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ON PSEUDO-RANDOM SEQUENCES AND THEIR RELATION TO A CLASS OF STOCHASTICAL LAWS

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A finite sequence is complex in the sense of Kolmogorov complexity approach if the length of its shortest generating program is greater than a prescribed lower bound. Infinite sequences, almost all initial segments of which are complex in this sense with lower bounds depending on the length of these segments, will be investigated in this paper. The case of interest is that one when such an infinite sequence satisfies some stochastical law depending on a function defining the lower bounds in question. Having a finite or countable collection of stochastical laws and corresponding collection of functions defining lower bounds one could be interested whether there is some other function defining lower bounds and characterizing, in a sense, simultaneously all stochastical laws from the given collection. Existence of such a function is proved and its properties are analyzed. The case of constructive functions defining lower bounds is investigated as well.

1. INTRODUCTION AND NOTATION

With the aim to introduce a notion of randomness attributable to particular sequences of results two approaches have been suggested and developed. The first one, more close to the classical paradigma of mathematical statistics, originates from the notion of particular tests for randomness and combines all such tests which are effectively computable into a universal one; a sequence (finite or infinite) is defined as random if it passes this universal test of randomness (cf., e. g., [1], [2], [5], [6]). The other approach defines the so called Kolmogorov (algorithmic) complexity of a finite sequence of outputs by the length of a shortest program generating this sequence on a universal Turing machine and proclaims a finite sequence as random if this length does not differ substantially from the length of the generated sequence itself (cf. [2], e. g.). This definition is stronger than the former one in the sense that sequences of high Kolmogorov complexity pass the Martin-Löf's universal test of randomness, but there are sequences of low complexity passing this test as well.

Instead of considering all the stochastical laws (laws of large numbers) comprimed altogether into the universal test of randomness we may be interested in infinite sequences obeying some particular stochastical laws and we may try to define, or at least to approximate, sets of such sequences in terms of Kolmogorov complexities of their initial segments. This has been done for the laws asserting the stability of relative frequencies of occurrences of particular events or strings of events in long run series of experiments [3] and for the law of iterated logarithm [4]. From a purely mathematical point of view, the results presented below deal with a possibility to define appropriate one-side approximations for countable joints of classes of infinite sequences of relatively high Kolmogorov complexity in the terms of this complexity. The results achieved in [3], [4], [7] show that certain particular cases of classes of infinite sequences which we shall investigate below can be intuitively but quite reasonably, interpreted by, or identified with, substantial parts of extensions of some well-known stochastical laws. (It means that any infinite sequence from a given class satisfies some stochastical law and, moreover, that majority of infinite sequences satisfying the stochastical law is contained in the class.) If we allow ourselves to extend such an interpretation or identification also to some other classes of infinite sequences investigated in this paper, we can say that the aim of this paper is to characterize, or to approximate, again in terms of Kolmogorov complexity and of lower tolerance bounds defined by a single function, the set of infinite sequences which are random with respect to a given set of stochastical laws. Sufficient conditions under which such approximations are possible and reasonable are found and corresponding assertions are proved.

The following notation will be used throughout this paper. $N = \{0, 1, 2, \ldots\}$ will denote the set of all nonnegative integers, $\mathcal{F}$ will denote the class of all total mappings taking N into $[0, \infty)$. Consider a finite set-alphabet Σ and let $c = \text{card}\ \Sigma \geq 2$. For each $n \in N$, Σ^n denotes the set of all n-tuples of elements of Σ (the set of all strings or words of the length n, in other words said). Set, moreover, $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$, hence, Σ^* is the set of all finite sequences (strings, words) over the alphabet Σ (here Σ^0 contains the empty string Λ as the only element). If $x \in \Sigma^*$, then $\ell(x)$ denotes the length of x, i. e., $\ell(x) = n$ iff $x \in \Sigma^n$. Finally, Σ^N will denote the set of all infinite sequences of elements of Σ; given $S \in \Sigma^N$ and $n \in N, S_n$ denotes the initial segment of the length n defined by S.

Let Ψ denote the class of all partial mappings from the Cartesian product $\Sigma^* \times \Sigma^*$ into Σ^*.

2. KOLMOGOROV COMPLEXITY OF FINITE AND INFINITE SEQUENCES

Let us begin with the following well-known definition.

