KYBERNETIKA — VOLUME 39 (2003), NUMBER 2, PAGES 137–141

SOLUBLE APPROXIMATION OF LINEAR SYSTEMS IN MAX–PLUS ALGEBRA

KATARÍNA CECHLÁROVÁ AND RAY A. CUNINGHAME–GREEN

We propose an efficient method for finding a Chebyshev-best soluble approximation to an insoluble system of linear equations over max-plus algebra.

Keywords: discrete-event dynamic systems, max-plus algebra, systems of linear equations, approximation

AMS Subject Classification: 93B25, 15A06, 06F05, 37M99

1. INTRODUCTION

It is well-known [1, 4] that the structure of many discrete-event dynamic systems may be represented by square matrices A over the *max-plus semiring*

$$\Re = (\{-\infty\} \cup R, \oplus, \otimes) = (\{-\infty\} \cup R, \max, +).$$

For example, if the initial event-times of such a system are represented by a vector $\mathbf{s}$, then the event-times after r stages are given by the rth term of the *orbit*

$$\{A^{(r)} \otimes \mathbf{s}(r = 1, 2, \dots)\} \quad \text{where} \quad A^{(r)} = A \otimes A \otimes \dots \otimes A(r\text{-fold}).$$

The *reachability problem* asks whether $\mathbf{s}$ can be chosen so that the orbit contains a given vector $\mathbf{b}$. Clearly, the answer is affirmative if and only if event-times $\mathbf{b}$ can be achieved after one stage from suitable previous event-times, so algebraically the reachability problem produces the linear-equations problem: to solve $A \otimes \mathbf{x} = \mathbf{b}$.

In a practical situation, the data may be such that an exact solution is not possible. In [4] it was shown how to find the maximum solution to the inequality $A \otimes \mathbf{x} \leq \mathbf{b}$ – the so-called *principal solution* – from which may be inferred the Chebyshev-least perturbation of $\mathbf{b}$ necessary to make the system $A \otimes \mathbf{x} = \mathbf{b}$ soluble. Some necessary facts relevant to this are reviewed in the next section.

In [5], the same problem was solved for the related algebraic system *fuzzy algebra.* The question of achieving solubility by modifying the matrix A was examined for fuzzy algebra in [2], while for both fuzzy algebra and $\Re$ the search for solubility by omitting equations was shown in [3] to lead to an NP-complete problem.

In the present paper, we consider how solubility may be achieved for a system $A \otimes \mathbf{x} = \mathbf{b}$ over $\Re$ if both A and $\mathbf{b}$ may be perturbed. Specifically, we seek a Chebyshev-least perturbation, consistent with solubility, of the matrix $[A, \mathbf{b}]$.

2. PRELIMINARIES

In the system $\Re$, we write $a^{(r)}$ to denote the r-fold product $a \otimes \ldots \otimes a$. Since the operation $\otimes$ represents arithmetical addition, $a^{(r)}$ has the value ra. $a^{(-1)}$ is the multiplicative inverse in $\Re$, hence $a^{(-1)} = -a$.

The system $\Re$ is embeddable in the self-dual system

$$\Im = (\{-\infty\} \cup R \cup \{+\infty\}, \oplus, \otimes, \oplus', \otimes') = (\{-\infty\} \cup R \cup \{+\infty\}, \max, +, \min, +)$$

where the operations $\otimes, \otimes'$, representing arithmetical addition, differ only in that

$$-\infty \otimes +\infty = -\infty, \qquad -\infty \otimes' +\infty = +\infty.$$

The set of all m by n matrices over $\Im$ will be denoted by $\Im(m,n)$, the set of all $m-$vectors by $\Im(m)$ and the operations $\oplus, \otimes$ and $\oplus', \otimes'$ are extended to matrix algebra in the usual way. Matrices will be denoted by upper-case italics and vectors by lower-case bold letters.

For any matrix $A = [a_{ij}] \in \Im(m,n)$, the *conjugate matrix* is $A^* = [-a_{ji}] \in \Im(n,m)$ obtained by negation and transposition. We shall use the following properties of conjugation (compare [4, p. 5])

$$(A^*)^* = A \text{ and } (A \otimes B)^* = B^* \otimes' A^*. \tag{1}$$

A set of linear inequalities $A \otimes \mathbf{x} \leq \mathbf{b}$ over $\Re$ always possesses a solution. The greatest is

$$\mathbf{x}^p(A, \mathbf{b}) = A^* \otimes' \mathbf{b}. \tag{2}$$

This *principal solution* is calculated in $\Im$ but lies in $\Re$. It is also the greatest solution of $A \otimes \mathbf{x} = \mathbf{b}$ if and only if any solution exists (see [4, p. 5] and [1, p. 112]).

