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On the Optimum Decision Rule for the Radar Signal Processing

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The optimum decision rule for processing of the primary search pulse radar information is derived and its statistical characteristics are determined in the present paper. In contrary to [1] the present derivation is based on the real assumption, that the starting value of radar signal phase is a random variable.

1. FORMULATION OF PROBLEM

In the given range interval the radar signal is an additive mixture of useful signal (echoes from aircraft) and jamming (noise and clutter), see e.g. [2]. This signal can be described by use of the finite random sequence of vectors $\eta_j = [d_j, e_j]$, where d_j, e_j are orthogonal components of η_j, and we can write

$$\eta_j = \xi_j + q\varkappa_j \tag{1}$$

where ξ_j is the vector of jamming, $\varkappa_j$ is the vector of echões from aircraft and q is some unknown quantity, the value of which is 1 or 0 depending on presence or absence of echoes from aircraft in the radar signal.

The sequence (1) contains the full information of radar signal in the given range interval. The purpose of this paper is to determine a decision rule, which would make possible the optimum (in some sense) choice of values of the quantity q. We can formulate this task as a hypothesis-testing problem. We define the observation space Γ as the set of all realizations of the finite random sequence γ, consisting of elements η_j, $j = 1, 2, \ldots, M$,

$$\gamma = [\eta_1, \eta_2, \ldots, \eta_M] \in \Gamma .$$

The quantity q expresses a nature strategy and the possible values of q are points in the parameter space. Depending on value of q there exist on Γ different conditional probability densities $w_\gamma(Z|q)$; $Z \in \Gamma$. As the parameter space consists only of two points 0 and 1, one of two single hypothesis H_0 or H_1 must be choosen in decision process. Thus the decision rule is a rule for dividing the observation space into two disjoint parts Γ_0 and Γ_1. Whenever an observation falls in Γ_0 we say H_0 (i.e. if $\gamma \in \Gamma_0$ we say "target is not present") and whenever an observation falls in Γ_1 we say H_1 (i.e. if $\gamma \in \Gamma_1$ we say "target is present").

We require that our chosen decision rule should be "better" than any other decision rule, i.e. that its decision should be the optimum one in some sense. Thus, first we have to define what the optimum decision criterion is. A number of well known decision criteria lead to likelihood ratio test. Likelihood ratio is the real function $L(Z)$, defined on Γ by expression

(2) $$L(Z) = \frac{w_\gamma(Z \mid 1)}{w_\gamma(Z \mid 0)}; \quad Z \in \Gamma .$$

The critical region Γ_1, optimal in the sense of likelihood ratio test, is defined on such a way that it consists of all elements $Z \in \Gamma$ for which

(3) $$L(Z) \geqq \Theta$$

where Θ is a previously chosen constant, so-called threshold of decision rule. Likelihood ratio test corresponds to several general criteria of optimality (e.g. Neyman-Pearson Criterion, that is often used in radar decision problems). Therefore we shall derive the optimum decision rule with the critical region Γ_1 defined by (3).

2. ASSUMPTIONS

In this section we shall specify properties of the sequence $\{\eta_j\}$ that we shall assume in the derivation of the optimum decision rule.

a. Orthogonal components of vector $\varkappa_j = [a_j, b_j]$ can be denoted by the following expressions

(4) $$a_j = J_j \cos(jF + \varphi), \quad b_j = J_j \sin(jF + \varphi),$$

where J_j is the known sequence of modulus, F is a known increment of phase and φ is an unknown random starting value of phase. The random variables φ and $\{\xi_j\}$ are statistically independent and the probability density of φ is uniform in $\langle 0, 2\pi)$.

b. Orthogonal components of vector $\xi_j = [\alpha_j, \beta_j]$ are two stationary random sequences [3] with elements α_j, β_j. Elements of both sequences are random variables with the normal joint probability density $w(\alpha_i, \alpha_{i+1}, \ldots, \beta_i, \beta_{i+1} \ldots)$; they are zero-mean

(5) $$m_1\{\alpha_i\} = m_1\{\beta_i\} = 0$$

and we know their covariance sequence

$$\varrho_{|i-j|+1} = \mathrm{m}_1\{\alpha_i\alpha_j\} = \mathrm{m}_1\{\beta_i\beta_j\} . \tag{6}$$

Furthermore we assume

$$\mathrm{m}_1\{\alpha_i\beta_j\} = 0 \tag{7}$$

for all i and j.