Definition 1. Let $x, w \in \Sigma^*$, let $\psi \in \Psi$. Then the *Kolmogorov (algorithmic) complexity $K_\psi(x|w)$ of the sequence x, with respect to the a priori information w and with respect to ψ*, is defined by

$$K_\psi(x|w) = \min \{\ell(p) : \ p \in \Sigma^*, \ \psi(p, w) = x\} \tag{1}$$

with $\min(\emptyset) = \infty$.

Let $\mathcal{W} = \langle w_0, w_1, \ldots \rangle$ be a sequence of finite strings over Σ (i. e., $w_i \in \Sigma^*$ for each i) such that w_n expresses some a priori information concerning the initial segments of the length n of the infinite sequences to be considered below. E.g., w_n contains an information about the length n of these segments or about the value of a function depending on this length. Let $\mathcal{T} = \mathcal{T}(w)$ denote the set of all triplets $\langle \psi, f, \mathcal{W} \rangle$ such that $\psi \in \Psi$, $f \in \mathcal{F}$.

Definition 2. Given $\langle \psi, f, \mathcal{W} \rangle \in \mathcal{T}$, a sequence $S \in \Sigma^N$ is called $\langle \psi, f, \mathcal{W} \rangle$-*complex*, if there exists $j \in N$ such that, for each $n \geq j$, the inequality

$$K_\psi(S_n \,|\, w_n) \geq n - f(n) \tag{2}$$

holds. Let $D(\psi, f, \mathcal{W})$ denote the set of all $\langle \psi, f, \mathcal{W} \rangle$-complex infinite sequences over Σ.

3. MARTIN–LÖF CONDITION

With the aim to generalize the well-known Martin–Löf's result (cf. [6]) we shall prove that $\sum_{n=0}^{\infty} c^{-f(n)} < \infty$ implies $D(\psi, f, \mathcal{W}) \neq \emptyset$; let us recall that $c = \text{card}\, \Sigma$.

Lemma 1. Let $w \in \Sigma^*$, $\psi \in \Psi$, $r \in [0, \infty)$. Then

$$\text{card}\,\{x \in \Sigma^n : \; K_\psi(x|w) < r\} \leq \left(c^{\lceil r \rceil} - 1\right)(c-1)^{-1}, \tag{3}$$

where $\lceil r \rceil = \min\{n : n \in N, \; n \geq r\}$.

Proof. Analogous to the proof of Lemma 2.7 in [1], p. 306. □

Let $\mathcal{B}$ be the minimal σ-field generated by the algebra $\mathcal{A}$ of cylinders over Σ^N, let P be the product probability measure generated on $\mathcal{B}$ by the uniform probability distribution over Σ (associating c^{-1} to each letter). Then, obviously,

$$P\left(\{S \in \Sigma^N : \; S_{\ell(w)} = w\}\right) = c^{-\ell(w)}. \tag{4}$$

A function $f \in \mathcal{F}$ *satisfies the Martin–Löf condition* (or: *posseses the Martin–Löf property*), if $\sum_{n=0}^{\infty} c^{-f(n)} < \infty$. Let $\text{ML} \subset \mathcal{F}$ denote the class of functions from $\mathcal{F}$ possessing the Martin–Löf property.

Theorem 1. Let $\langle \psi, f, \mathcal{W} \rangle \in \mathcal{T}$. Then $D(\psi, f, \mathcal{W})$ is measurable. Moreover, if $f \in \text{ML}$, then $P(D(\psi, f, \mathcal{W})) = 1$, hence, $D(\psi, f, \mathcal{W}) \neq \emptyset$.

Proof. Definition 2 immediately implies that

$$\Sigma^N \setminus D(\psi, f, \mathcal{W}) = \bigcap_{j=0}^{\infty} \bigcup_{n=j}^{\infty} \bigcup_{x \in \Sigma^n} \{S \in \Sigma^N : \; S_n = x, K_\psi(x|w_n) < n - f(n)\}. \tag{5}$$