For brevity, in what follows, the symbol $[A, \mathbf{b}]$ for $A \in \Im(m,n), \mathbf{b} \in \Im(m)$ represents the $m \times (n+1)$ matrix obtained by appending column $\mathbf{b}$ as column $n+1$ to matrix A.

Definition 1. Given two matrices $P, Q \in \Im(m,n)$, their Chebyshev distance will be denoted by $\Delta(P,Q) = \max_{i,j} |p_{ij} - q_{ij}|$.

Definition 2. For two given integers m, n denote the family of all soluble max-plus linear systems with n unknowns and m equations by

$$\mathcal{S}(m,n) = \{(A, \mathbf{b}); A \in \Im(m,n), \mathbf{b} \in \Im(m); \text{ system } A \otimes \mathbf{x} = \mathbf{b} \text{ is soluble}\}.$$

A Chebyshev-best soluble approximation of an insoluble system

$$A \otimes \mathbf{x} = \mathbf{b}, A \in \Im(m,n), \mathbf{b} \in \Im(m)$$

is a pair $A' \in \Im(m,n), \mathbf{b}' \in \Im(m)$ such that $(A', \mathbf{b}') \in \mathcal{S}(m,n)$ and

$$\Delta([A', \mathbf{b}'], [A, \mathbf{b}]) \le \Delta([A'', \mathbf{b}''], [A, \mathbf{b}])$$

for each pair $(A'', \mathbf{b}'') \in \mathcal{S}(m,n)$.

Let us denote by

$$\delta^+(B \otimes \mathbf{x}; \mathbf{b}) = \max_i \{(B \otimes \mathbf{x})_i - b_i\}$$

and by

$$\delta^-(B \otimes \mathbf{x}; \mathbf{b}) = \min_i \{(B \otimes \mathbf{x})_i - b_i\}$$

the extreme positive and the extreme negative deviation of $B \otimes \mathbf{x}$ from $\mathbf{b}$, respectively. In notation of max-plus algebra

$$\delta^+(B \otimes \mathbf{x}; \mathbf{b}) = \mathbf{b}^* \otimes (B \otimes \mathbf{x})$$

and

$$\delta^-(B \otimes \mathbf{x}; \mathbf{b}) = \mathbf{b}^* \otimes' (B \otimes \mathbf{x}).$$

Note that if $\hat{\mathbf{x}} = \mathbf{x}^p(B, \mathbf{b})$ then $\delta^+(B \otimes \hat{\mathbf{x}}; \mathbf{b}) = 0$ and $\delta^-(B \otimes \hat{\mathbf{x}}; \mathbf{b}) \le 0$, moreover $\delta^-(B \otimes \hat{\mathbf{x}}, \mathbf{b}) = 0$ if and only if the system $B \otimes \mathbf{x} = \mathbf{b}$ is soluble.

Theorem 1. Let $A \in \Im(m,n)$ and $\mathbf{b} \in \Im(m)$ be such that $(A, \mathbf{b}) \notin \mathcal{S}(m,n)$; let us define

$$\delta = (\delta^-(A \otimes \mathbf{x}^p(A, \mathbf{b}); \mathbf{b}))^{(1/4)}. \tag{3}$$

If $B \in \Im(m,n)$ is such that $\Delta(A, B) \le \delta$, i.e.

$$\delta^{(-1)} \otimes A \le B \le \delta \otimes A,$$

then $\Delta(B \otimes \mathbf{x}, \mathbf{b}) \ge \delta$ for each $\mathbf{x} \in \Im(n)$, with equality only if $(\mathbf{x}^p(A, \mathbf{b}))^* \otimes \mathbf{x} = \delta^{(2)}$.