For physical interpretation of these assumptions see [2].

3. A DERIVATION OF THE OPTIMUM DECISION RULE

The sequence of M values of jamming vector ξ_i we denote by a column matrix $\mathbf{X}$, which contains $N = 2M$ elements x_i

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_M \\ \beta_1 \\ \vdots \\ \beta_M \end{bmatrix}$$

where

$$x_i = \alpha_i , \qquad i \in \langle 1, M \rangle ,$$

$$x_i = \beta_{i-M} , \quad i \in \langle M + 1, N \rangle ,$$

and α_i, β_i are the orthogonal components of jamming vector with properties defined in Section 2. All covariance coefficients $r_{ij} = \mathrm{m}_1\{x_i x_j\}$ can be described by $N \times N$ matrix $\mathbf{R}$ and using (6) and (7), we have

$$\mathbf{R} = \begin{bmatrix} r_{11} & \dots & r_{1N} \\ \vdots & & \vdots \\ r_{N1} & \dots & r_{NN} \end{bmatrix} = \begin{bmatrix} \varrho_1 & \dots & \varrho_M & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \varrho_M & \dots & \varrho_1 & 0 & \dots & 0 \\ 0 & \dots & 0 & \varrho_1 & \dots & \varrho_M \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \varrho_M & \dots & \varrho_1 \end{bmatrix} .$$

In accord with the assumptions specified in Section 2 the joint probability density $w(\mathbf{X}) = w(x_1, x_2, \dots, x_N)$ is normal and can be written

$$w(\mathbf{X}) = \frac{\exp\left(-\frac{1}{2}\,\mathbf{X}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{X}\right)}{(2\pi)^M \sqrt{(\det \mathbf{R})}} \tag{8}$$

where $\mathbf{X}^T$ is the transposed matrix of $\mathbf{X}$, $\mathbf{R}^{-1}$ is the inverse matrix of $\mathbf{R}$ and $(\det \mathbf{R})$ is the determinant of $\mathbf{R}$.

The sequence of M values of useful signal vectors we express as a column matrix $\mathbf{S}$, consisting of $N = 2M$ elements s_i

$$\mathbf{S} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ \\ s_N \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_M \\ b_1 \\ \vdots \\ b_M \end{bmatrix}$$

where

(9)
$$s_i = a_i\,, \qquad i \in \langle 1, M \rangle\,,$$
$$s_i = b_{i-M}\,, \quad i \in \langle M+1, N \rangle\,,$$

a_i, b_i are orthogonal components of vector $\varkappa_i$ given by expressions (4). The relations (4) imply that $\mathbf{S} = \mathbf{S}(\varphi)$.

The sequence of signal vector values, that contain jamming and useful signal, we denote by column matrix $\mathbf{Y}$, consisting of $N = 2M$ elements y_i

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

where according to (1) for $q = 1$ is $\mathbf{Y} = \mathbf{X} + \mathbf{S}(\varphi)$ thus $\mathbf{Y} = \mathbf{Y}(\varphi)$. Elements of the matrix $\mathbf{Y}$ are the random variables, that have been formed from elements of matrix $\mathbf{X}$ by transformation $y_i = x_i + s_i$.

The conditional joint probability density $w(\mathbf{Y}|\varphi) = w(y_1, \ldots, y_N \mid \varphi)$ can be expressed as

(10)
$$w(\mathbf{Y} \mid \varphi) = \frac{\exp\{-\frac{1}{2}\,[\mathbf{Y}^T - \mathbf{S}^T(\varphi)]\mathbf{R}^{-1}\,[\mathbf{Y} - \mathbf{S}(\varphi)]\}}{(2\pi)^M \sqrt{(\det \mathbf{R})}}$$

where $\mathbf{Y}^T$ and $\mathbf{S}^T(\varphi)$ are transposed matrices of $\mathbf{Y}$ and $\mathbf{S}$. As the random variable φ is statistically independent on $\{\xi_i\}$ and its probability density is uniform $\langle 0, 2\pi)$, we can write