For each $n \in N$, n fixed, $x \in \Sigma^N$, the set

$$B_{x,n} = \left\{S \in \Sigma^N : \; S_n = x, K_\psi(x|w_n) < n - f(n)\right\} \tag{6}$$

is a cylinder, hence, $\Sigma^N \setminus D(\psi, f, \mathcal{W})$ is measurable due to (5). Moreover,

$$P\left(\bigcup_{x\in\Sigma^n} B_{x,n}\right) = c^{-n}\,\mathrm{card}\,\{x \in \Sigma^n : K_\psi(x|w_n) < n - f(n)\}, \tag{7}$$

so that, using Lemma 1 and the fact that $c \geq 2$, we obtain that

$$P\left(\bigcup_{x\in\Sigma^n} B_{x,n}\right) \leq c^{-n}\left(c^{\lceil n-f(n)\rceil} - 1\right)(c-1)^{-1} < c^{-f(n)+1}. \tag{8}$$

Due to (5), for each $j \in N$,

$$P\left(\Sigma^N \setminus D(\psi, f, \mathcal{W})\right) \leq \sum_{n=j}^{\infty} P\left(\bigcup_{x\in\Sigma^n} B_{x,n}\right), \tag{9}$$

hence, for each $j \in N$,

$$P\left(\Sigma^N \setminus D(\psi, f, \mathcal{W})\right) < c\sum_{n=j}^{\infty} c^{-f(n)}. \tag{10}$$

But $f \in \mathrm{ML}$, so that $\inf\left\{\Sigma_{n=j}^{\infty} c^{-f(n)} : j \in N\right\} = 0$, consequently,

$$P\left(\Sigma^N \setminus D(\psi, f, \mathcal{W})\right) = 0, \tag{11}$$

and the assertion immediately follows. □

For two total mappings f, g taking N into real numbers, define mappings $f\dot{+}g$, $f\dot{-}g$ by setting

$$(f\dot{+}g)(n) = \max\{f(n) + g(n), 0\}, \qquad (f\dot{-}g)(n) = \max\{f(n) - g(n), 0\}.$$

The following assertion is obvious.

Lemma 2. If $f \in \mathrm{ML}$, $g : N \to R$ is total, and if $|g(n)| \leq K$ holds for a constant K and for all $n \in N$, then $(f\dot{+}g) \in \mathrm{ML}$, $(f\dot{-}g) \in \mathrm{ML}$.

4. ON CLASSES OF COMPLEX SEQUENCES

Henceforth, ψ and $\mathcal{W}$ are fixed so that we may write $D(f)$ instead of $D(\psi, f, \mathcal{W})$ if no misunderstanding menaces. If $\mathcal{G} \subset \mathcal{F}$, set $D(\mathcal{G}) = \bigcap_{g\in\mathcal{G}} D(g)$. For $f, g \in \mathcal{F}$, $f \prec g$ means that f is smaller than or equal to g almost everywhere, i.e.

$$f \prec g \Leftrightarrow (\exists j \in N)\ (\forall n \geq j)\ (f(n) \leq g(n)). \tag{12}$$

If $f \in \mathcal{F}$, $\mathcal{G} \subset \mathcal{F}$, then $f \prec \mathcal{G}$ means that $f \prec g$ for each $g \in \mathcal{G}$.

Lemma 3. If $f \prec g$, then $D(f) \subset D(g)$.

Proof. If $D(f) = \emptyset$, the assertion is trivial, so let $f \prec g$ and let $S \in D(f)$. Hence, there exists $j_0 \in N$ such that, for all $n \geq j_0$, $f(n) \leq g(n)$ holds, at the same time, $S \in D(f)$ implies that there exists $j_1 \in N$ such that $K_\psi(S_n \,|\, w_n) \geq n - f(n)$ for each $n \geq j_1$. So, $n \geq \max\{j_0, j_1\}$ implies $K_\psi(S_n \,|\, w_n) \geq n - g(n)$, hence, $S \in D(g)$ and the assertion is proved. □

The following corollary immediately follows.

Corollary 1. Let $f \in \mathcal{F}$, $\mathcal{G} \subset \mathcal{F}$. If $f \prec \mathcal{G}$, then $D(f) \subset D(\mathcal{G})$. If, moreover, $f \in ML$, then $P(D(f)) = 1$ and $P(D(\mathcal{G})) = 1$.

Under the interpretation suggested and briefly discussed in the introductory part of this paper, each set $D(g)$, $g \in \mathcal{G}$, corresponds to some "stochastical law", or "law of large numbers", hence, if $f \prec \mathcal{G}$, then each sequence from $D(f)$ obeys all these laws corresponding to functions $g \in \mathcal{G}$. In other words, the problem to describe or at least to approximate, in an effective way, the set of infinite sequences obeying all the laws corresponding to functions from $\mathcal{G}$ is reduced to that of finding a function f such that $f \prec \mathcal{G}$ and $f \in$ ML.