Proof. Let $(\mathbf{x}^p(A, \mathbf{b}))^* \otimes \mathbf{x} = \varepsilon^{(2)}$. This means that $\max_j\{x_j - (\mathbf{x}^p(A, \mathbf{b}))_j\} = \varepsilon^{(2)}$, hence for each j $x_j \le \varepsilon^{(2)} + (\mathbf{x}^p(A, \mathbf{b}))_j$; or in max-plus algebra notation $\mathbf{x} \le \varepsilon^{(2)} \otimes \mathbf{x}^p(A, \mathbf{b})$. Two cases arise:

1. $\varepsilon \ge \delta$. Since $B \ge \delta^{(-1)} \otimes A$, we have

$$\begin{aligned}
\delta^+(B \otimes \mathbf{x}, \mathbf{b}) &= \mathbf{b}^* \otimes (B \otimes \mathbf{x}) \ge \\
&\ge \delta^{(-1)} \otimes \mathbf{b}^* \otimes (A \otimes \mathbf{x}) = \\
&= \delta^{(-1)} \otimes (A^* \otimes' \mathbf{b})^* \otimes \mathbf{x} = \quad \text{(by (1) and associativity of } \otimes) \\
&= \delta^{(-1)} \otimes (\mathbf{x}^p(A, \mathbf{b}))^* \otimes \mathbf{x} = \quad \text{(by (2))} \\
&= \delta^{(-1)} \otimes \varepsilon^{(2)} \ge \delta.
\end{aligned}$$

2. $\varepsilon < \delta$. Since $B \le \delta \otimes A$ and $\mathbf{x} \le \varepsilon^{(2)} \otimes \mathbf{x}^p(A, \mathbf{b})$, we have

$$\begin{aligned} \delta^-(B \otimes \mathbf{x}, \mathbf{b}) &= \mathbf{b}^* \otimes' (B \otimes \mathbf{x}) \le \\ &\le \mathbf{b}^* \otimes' (\delta \otimes A \otimes \varepsilon^{(2)} \otimes \mathbf{x}^p(A, \mathbf{b})) = \\ &= \delta \otimes \varepsilon^{(2)} \otimes \mathbf{b}^* \otimes' (A \otimes \mathbf{x}^p(A, \mathbf{b})) = \text{(by commutativity of scalar multiplication)} \\ &= \delta \otimes \varepsilon^{(2)} \otimes \delta^{(-4)} < \text{(by (3))} \\ &< \delta^{(-1)}. \end{aligned}$$

Hence either $\delta^+(B \otimes \mathbf{x}, \mathbf{b}) \ge \delta$ or $\delta^-(B \otimes \mathbf{x}, \mathbf{b}) < \delta^{(-1)}$ and so $\Delta(B \otimes \mathbf{x}; \mathbf{b}) \ge \delta$. □

3. ALGORITHM APPROXIMATION

Input: Matrix $A \in \Im(m, n)$, vector $\mathbf{b} \in \Im(m)$.

Output: A pair $(A', \mathbf{b}') \in \mathcal{S}(m, n)$ with $\Delta([A, \mathbf{b}], [A', \mathbf{b}'])$ smallest possible.

Step 1. Find the principal solution $\mathbf{x}^p(A, \mathbf{b})$ and $\delta := (\Delta(A \otimes \mathbf{x}^p(A, \mathbf{b}), \mathbf{b}))^{(1/4)}$.

Step 2. $\hat{\mathbf{x}} := \delta^{(2)} \otimes \mathbf{x}^p(A, \mathbf{b})$.

Step 3. **For each** row i **with** $b_i^* \otimes' (A \otimes \hat{\mathbf{x}})_i = \varepsilon_i^{(2)}$ **do** (comment $|\varepsilon_i| \le \delta$)
begin $b_i' := \varepsilon_i \otimes b_i$; **for all** j **do** $a_{ij}' = \varepsilon_i^{(-1)} \otimes a_{ij}$ **end.**

Example. Suppose the following matrix A and vector $\mathbf{b}$ are given.

$$A = \begin{pmatrix} 10 & -1 & 11 \\ 9 & 11 & 5 \\ 5 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}; \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}.$$

We compute successively

$$\mathbf{x}^p(A, \mathbf{b}) = \begin{pmatrix} -10 & -9 & -5 & -1 \\ 1 & -11 & 0 & 2 \\ -11 & -5 & -2 & 0 \end{pmatrix} \otimes' \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -8 \\ -9 \end{pmatrix}; A \otimes \mathbf{x}^p(A, \mathbf{b}) = \begin{pmatrix} 2 \\ 3 \\ -3 \\ -7 \end{pmatrix}$$

so the Chebyshev error is $\Delta(A \otimes \mathbf{x}^p(A, \mathbf{b}), \mathbf{b}) = \delta^{(4)} = 8$ and it is achieved in row 4. Now,

$$\hat{\mathbf{x}} = \begin{pmatrix} -4 \\ -4 \\ -5 \end{pmatrix}; A \otimes \hat{\mathbf{x}} = \begin{pmatrix} 6 \\ 7 \\ 1 \\ -3 \end{pmatrix}; \varepsilon^{(2)} = \begin{pmatrix} 4 \\ 4 \\ 0 \\ -4 \end{pmatrix}; A' = \begin{pmatrix} 8 & -3 & 9 \\ 7 & 9 & 3 \\ 5 & 0 & 2 \\ 3 & 0 & 2 \end{pmatrix}; \mathbf{b}' = \begin{pmatrix} 4 \\ 5 \\ 1 \\ -1 \end{pmatrix}.$$

