)^T \ \text{ and } \ \mathcal{K}^0 = \left\{ (v_1, v_2, v_3) \in \mathbb{R}^3 \,\middle|\, v_1 + v_2 + v_3 \le 0 \right\}.$$

Due to the aforementioned relationships

$$\mathcal{K}^0 \times \mathcal{K} = \widehat{N}_{\text{gph} N_{\mathcal{K}}}(0,0) \subset N_{\text{gph} N_{\mathcal{K}}}(0,0),$$

we take, e. g., $\bar{w} = (-1,-1,-1) \in -\mathcal{K}$, $\bar{z} \in (1,0,-1) \in \mathcal{K}^0$ and get that

$$\langle \bar{w}, \nabla^2_{xx}\mathcal{L}(\bar{x},\bar{\lambda})\bar{w}\rangle + \langle \bar{z},\bar{w}\rangle = 0.$$

Thus we conclude by Theorem 3.4 that $\bar{x}$ is *not* a tilt-stable local minimizer in the nonlinear program (4.1).

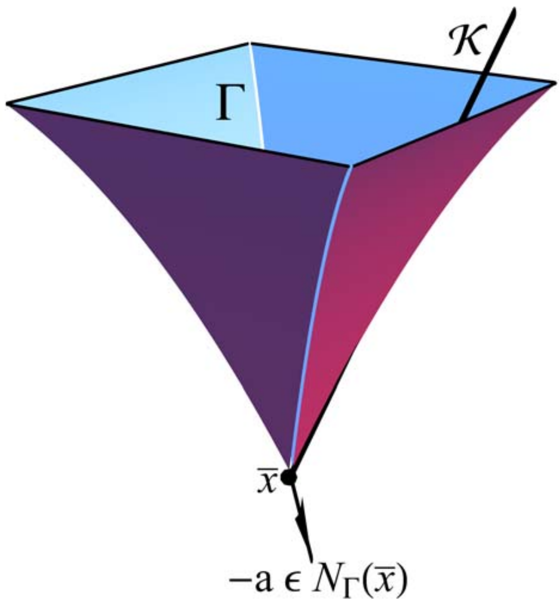

Fig. 2. Illustration of the feasible set Γ defined in (4.1) and the critical cone $\mathcal{K}$.

Our next example illustrates the usage of Theorem 3.5 for confirming tilt stability without the validity of LICQ.

Example 4.2. (tilt stability from SSOSC) Consider the following program:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x-a\|^2 \\ \text{subject to} & \\ & -\frac{1}{2}x_1^2 + x_1 - x_3 \le 0, \\ & -\frac{1}{2}x_1^2 - x_1 - x_3 \le 0, \\ & -\frac{1}{2}x_2^2 + x_2 - x_3 \le 0, \\ & -\frac{1}{2}x_2^2 - x_2 - x_3 \le 0, \end{array} \tag{4.2}$$

where $a = (1,0,-1)$. Observe that $\bar{x} = 0$ is a solution to (4.2), both MFCQ and CRCQ are satisfied at $\bar{x} = 0$, and

$$\Lambda = \{\widetilde{\lambda}\} \quad \text{with} \quad \widetilde{\lambda} = (1,0,0,0).$$

In this case we easily calculate that

$$\nabla^2_{xx}\mathcal{L}(\bar{x},\widetilde{\lambda}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and that the kernel of the gradient of the first inequality in the constraints at $\bar{x}$ amounts to the linear subspace $L = \{(w_1, w_2, w_3)|\ w_1 = w_3\}$. Since for $w \in L \setminus \{0\}$ we have

$$\langle w, \nabla^2_{xx}\mathcal{L}(\bar{x},\bar{\lambda})w\rangle = w_2^2 + w_3^2 > 0,$$

it follows from Theorem 3.5 that $\bar{x}$ is a tilt-stable minimizer in program (4.2).