(11)
$$w(\mathbf{Y}) = w(y_1, y_2, \ldots, y_N) = \frac{1}{2\pi}\int_0^{2\pi} w(\mathbf{Y} \mid \varphi)\,\mathrm{d}\varphi =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp\{-\frac{1}{2}[\mathbf{Y}^T - \mathbf{S}^T(\varphi)]\,\mathbf{R}^{-1}[\mathbf{Y} - \mathbf{S}(\varphi)]\}}{(2\pi)^M \sqrt{(\det \mathbf{R})}}\,\mathrm{d}\varphi\,.$$

We express the sequence of M values of signal vectors received in the given range interval by the column matrix $\mathbf{Z}$, consisting of $N = 2M$ elements z_i

$$\mathbf{Z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ \\ z_N \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_M \\ e_1 \\ \vdots \\ e_M \end{bmatrix}$$

where

(12)
$$z_i = d_i\,, \qquad i \in \langle 1, M \rangle\,,$$
$$z_i = e_{i-M}\,, \quad i \in \langle M + 1, N \rangle$$

d_i and e_i are orthogonal components of received signal vectors η_i. It is clear, that the joint probability density $w_\gamma(\mathbf{Z} \mid q) = w_\gamma(z_1, \ldots, z_N \mid q)$ depends on the value of q in (1). For $q = 0$ is $\mathbf{Z} = \mathbf{X}$ and then

(13)
$$w_\gamma(\mathbf{Z} \mid 0) = \frac{\exp\left(-\frac{1}{2}\,\mathbf{Z}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{Z}\right)}{(2\pi)^M \sqrt{(\det \mathbf{R})}}$$

for $q = 1$ is $\mathbf{Z} = \mathbf{Y}$ thus

(14)
$$w_\gamma(\mathbf{Z} \mid 1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp\left\{-\frac{1}{2}[\mathbf{Z}^{\mathrm{T}} - \mathbf{S}^{\mathrm{T}}(\varphi)]\,\mathbf{R}^{-1}[\mathbf{Z} - \mathbf{S}(\varphi)]\right\}}{(2\pi)^M \sqrt{(\det \mathbf{R})}}\,\mathrm{d}\varphi\,.$$

The probability density on Γ under H_0 or H_1 is defined by expressions (13) or (14).

To simplify notation we denote matrix $\mathbf{R}^{-1}$ by $\mathbf{C}$ and its elements by c_{ij}. In this notation and using (2), (13) and (14) and after arrangement we have the following expression for the likelihood ratio

(15)
$$L(\mathbf{Z}) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{\tfrac{1}{2}\mathbf{Z}^{\mathrm{T}}\mathbf{C}\mathbf{Z} - \tfrac{1}{2}[\mathbf{Z}^{\mathrm{T}} - \mathbf{S}^{\mathrm{T}}(\varphi)]\,\mathbf{C}[\mathbf{Z} - \mathbf{S}(\varphi)]\right\}\mathrm{d}\varphi =$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \exp\left[W(\varphi)\right]\mathrm{d}\varphi\,.$$

Furthermore to enable to find a solution of the integral in (15), we shall arrange $W = W(\varphi)$ to a suitable form. Using the symmetry of $\mathbf{R}$ we have

(16)
$$W = \mathbf{Z}^{\mathrm{T}}\mathbf{C}\mathbf{S} - \tfrac{1}{2}\mathbf{S}^{\mathrm{T}}\mathbf{C}\mathbf{S}\,.$$

Let us partition $\mathbf{R}$ into four fields, consisting of four $M \times M$ matrices from $\mathbf{R}_1$ to $\mathbf{R}_4$, $M = N/2$.