There is an analogy between this situation and that occurring when the so called universal Martin–Löf's tests are applied. With the aim to develop this analogy in more detail and to express the properties of the sets $D(f)$ and $D(\mathcal{G})$ in terms of recursion theory we introduce and prove the following two lemmas and a theorem resulting from them.

Let $\mathcal{F}_{\rm sum} = \{f \in \mathcal{F} : \sum_{n=0}^{\infty} f(n) < \infty\}$ be the class of all summable functions from $\mathcal{F}$.

Lemma 4. Let $\mathcal{H} \subset \mathcal{F}_{\rm sum}$ be finite or countable. Then there exists $k \in \mathcal{F}_{\rm sum}$ such that $h \prec k$ for each $h \in \mathcal{H}$ and $0 < k(n) \leq 1$ for each $n \in N$.

Proof. Let us arrange the set $\mathcal{H}$ into a sequence $\mathcal{H} = \langle h_1, h_2, \ldots \rangle$, let $h_0 \in \mathcal{F}_{\rm sum}$ be such that $0 < h_0(n)$ for each $n \in N$ and $\sum_{n=0}^{\infty} h_0(n) < 1$ and let us join h_0 to $\mathcal{H}$. Set, for each $m \in N$,

$$\ell_m = \max\{h_0, h_1, \ldots, h_m\}. \tag{13}$$

Obviously, for each $m \in N$,

$$\sum_{n=0}^{\infty} \ell_m(n) \leq \sum_{n=0}^{\infty} \left(\sum_{j=0}^{m} h_j(n) \right) < \infty, \tag{14}$$

so that $\{\ell_0, \ell_1, \ldots\} \subset \mathcal{F}_{\rm sum}$. Moreover, for each $m \in N$, $h_m \prec \ell_m$. Hence, in order to prove the first part of the assertion, if suffices to find $k \in \mathcal{F}_{\rm sum}$ such that $\ell_m \prec k$ for each $m \in N$.

Now, let us construct, by induction, an increasing sequence $n_0, n_1, \ldots$ of integers such that, for each $m \in N$,

$$\sum_{n=n_m}^{\infty} \ell_m(n) \leq 2^{-m}. \tag{15}$$

In fact, take $m = 0$ and set $n_0 = 0$. Then $\ell_0 = h_0$ and (15) immediately follows from the declared properties of h_0. Having chosen n_{m-1}, there exists $j \in N$ such that $\sum_{n=j}^{\infty} \ell_m(n) \leq 2^{-m}$ holds, hence, set $n_m = \max\{j, n_{m-1}+1\}$. Now, (15) evidently holds. Setting $k(n) = \ell_m(n)$ for each $m \in N$, $n_m \leq n \leq n_{m+1} - 1$, we easily find that

$$\sum_{n=0}^{\infty} k(n) = \sum_{m=0}^{\infty} \sum_{n=n_m}^{n_{m+1}-1} \ell_m(n) \leq 1, \tag{16}$$

i.e., $k \in \mathcal{F}_{\text{sum}}$, $k(n) \leq 1$ for each $n \in$ N. As $h_0(n) = \ell_0(n) \leq k(n), 0 < k(n)$ holds for each $n \in$ N.

Take $m \in N$, $n \geq n_m$. Then there exists $q \in N$ such that $n_q \leq n \leq n_{q+1} - 1$. Evidently, $q \geq m$, so that

$$\begin{aligned} k(n) = \ell_q(n) &= \max\{h_0(n), h_1(n), \ldots, h_q(n)\} \geq & (17)\\ &\geq \max\{h_0(n), h_1(n), \ldots, h_m(n)\} = \ell_m(n), & (18)\end{aligned}$$

hence, for each $n \geq n_m$, $\ell_m(n) \leq k(n)$, so that $\ell_m \prec k$ holds. The assertion is proved. □

Lemma 5. Let $\mathcal{G} \subset$ ML be countable. Then there exists $f \in$ ML such that $f \prec \mathcal{G}$.