As mentioned above, LICQ does not hold at the solution points in both nonlinear programs (4.1) and (4.2) in our Examples 4.1 and 4.2. Observe that the perturbed KKT system associated with (4.2) has the form (3.13) with

$$\Phi(x,\lambda) = \begin{bmatrix} x - a + \sum\limits_{i=1}^{4} \lambda_i \nabla q_i(x) \\ -q(x) + N_{\mathbb{R}^4_+}(\lambda) \end{bmatrix}$$

with the corresponding constraint functions $q_i,\ i = 1,\ldots,4$, defined in (4.2). By Theorem 3.6 we claim that there is a neighborhood $\mathcal{V}$ of $(\bar{x},\widetilde{\lambda})$, a neighborhood $\mathcal{U}$ of $(0_{\mathbb{R}^3} \times 0_{\mathbb{R}^4})$ and a number $\ell \geq 0$ such that

$$\|x\| + \|\lambda - \widetilde{\lambda}\| \leq \ell\|(a,b)\| \ \text{ for all } \ (x,\lambda,a,b) \in \operatorname{gph}\Phi, (x,\lambda) \in \mathcal{V} \ \text{ and } \ (a,b) \in \mathcal{U}.$$

It means in that this perturbed KKT system exhibits the *isolated calmness* property at $(\bar{x},\widetilde{\lambda})$, which is even stronger than (3.15); see, e.g., [5, Section 3.I].

5. CONCLUDING REMARKS

It is shown in Theorem 3.2 that tilt stability of local minimizers for C^2 nonlinear programs (2.1) can be equivalently described, under MFCQ, via strong metric regularity of the auxiliary mapping Ψ defined in (3.1). This mapping and its strong metric regularity property do not include Lagrange multipliers and are not directly connected with the conventional KKT system in nonlinear programming. Our intention to describe tilt stability via Lagrange multiplies and eventually via second-order conditions in nonlinear programming has led us to sufficient conditions of tilt stability obtained in Theorem 3.3 and necessary conditions for this property given in Theorem 3.4 via second-order generalized differential constructions of variational analysis. Under the classical LICQ both of these conditions reduce to the well-known SSOSC and thus allow us to give another proof of the SSOSC characterization of tilt stability recently obtained in [25]. However, it is not the case in the absence of LICQ, even when we replace it by the combination of MFCQ and CRCQ. From the viewpoint of second-order subdifferential calculus this gap is largely due to possible proper inclusions in second-order chain rule (3.5) as well as in formula (3.9). It is strongly desirable to close this gap by adding some additional conditions on the problem data. In [25, Theorem 4.3] the exact (equality type) second-order chain rule has been obtained in the general case of strongly amenable compositions with piecewise linear outer functions. However, in the particular case of nonlinear programming considered in this paper the second-order qualification condition

$$D^*N_{\mathbb{R}^s_-}\big(q(\bar{x}),\lambda\big)(0)\cap\ker\big(\nabla q\big(\bar{x}\big)\big)^T=\{0\}$$

$$\text{whenever}\quad \lambda\in N_{\mathbb{R}^s_-}\big(q(\bar{x})\big)\quad\text{with}\quad (\nabla q(\bar{x}))^T\lambda=\bar{v},$$

imposed in [25], reduces to LICQ in our framework. Thus workable characterizations of tilt stability in the absence of LICQ (and possibly in the absence of CRCQ as well) are still very challenging even for standard C^2 nonlinear programs.

Our other lines of research in this direction include the development of the methods proposed in this paper to the study of *full stability* of local minimizers in nonlinear programming, which is a general and highly important stability concept in optimization introduced in [17]. Furthermore, we plan to extend this research to problems of *conic programming*; in particular, those related to descriptions via products of *second-order/Lorentz/ice-cream cones*. Some second-order chain rules have been recently derived in [27] for such settings; they can be useful in the frameworks of tilt and full stability.

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