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Learning from Mistakes: Constructing Knowledge in Late Antique Mathematical Texts

Abstract: This paper analyzes problems in practical mathematics compiled in late antiquity from two sources: the newly published *P. Math.* (Bagnall/Jones 2019) and the *Stereometrica* associated with Hero of Alexandria. These texts are often far from orderly, and often far from innovative at a first glance: tangles of algorithmic problem-solving techniques, most of uncertain authorship, and often plagued by scribal, mathematical, and conceptual errors. Yet the very features that make these texts so difficult to assess cleanly also mean they are fascinating windows onto a rougher stage of "knowledge construction."

1 Introduction

The surviving corpus of metrological texts (i.e., mathematical texts containing problems on measuring areas and volumes of objects and unit conversions, often practically oriented) takes a marginal role in the study of Greek mathematics, edged out of the spotlight by the justifiably intense focus by most scholars on geometry. Euclid's orderly, rigorous cascades of proofs and Archimedes' dazzling innovations are indeed a compelling field of study. The metrological texts, on the other hand, are far from orderly, and often far from innovative at a first glance: tangles of algorithmic techniques for solving measuring problems, most of uncertain authorship, and often plagued by scribal, mathematical, and conceptual errors. Unlike geometrical texts like Euclid's, where later proofs often rely explicitly on earlier ones and so tend to preserve the wholeness of the work, the metrological texts are by nature much more "discrete" (to use Markus Asper's term), and in the surviving texts their problems are mixed and matched with abandon.[1] Yet the very features that make these texts so difficult to assess cleanly also mean they are fascinating windows onto a rougher stage of "knowledge construction." The manuscript codices of problem-sets assembled and reassembled in late antiquity, the pedagogical papyri that preserve students' errors among their efforts to learn by imitation — these show us

1 Asper 2007, 198.

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knowledge in the making at the level of the discipline and the individual. In what follows I will use recent scholarship on mathematics pedagogy to show how the very errors found in these texts can open up that learning process for analysis.

Perhaps "corpus" is not quite the right term here: this term suggests something more cohesive than the actual state of affairs. Metrological texts survive as problem collections in papyri and manuscript codices, texts often accompanied (or meant to be supplemented) by diagrams, tables of fractions and unit conversions, and other aids to visualization and calculation. Several of the texts compiled into Byzantine manuscripts are associated with the name of Hero of Alexandria, but most should not be attributed to the historical Hero, a figure likely from the 1st or 2nd century CE who composed works on geometry, theoretical mechanics, and practical applications of mechanics like catapult design or the construction of automatic puppet theaters.[2]

Hero did compose a *Metrica* whose three books include one on geometrical and arithmetical techniques for calculating surface areas of a wide range of geometrical objects, one on calculating their volumes, and a third on methods for dividing up those objects in set proportions. The *Metrica* is a fascinating text in its own right for the way it blends the techniques and language associated with geometrical methods (usually abstract, general, and highly privileged) and those characteristic of arithmetical calculation methods (which by contrast are typically focused on specific concrete problems and are less privileged). It also served as the foundation for a centuries-long tradition of Greek metrological texts, many associated with Hero's name, like the *Geometrica*, *Geodaisia*, and *Stereometrica*. These texts incorporate some problems and techniques from the *Metrica* alongside a host of new types of problems, many of them creatively reworked and reorganized in the texts' several recensions.

Vitrac has made a detailed codicological study of the resulting "corpus" of metrological texts, updating previous studies by Hultsch and Heiberg, focusing more on the complexities of their collection and propagation rather than trying to establish a single authoritative text, as Heiberg of necessity did in pursuit of a definitive edition.[3] Amid the variations in structure, content, and apparent audience between the various texts of the metrological corpus, Vitrac nevertheless identifies some common features, like the inclusion of Euclidean-style defini-

2 For an excellent overview of these metrological texts with detailed consideration of questions about authorship, see Hero 2014, 429–533. On Hero's works more generally, see Cuomo 2002; Tybjerg 2003; 2004.

3 Vitrac 2010.

tions of the objects under study, tables of metrical units and equivalences including a wide range of geographical variations, and the assimilation of real-world objects to geometrical objects and diagrams. Vitrac notes further that the problems in the metrological texts other than the *Metrica* draw almost exclusively on the "algorithmic" tradition of practical mathematics rather than the demonstrative tradition of geometrical texts. And indeed, Hero's persistent focus on the practical applications of science and mathematics, even as he continues to engage with the "demonstrative" tradition in his own texts, probably encouraged his association with the metrological texts, which are overwhelmingly practical in their focus.

The practicality of the metrological texts might seem pedestrian compared to Archimedes' flights of logic or Apollonius's elegant curves, but the mathematics of the "real world" has its own kind of beauty. Gregory of Nazianzus eloquently praised the hexagonal precision of honeycombs, the complex webs woven by spiders, and the effortless flying formations of cranes, contrasting them with Euclid, whom he characterizes as "finding philosophy in nonexistent lines and exhausting himself in his demonstrations."[4] He critiques the efforts of geometers and tactical theorists as empty labor that blinds them to the order already present in the natural world. The metrological authors are, to be sure, concerned with the study of human artifacts like granaries, theaters, and taxation systems rather than the natural world. But one can imagine a comparable frustration with Euclid's "nonexistent lines" fueling their commitment to developing techniques to apprehend the concrete, the measurable, and the marketable. Metrological texts often bridge the gap between geometry and the "real world" representations featured in many technical texts by blending problems dealing with measuring purely geometrical objects with problems in measuring objects, like buildings or wells, that can be approximated by geometrical objects like cylinders or rectangular prisms. These problems can take various forms, from proofs to algorithmic problem-solving routines, with accompanying variations in their formal characteristics like forms of address, use of letter labels and numerical quantities, and the way the text interacts with visual elements like diagrams and tables.