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_3 \\ \mathbf{R}_2 & \mathbf{R}_4 \end{bmatrix}$$

where

$$\mathbf{R}_1 = \mathbf{R}_4 = \begin{bmatrix} \varrho_1 & \cdots & \varrho_M \\ \vdots & & \vdots \\ \varrho_M & \cdots & \varrho_1 \end{bmatrix}, \tag{17}$$

$$\mathbf{R}_2 = \mathbf{R}_3 = \mathbf{0}\,. \tag{18}$$

Similarly let us partition $\mathbf{C}$ into four fields, consisting of four $M \times M$ matrices from $\mathbf{C}_1$ to $\mathbf{C}_4$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_3 \\ \mathbf{C}_2 & \mathbf{C}_4 \end{bmatrix}$$

According to the rule of multiplying of partitioned to field matrices [4] and using properties (17) and (18) we can show, that

$$\begin{aligned} \mathbf{C}_1 &= \mathbf{C}_4 = \mathbf{R}_1^{-1} \\ \mathbf{C}_2 &= \mathbf{C}_3 = \mathbf{0}\,. \end{aligned} \tag{19}$$

Properties (19) will enable us further to reduce the term W. After carrying out the matrix operations (16) we shall express W using the elements of matrices $\mathbf{Z}$, $\mathbf{C}, \mathbf{S}$.

$$W = \sum_{i=1}^{M} \sum_{j=1}^{M} c_{ij}(z_i - \tfrac{1}{2}s_i)\, s_j + \sum_{i=M+1}^{N} \sum_{j=M+1}^{N} c_{ij}(z_i - \tfrac{1}{2}s_i)\, s_j\,.$$

Using (9) and (12)

$$W = \sum_{i=1}^{M} \sum_{j=1}^{M} c_{ij}(a_j d_i + b_j e_i) - \tfrac{1}{2}\sum_{i=1}^{M} \sum_{j=1}^{M} c_{ij}(a_i a_j + b_i b_j)\,.$$

In accord with (4)

$$a_i a_j + b_i b_j = J_i J_j \cos\left[F(j - i)\right]$$

and then

$$W = \sum_{i=1}^{M} \sum_{j=1}^{M} c_{ij} J_j\left[d_i \cos(jF + \varphi) + e_i \sin(jF + \varphi)\right] - $$

$$- \tfrac{1}{2}\sum_{i=1}^{M} \sum_{j=1}^{M} c_{ij} J_i J_j \cos\left[F(j - i)\right].$$

Let us denote

$$W = W_a(\varphi) + W_b \tag{20}$$

where

$$W_a(\varphi) = \sum_{i=1}^{M} \sum_{j=1}^{M} c_{ij} J_j\left[d_i \cos(jF + \varphi) + e_i \sin(jF + \varphi)\right], \tag{21}$$

$$W_b = -\tfrac{1}{2}\sum_{i=1}^{M} \sum_{j=1}^{M} c_{ij} J_i J_j \cos\left[F(j - i)\right]. \tag{22}$$

The expression on the right side of equality (22) consists only of factors that according to assumptions in Section 2 are known constants. Therefore $W_b = \text{const}$, too. After easy arrangements of (21) we obtain

(23) $$W_a = \sum_{j=1}^{M} [T_j \cos(jF + \varphi) + U_j \sin(jF + \varphi)]$$

where

(24) $$T_j = J_j \sum_{i=1}^{M} c_{ij} d_i ,$$

$$U_j = J_j \sum_{i=1}^{M} c_{ij} e_i .$$

Denote

(25) $$T_j = V_j \cos \psi_j ,$$

$$U_j = V_j \sin \psi_j$$

where V_j, ψ_j are variables, given by expressions

$$V_j = \sqrt{(T_j^2 + U_j^2)} ,$$

$$\psi_j = \operatorname{arctg} \frac{U_j}{T_j} - \frac{\pi}{2} (\operatorname{sign} T_j - 1) ; \quad T_j \neq 0 ,$$

$$= \frac{\pi}{2} \operatorname{sign} U_j ; \qquad T_j = 0 .$$

Then substituing (25) into (23), using basic trigonometric theorems and denoting

(26) $$m_j = jF - \psi_j$$

we obtain

(27) $$W_a = \cos \varphi \sum_{j=1}^{M} V_j \cos m_j - \sin \varphi \sum_{j=1}^{M} V_j \sin m_j .$$

Denote in (27)