Proof. Arrange $\mathcal{G}$ into a sequence $\mathcal{G} = \langle g_0, g_1, \ldots\rangle$ and set, for each $m, n \in N$, $h_m(n) = c^{-g_m(n)}$. The set $\mathcal{H} = \{h_0, h_1, \ldots\}$ satisfies the condition of Lemma 4, hence, there exists $k \in \mathcal{F}_{\text{sum}}$ satisfying the assertion of Lemma 4. Set $f(n) = -\log_c(k(n))$, then $0 \leq f(n)$ for each $n \in N$ and $\sum_{n=0}^{\infty} c^{-f(n)} = \sum_{n=0}^{\infty} k(n) < \infty$, so that $f \in$ ML.

Let $g \in \mathcal{G}$, let $g = g_m$, recalling the proof of Lemma 4 above we can easily see that there exists $j \in N$ such that, for all $n \geq j$, $h_m(n) \leq k(n)$. Consequently, $g(n) = g_m(n) = -\log_c h_m(n) \geq -\log_c k(n) = f(n)$ holds for each $n \geq j$ as well, so that $f \prec g$ and $f \prec \mathcal{G}$ follow. □

The following statement immediately follows from Corollary 1 and Lemma 5.

Theorem 2. Let $\mathcal{G} \subset$ ML be countable. Then there exists $f \in$ ML such that $D(f) \subset D(\mathcal{G})$ and $P(D(f)) = 1$.

Definition 3. For each $m \in N$, let $d_m(n) = m$ for all $n \in N$. A class $\mathcal{G} \subset \mathcal{F}$ is called *d-closed*, if $g \dot{-} d_m \in \mathcal{G}$ for each $g \in \mathcal{G}$, $m \in N$.

E.g., the class ML is d-closed.

Theorem 3. Let $\mathcal{G} \subset \mathcal{F}$ be d-closed and $g \in \mathcal{G}$. If $g \in \mathcal{G}$ is not bounded, then for each $S \in D(\mathcal{G})$

$$\limsup_{n\to\infty} \{K_\psi(S_n \,|\, w_n) - (n - g(n))\} = \infty. \tag{18}$$

If $\lim_{n\to\infty} g(n) = \infty$, then for each $S \in D(g)$,

$$\lim_{n\to\infty} \{K_\psi(S_n \,|\, w_n) - (n - g(n))\} = \infty. \tag{19}$$

Proof. Let $g \in \mathcal{G}$ be unbounded. Then there exists an increasing infinite sequence $n_0, n_1, \ldots$ of integers such that $\lim_{i\to\infty} g(n_i) = \infty$. Fix $m \in N$, there exists $i_0 \in N$ such that, for each $i \geq i_0$, $(g \dot{-} d_m)(n_i) = g(n_i) - m$. Moreover, $g \dot{-} d_m \in \mathcal{G}$, so that there exists $j_1 \in N$ such that, for each $n \geq j_1$,

$$K_\psi(S_n \,|\, w_n) \geq n - (g \dot{-} d_m)(n). \tag{20}$$

Let i_1 be such that $j_1 \leq n_{i_1}$. Take $i \geq \max(i_0, i_1)$, then

$$\begin{aligned} & K_\psi(S_{n_i} \,|\, w_{n_i}) - (n_i - (g(n_i)) = \\ = \; & K_\psi(S_{n_i} \,|\, w_{n_i}) - (n_i - (g \dot{-} d_m)(n_i)) + m \geq m, \end{aligned} \tag{21}$$

and (18) is proved. The case when $\lim_{n\to\infty} g(n) = \infty$ can be converted into the just proved one by setting $n_i = i$ for each $i \in N$. □

The importance of the assertion just proved consists in the fact that, as can be easily seen, the condition of d-closeness is satisfied by a number of theoretically and practically important classes of functions, e.g., by the class of all recursive functions, by the class of primitive recursive functions, or by the class of elementary functions.

Lemma 6. Let $\mathcal{G} \subset \mathrm{ML}$ be d-closed. Then for no $f \in \mathcal{G}$ the relation $f \prec \mathcal{G}$ holds.

Proof. Let $f \in \mathcal{G}$, then $\sum_{n=0}^{\infty} c^{-f(n)} < \infty$, so that there exists $m \in N$ such that $f(n) \geq 1$ for each $n \geq m$. Consequently, $(f \dot{-} d_1)(n) = f(n) - 1 < f(n)$, hence, $f \prec f \dot{-} d_1$ does not hold. However, $\mathcal{G}$ is d-closed so that $f \dot{-} d_1 \in \mathcal{G}$, hence, $f \prec \mathcal{G}$ cannot hold.