Theorem 2. Algorithm APPROXIMATION correctly finds in $O(mn)$ steps a Chebyshev-best soluble approximation of system $A \otimes \mathbf{x} = \mathbf{b}, A \in \Im(m, n), \mathbf{b} \in \Im(m)$ over max-plus algebra.

P r o o f. Notice, that for $\hat{x}$ defined in the second step of the algorithm, $\delta^+(\delta^{(2)} \otimes A \otimes x^p(A, b); b) = \delta^{(2)}$, $\delta^-(\delta^{(2)} \otimes A \otimes x^p(A, b); b) = \delta^{(-2)}$, and hence $\Delta(A\hat{x}, b) = \delta^{(2)}$.

Then, system $A' \otimes \mathbf{x} = \mathbf{b}'$ is soluble, $\hat{\mathbf{x}}$ being a solution. Further, $\Delta([A, \mathbf{b}], [A', \mathbf{b}']) \leq \delta$. Moreover, Theorem 1 shows that it is impossible to find a soluble system $A'' \otimes \mathbf{x} = \mathbf{b}''$ with Chebyshev error $\Delta([A, b], [A'', b''])$ smaller than δ.

The complexity bound is trivial. □

In conclusion, we recall [4, p. 5] the important property of $\mathbf{x}^p(A, \mathbf{b})$ that no $\mathbf{x}$ can have both

$$\delta^+(A \otimes \mathbf{x}, \mathbf{b}) \leq 0 \text{ (i.e. } A \otimes \mathbf{x} \leq \mathbf{b})$$

and

$$\delta^-(A \otimes \mathbf{x}, \mathbf{b}) > \delta^-(A \otimes \mathbf{x}^p(A, \mathbf{b}), \mathbf{b}) = \delta^{(-4)}.$$

Setting $\mathbf{x} = \delta^{(-2)} \otimes \mathbf{y}$, it follows that no $\mathbf{y}$ can have $\Delta(A \otimes \mathbf{y}, \mathbf{b}) < \delta^{(-2)}$ (see also [6]). In other words, to produce a soluble approximation if only $\mathbf{b}$ may be perturbed incurs at best a Chebyshev error double that incurred at best if both A and $\mathbf{b}$ may be perturbed.

ACKNOWLEDGEMENT

This work was supported by the Slovak Grant Agency for Science, contract #1/7465/20 "Combinatorial Structures and Complexity of Algorithms".

(Received April 5, 2002.)

REFERENCES

[1] F. L. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat: Synchronization and Linearity, An Algebra for Discrete Event Systems. Wiley, Chichester 1992.

[2] K. Cechlárová: A note on unsolvable systems of max-min (fuzzy) equations. Linear Algebra Appl. *310* (2000), 123–128.

[3] K. Cechlárová and P. Diko: Resolving infeasibility in extremal algebras. Linear Algebra Appl. *290* (1999), 267–273.

[4] R. A.Cuninghame-Green: Minimax Algebra (Lecture Notes in Economics and Mathematical Systems 166). Springer, Berlin 1979.

[5] R. A. Cuninghame-Green and K. Cechlárová: Residuation in fuzzy algebra and some applications. Fuzzy Sets and Systems *71* (1995), 227–239.

[6] B. De Schutter: Max-Algebraic System Theory for Discrete Event Systems. Thesis. Katholieke Universiteit Leuven, Belgium 1996.

Doc. RNDr. Katarína Cechlárová, CSc., Institute of Mathematics, Faculty of Science, P. J. Šafárik University, Jesenná 5, 041 54 Košice. Slovakia.
e-mail: cechlarova@science.upjs.sk

Prof. Dr. Ray A. Cuninghame-Green, School of Mathematics and Statistics, The University of Birmingham, Edgbaston, BirminghamB15 2TT. United Kingdom.
e-mail: R.A.Cuninghame-Greene@bham.ac.uk