(28) $$\sum_{j=1}^{M} V_j \cos m_j = Q \cos \mu ,$$

$$\sum_{j=1}^{M} V_j \sin m_j = Q \sin \mu$$

then we can write

(29) $$W_a = Q \cos (\varphi + \mu) .$$

Substituing (20), (22) and (29) into (15) we obtain the following expression of likelihood ratio

$$L(\mathbf{Z}) = \frac{\exp(W_b)}{2\pi} \int_0^{2\pi} \exp[Q \cos(\varphi + \mu)] \, d\varphi$$

this implies that

(30) $$L(\mathbf{Z}) = \exp(W_b) \, . \, I_0(Q)$$

where W_b is constant given by (22), I_0 is a modified Bessel function of the first kind. According to (3) we can determine the optimum Γ_1 by the following inequality

$$\exp(W_b) \, I_0(Q) \geqq \Theta \, .$$

As W_b is a constant and I_0 is a strictly monotonic function, the last inequality is equivalent to inequality

(31) $$Q \geqq \vartheta$$

where ϑ is the modified threshold value, which is connected with Θ by expression

$$I_0(\vartheta) = \frac{\Theta}{\exp(W_b)} \, .$$

From (26) and (28) follows that

$$Q^2 = [\sum_{j=1}^{M} V_j \sin(jF - \psi_j)]^2 + [\sum_{j=1}^{M} V_j \cos(jF - \psi_j)]^2 \, .$$

With use of (24) and (25) we can write

(32) $$Q^2 = [\sum_{i=1}^{M} \sum_{j=1}^{M} J_j c_{ij} d_i \sin jF - \sum_{i=1}^{M} \sum_{j=1}^{M} J_j c_{ij} e_i \cos jF]^2 + + [\sum_{i=1}^{M} \sum_{j=1}^{M} J_j c_{ij} d_i \cos jF + \sum_{i=1}^{M} \sum_{j=1}^{M} J_j c_{ij} e_i \sin jF]^2 \, .$$

Denote

(33) $$\sum_{j=1}^{M} J_j c_{ij} \sin jF = A_i \, ,$$

$$\sum_{j=1}^{M} J_j c_{ij} \cos jF = B_i \, ,$$

where A_i, B_i are evidently constants, because the terms on the left sides of equalities (33) consist only of factors that are known constants according to the assumptions specified in Section 2. From (32) and (33) follows that

(34) $$Q = \sqrt{\{[\sum_{i=1}^{M} (A_i d_i - B_i e_i)]^2 + [\sum_{i=1}^{M} (B_i d_i + A_i e_i)]^2\}} \, .$$

Q in (31) is then the random variable that have been formed from the elements $\mathbf{Z} \in \Gamma$ by functional transformation defined by (34). The block diagram representing the optimum decision rule is shown in Fig 1. The weight coefficients A_i, B_i in this figure are given by (33).

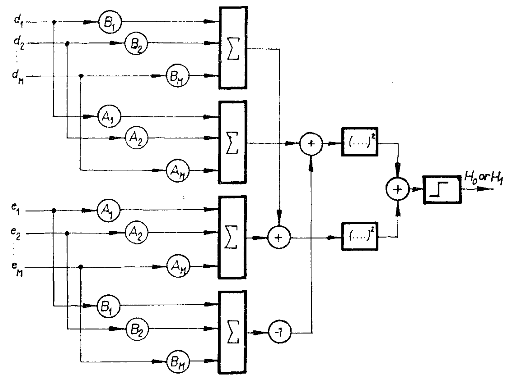

Fig. 1.

4. PROBABILITY DENSITY OF RANDOM VARIABLE Q

In this section we determine the probability density $w_Q(x \mid q)$ of random variable Q. The covariance sequence of random sequences $\{\alpha_i\}$ and $\{\beta_i\}$ for $i \in \langle 1, M \rangle$ can be expressed by canonical expansion with coordinate coefficients k_{ij}; $i, j \in \langle 1, M \rangle$. Therefore the random sequences $\{\alpha_i\}$ and $\{\beta_i\}$ can be expressed by canonical expansion with the same coordinate coefficients [5],

$$\alpha_i = \sum_{j=1}^{M} u_j k_{ij}\,, \quad \beta_i = \sum_{j=1}^{M} v_j k_{ij}\,,$$

where u_j, v_j are independent, zero-mean Gaussian random variables with covariances equal to 1. As to the method of determining of coordinate coefficients k_{ij}, see e.g. [5]. Then we can write

$$d_i = \sum_{j=1}^{M} u_j k_{ij} + q a_i\,, \quad e_i = \sum_{j=1}^{M} v_j k_{ij} + q b_i\,. \tag{35}$$