The following assertion is obvious.

Lemma 7. If $g \in \mathrm{ML}$, then $\lim_{n\to\infty} g(n) = \infty$.

Theorem 4. Let $\mathcal{G} \subset \mathrm{ML}$ be d-closed and countable. Then
(a) there exists $f \in \mathrm{ML}$ such that $D(f) \subset D(\mathcal{G})$ and $P(D(f)) = 1$, $P(D(\mathcal{G})) = 1$,
(b) there is no $f \in \mathcal{G}$ such that $f \prec \mathcal{G}$,
(c) if $g \in \mathcal{G}$ and $S \in D(\mathcal{G})$, then (19) holds.

Proof. The theorem just cumulates the assertions proved above. □

5. RECURSION-THEORETICAL PROPERTIES OF CLASSES $D(\mathcal{G})$

The ideas and results of this chapter take profit of the classical theory of recursive functions and can be easily relativized by introducing an oracle A.

Let us denote by MLR the set of recursive rational-valued functions possessing the Martin–Löf property. The following two assertions are evident.

Lemma 8. MLR is a d-closed and countable subset of ML.

Corollary 2. All the three assertions of Theorem 4 hold for $\mathcal{G} =$ MLR.

A short reconsideration of the result just achieved from the viewpoint of the theory of Martin–Löf tests seems to be useful. A more detailed insight into our model shows, that this model is equivalent to a generalized notion of Martin–Löf test resulting when the condition $P(V_m) < 2^{-m}$ for each component V_m of the test is replaced by a more general condition $P(V_m) < 2^{-f(m)}$. An analysis of the proof of the assertion claiming the existence of a universal Martin–Löf test [1] yields that under the generalization just introduced such a universal test exists, if the function f belongs to the class MLRE of functions which will be defined below. Theorem 4 and Corollary 2 express the fact just mentioned in the terms used in this paper.

A function $f \in \mathcal{F}_{\text{sum}}$ is called *effectively summable* if there exists a recursive function ascribing to each positive rational number r an integer m such that $\sum_{n=m}^{\infty} f(n) < r$. For each $f \in \mathcal{F}$ define the function C_f by $C_f(n) = c^{-f(n)}$. By MLRE denote the class of all recursive rational-valued functions f such that C_f is effectively summable. As can be easily seen, if $f \in$ MLRE, then f possesses the Martin–Löf property. The following two assertions are obvious.

Lemma 9. MLRE is a countable and d-closed subset of ML.

Corollary 3. All the three assertions of Theorem 4 hold for $\mathcal{G} =$ MLRE.

Theorem 5. Let $\mathcal{G} \subset$ MLRE be an indexed set of functions, let there exist a recursive function $G(\cdot,\cdot)$ such that, for each $g \in \mathcal{G}$, there exists $i \in N$ with the property $G(i,\cdot) = g$. Then a function $f \in$ MLRE such that $f \prec \mathcal{G}$ can be effectively found.

Proof. For each $g \in \mathcal{G}$ an integer-valued recursive function $\tilde{g}$ can be effectively found such that $|g(n) - \tilde{g}(n)| \leq 2$ for each $n \in N$. Evidently, each $\tilde{g}$ is effectively summable. The set $\{C_{\tilde{g}} : g \in \mathcal{G}\}$ is a recursive set of effectively summable functions and we may apply the proof of Lemma 5 above to this set and to the function $h_0(n) = c^{-n-1}$; all steps of this proof are effective. So we construct a function $k \in \mathcal{F}_{\text{sum}}$ such that $C_{\tilde{g}} \prec k$ for each $g \in \mathcal{G}$, k is a recursive and effectively summable rational-valued function and, for each $n \in N$, $k(n) = c^{-\xi(n)}$ where ξ is a recursive function. Setting $f = \xi \dot{-} d_2$ we can easily see that f is a recursive function such that C_f is effectively summable and $f \prec \mathcal{G}$. □

Corollary 4. Let $\mathcal{G}$ be as in Theorem 5. Then a function $f \in$ MLRE can be effectively found such that $D(f) \subset D(\mathcal{G})$ and $P(D(f)) = 1$, $P(D(\mathcal{G})) = 1$.

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