Let us denote in (34)

$$G = \sum_{i=1}^{M} (A_i d_i - B_i e_i)\,, \quad H = \sum_{i=1}^{M} (B_i d_i + A_i e_i)\,, \tag{36}$$

where G and H are new auxiliary variables. Using (35) in (36) and substituting

(37) $$\mathscr{A}_j = \sum_{i=1}^{M} A_i k_{ij}\,, \quad \mathscr{B}_j = \sum_{i=1}^{M} B_i k_{ij}$$

we obtain

(38) $$G = \sum_{j=1}^{M} (\mathscr{A}_j u_j - \mathscr{B}_j v_j) + q \sum_{i=1}^{M} (a_i A_i - b_i B_i)\,,$$

$$H = \sum_{j=1}^{M} (\mathscr{B}_j u_j + \mathscr{A}_j v_j) + q \sum_{i=1}^{M} (a_i B_i + b_i A_i)\,.$$

The random variable G is given as a sum of M independent zero-mean Gaussian random variables u_j resp. v_j with variances $\mathscr{A}_j^2$, $\mathscr{B}_j^2$ and of the quantity $q \sum_{i=1}^{M} (a_i A_i - b_i B_i)$. Thus the random variable G is also Gaussian, its mean is

(39) $$\mathrm{m}_1\{G\} = q \sum_{i=1}^{M} (a_i A_i - b_i B_i) = q E_G$$

and variance

(40) $$\mathrm{M}_2\{G\} = \sum_{j=1}^{M} (\mathscr{A}_j^2 + \mathscr{B}_j^2) = \sigma^2\,.$$

Similarly we can show, that H is Gaussian random variable with mean

$$\mathrm{m}_1\{H\} = q \sum_{i=1}^{M} (a_i B_i + b_i A_i) = q E_H$$

and with variance

$$\mathrm{M}_2\{H\} = \mathrm{M}_2\{G\} = \sigma^2\,.$$

From independence of u_j and v_j follows independence of G and H. Substituing (4) into (39) and after arrangement we can write

$$E_G = \cos\varphi \sum_{i=1}^{M} J_i[A_i \cos(iF) - B_i \sin(iF)] - \sin\varphi \sum_{i=1}^{M} J_i[A_i \sin(iF) + B_i \cos(iF)]\,.$$

Denote

(41) $$D\cos\tau = \sum_{i=1}^{M} J_i[A_i \cos(iF) - B_i \sin(iF)]\,,$$

$$D\sin\tau = \sum_{i=1}^{M} J_i[A_i \sin(iF) + B_i \cos(iF)\,.$$

Then

(42) $$E_G = D\cos(\varphi + \tau)\,.$$

 Similarly we can show that

(43) $$E_H = \mathrm{D}\sin(\varphi + \tau)$$

where from (41) follows that

(44) $$D = \sqrt{\{[\sum_{i=1}^{M} J_i(A_i \cos iF - B_i \sin iF)]^2 + [\sum_{i=1}^{M} J_i(A_i \sin iF + B_i \cos iF)]^2\}}$$

In accord with (34) and (36)

$$Q = \sqrt{(G^2 + H^2)}\,.$$

The random variable Q is the length of the radius vector of the point, the Cartesian coordinates of which are the independent Gaussian random variables G and H with means $qD\cos(\varphi + \tau)$ and $qD\sin(\varphi + \tau)$ and with equal variances σ^2, see (40). This implies, that the random variable Q has the Rayleigh-Rice probability density

(45) $$w_Q(x \mid q) = \frac{x}{\sigma^2}\exp\left(\frac{-x^2 + qD^2}{2\sigma^2}\right) I_0\left(\frac{qDx}{\sigma^2}\right); \quad x > 0$$

for $q = 0$ (i.e. under H_0) (45) reduces to Rayleigh probability density.

The performace of a decision process can be described by conditional probabilities of the type I and type II errors or by the probabilities of contradictory events. Therefore the performace is often expressed by the following probabilities.

(46) $$P_f = P\{Q \geqq \vartheta \mid q = 0\} = \int_{\vartheta}^{\infty} w_Q(x \mid q = 0)\,dx\,,$$

(47) $$P_d = P\{Q \geqq \vartheta \mid q = 1\} = \int_{\vartheta}^{\infty} w_Q(x \mid q = 1)\,dx$$

where P_f is probability of false alarm and P_d is probability of target detection. The solution of (46) is

(48) $$P_f = \exp\left(-\frac{\vartheta^2}{2\sigma^2}\right)$$

Integral (47) cannot be expressed by a finite expression consisting of elementary functions. Nevertheless, the values of this integral are tabulated in detail e.g. in [6]. Relations from (45) to (48) enable the easy calculation of the decision process performance.

5. CONCLUSION

The problem of synthesis of optimum decision rule for processing of radar signal information has also been solved in [1]. In that reference a decision rule has been derived which we will refer to as the Wirth decision rule, according to the name of the author. A derivation of the Wirth decision rule has been based on the same assumptions as were specified in Section 2 of the present paper with except of the assumption concerning the phase φ starting value properties given by (4). Assumption of the phase starting value has been expressed in [1] by the following expressions

$$ {}^{W}a_j = J_j \cos jF \,, \quad {}^{W}b_j = J_j \sin jF \,. \tag{49} $$

Expressions (49) are a special case of (4) for

$$ \varphi = 0 \tag{50} $$

Under these assumptions the Wirth decision rule is the optimum one in the sense of likelihood ratio test.

It is obvious that in radar signal the assumption (50) cannot be generally satisfied. We shall discuss what influence has this circumstance on properties of Wirth decision rule when the real radar signal is processed. Wirth decision rule is defined by the expression

$$ {}^{W}Q = \sum_{j=1}^{M} (d_j \, {}^{W}A_j + e_j \, {}^{W}B_j) \gtreqqless {}^{W}\vartheta \tag{51} $$

where d_j, e_j are orthogonal components of vector η_j, see (1), ${}^{W}\vartheta$ is the threshold of Wirth decision rule and ${}^{W}A_j$, ${}^{W}B_j$ are constants given by

$$ {}^{W}A_j = \sum_{i=1}^{M} c_{ij} \, {}^{W}a_i \,, \quad {}^{W}B_j = \sum_{i=1}^{M} c_{ij} \, {}^{W}b_i \,. $$

Let us determine the probability density $w_{{}^{W}Q}(Z \mid q)$ of random variable ${}^{W}Q$. According to Section 4 we can express the orthogonal components of vector η_i by (35). This implies that d_i, e_i are Gaussian random variables with means qa_i and qb_i. Then ${}^{W}Q$ is Gausian random variable with a variance ${}^{W}\sigma^2$ and with a mean

$$ \mathrm{m}_1\{{}^{W}Q\} = q \sum_{j=1}^{M} ({}^{W}A_j a_j + {}^{W}B_j b_j) \,. \tag{52} $$

Under H_0, i.e. when $q = 0$, $\mathrm{m}_1\{{}^{W}Q\} \equiv 0$ and the conditional probability density is

$$ w_{{}^{W}Q}(Z \mid q = 0) = \frac{1}{\sqrt{(2\pi \, {}^{W}\sigma^2)}} \exp\left(- \frac{Z^2}{2 {}^{W}\sigma^2} \right) . \tag{53} $$

Further let us discusse the case when $q = 1$. Substituting (4) into (52) and arranging we find

$$\mathrm{m}_1\{^{\mathrm{W}}Q\} = \sin\varphi \sum_{j=1}^{M} J_j(^{\mathrm{W}}B_j \cos jF - {}^{\mathrm{W}}A_j \sin jF) +$$

$$+ \cos\varphi \sum_{j=1}^{M} J_j(^{\mathrm{W}}B_j \sin jF + {}^{\mathrm{W}}A_j \cos jF) = {}^{\mathrm{W}}U \sin\varphi + {}^{\mathrm{W}}T \cos\varphi \ .$$

where ${}^{\mathrm{W}}U$, ${}^{\mathrm{W}}T$ are obviously constants. The expression on the right side can be written as

$$\mathrm{m}_1\{^{\mathrm{W}}Q\} = {}^{\mathrm{W}}V \sin(\nu + \varphi)\,.$$

It is obvious that

$$w_{^{\mathrm{W}}Q}(Z \mid \varphi) = \frac{1}{\sqrt{(2\pi\, {}^{\mathrm{W}}\sigma^2)}} \exp\left\{-\frac{1}{2\, {}^{\mathrm{W}}\sigma^2}\left[Z - {}^{\mathrm{W}}V \sin(\nu + \varphi)\right]^2\right\}.$$

In accord with Section 2 we assume, that φ is uniform random variable in $\langle 0, 2\pi)$ and is independent of $\{\xi_j\}$. Then

$$(54) \quad w_{^{\mathrm{W}}Q}(Z \mid q = 1) = \frac{1}{(2\pi)^{3/2}\, {}^{\mathrm{W}}\sigma} \int_0^{2\pi} \exp\left\{-\frac{1}{2\, {}^{\mathrm{W}}\sigma^2}\left[Z - {}^{\mathrm{W}}V \sin(\nu + \varphi)\right]^2\right\} \mathrm{d}\varphi$$

and we find, that

$$(55) \qquad w_{^{\mathrm{W}}Q}(Z \mid q = 1) = w_{^{\mathrm{W}}Q}(-Z \mid q = 1)\,.$$

From (53) and (55) follows that ${}^{\mathrm{W}}Q$ is zero-mean random variable and its conditional probability density is symmetric about the mean regardless of the value of q. This implies that if we require to be $P_f < 0{\cdot}5$ then it is impossible to reach the higher value of P_d then 0·5. Thus, we can see, that performance of the Wirth decision rule is quite unsuitable for the radar signal processing.

On the other hand the results of numerical calculations of values P_f and P_d show us, that with use of the optimum decision rule (31), (33), (34) derived in this paper we can reach the good performance. The values of P_f and P_d were calculated by (46) and (47) for typical properties of jamming and useful signal echoes. According to [2] the following properties of jamming and useful signal were considered

$$(56) \qquad \begin{aligned} \varrho_{|i-j|+1} &= \Lambda^2 \exp\left[-\Omega(i-j)^2\right]; \quad && i \neq j\,, \\ &= 1 + \Lambda^2\,; && i = j\,, \end{aligned}$$

where Λ^2 is the variance of clutter to variance of noise ratio and Ω is a positive constant characterizing the spectrum width of clutter (Ω depends e.g. on the wind velocity).

$$J_j = \lambda \left(\frac{\sin \mu_j}{\mu_j} \right)^2 ; \qquad j \in \langle 1, M \rangle ,$$
$$= 0 ; \qquad j \notin \langle 1, M \rangle , \tag{57}$$

and

$$\mu_j = (2j - M - 1) \frac{\pi}{M + 1} \quad \text{for odd } M ,$$

$$= (2j - M - 1) \frac{\pi}{M} \quad \text{for even } M .$$

By use of (56) and (57) typical properties of jamming covariance and typical shape of radar antenna radiation pattern can be approximated.

Results of calculations show us that e.g. when we consider $P_f = 10^{-3}$, $\Omega < 5 \,.\, 10^{-4}$ (i.e. the wind velocity < 30 km/hod), $M = 11$, $\Lambda/\lambda < 10^3$ and $f \in \langle \pi/6, 2\pi - \pi/6 \rangle$ where $F = f \pm 2k\pi$ then we can reach $P_d \geqq 0.99$. This implies, that with use of the optimum decision rule we can obtain good performance of radar signal processing nearly in the whole practically important range of signal properties. Even in such cases, when value of jamming many times exceeds value of useful signal.

Synthesis of the optimum decision rule gives a solution of important problem from radar information processing domain. In present the technical utilization of optimum decision rule in the new radar systems is in preparation.

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