But it is not only human artifacts as objects of study, but humans as learning *subjects*, that the metrological texts illuminate with particular clarity. Their focus on algorithmic problem-solving is in some ways conceptually rigid: one identifies the problem-type, chooses an appropriate algorithm for the type, and follows the steps of calculation. In practice, however, the surviving texts offer

4 Gregorius Nazianzenus, *De theologia* 25.

glimpses into the human behind the algorithm: their selection of problem-solving techniques, the diagrams, calculations, and other aids used to help them along the way, and even the errors and missteps that the journey to a solution may entail. Some of these errors are simple scribal or arithmetic errors, but others are conceptual errors that can, from a perspective focused on mathematical pedagogy, illuminate some aspects of the processes of learning in antiquity that are otherwise so difficult to piece together from the surviving evidence.

Those learning processes are captured still more vividly in the mathematical papyri, another class of texts dealing with practical mathematics. A long Egyptian tradition of mathematical papyri written first in hieroglyphics and later in demotic Egyptian were eventually augmented by Greek mathematical papyri beginning in the Hellenistic period. While the Greek papyri do include some fragments of geometrical texts like Euclid's *Elements*, both the Egyptian and Greek papyri more commonly featured problems familiar from the long Egyptian and Mesopotamian traditions of arithmetical problems focused on techniques for measuring and manipulating real-world objects.[5] Quite often these problems are framed as "model" problems for a technique, including a formulaic statement that similar problems may be solved with the same method. The whole population of mathematical papyri is of course quite diverse, including formal geometrical and algebraic texts, astronomical texts, and less formal problem collections that seem to have served a pedagogical purpose.[6] Most interesting here will be the tradition of papyri that seem for various reasons to have been designed for a teaching context.

In this paper I will compare the collections of metrological problems edited by Heiberg as the pseudo-Heronian *Stereometrica* to the problem-solving approaches taken in *P. Math.*, a mathematical papyrus recently published by Bagnall and Jones, likely dating from 4th-century Oxyrhynchus.[7] Diverse as these texts are, they are linked by common threads of problem types and algorithms, so I will begin by sketching out these links, emphasizing in particular their situations within particular cultures. I will then explore a few case studies of missteps in problem-solving in *P. Math.*, including conceptual and algorithmic errors with mentally assembling complex objects, and related struggles on the solver's part to visualize and diagram elements of his problems in a way that facilitates correct solution.

5 For wide-ranging analyses of these traditions, see Høyrup 1994; Imhausen 2016. For an excellent review of the papyrus evidence for Greek geometrical texts, see Fowler 1999, 204–217.

6 An overview of the mathematical papyri is found in Jones 2009.

7 Bagnall/Jones 2019.

Given that we know so little about the details of pedagogy of any kind in antiquity, but especially technical education such as the *P. Math.* solver seems to have been engaged in, it might seem fruitless to understand these errors as anything but the random vagaries of a novice student. However, I hope that I can extract some insight into the solver's experience both by comparing his missteps to more successful solutions of comparable problems in the differently organized *Stereometrica*, and also by engaging with contemporary scholarship on cognition and mathematics learning. Here I will give particular attention to the importance of what Davis calls problem-solving "frames" and the conceptual "framing errors" that manifest in some of the solutions in *P. Math.* and other texts.[8] Both the correct solutions and errors open up a picture of what Lave calls "cognition in practice," mathematics performed in and on a real world that offers pedagogical and conceptual opportunities quite different from "pure" mathematical exercises like those we associate in antiquity with Greek geometrical texts.[9]

2 Learning Mathematics

The aims and practices of ancient education remain, for the most part, tantalizingly out of reach, all the more so for the specific case of education in technical or scientific subjects. Cribiore touches on numeracy in her foundational study of ancient education, while Fowler considers the mathematics curriculum suggested by Plato's *Republic* alongside some surviving material evidence for geometry teaching in antiquity.[10] Nevertheless, more questions than answers remain about how mathematics was taught in antiquity and the differences that might have separated the teaching of geometry from "practical" mathematics education. The metrological texts in the "Heronian" corpus and several of the surviving mathematical papyri do seem to have been composed with pedagogical aims in mind, though the signals of these aims differ considerably between the two types of text.

The *Metrica* is a model for many of the later compilations of metrological problems, and it is structured quite rigorously in a way that facilitates learning from the ground up, not unlike Euclid's *Elements*. For example, the first book of

8 Davis 1984.

9 Lave 1988.

10 Cribiore 2001, 180–183; Fowler 1999, 103–151.

the *Metrica* is dedicated to techniques for measuring plane figures. Hero begins with the trivial case of the rectangle, and indeed this first passage seems to be as much a continuation of the argument from the preface in favor of the use of abstract "units" (*monades*) rather than concrete units like cubits or feet as it is about the rectangle itself. As the book proceeds, Hero goes on to deliver a series of highly standardized problems in finding the areas of equilateral polygons from the triangle to the dodecagon. Each of these problems is structured in the same way, beginning with the same formulaic language stipulating that each side of every polygon is 10 units. Each begins its solution with a geometrical construction that closely follows the "prototypical" Euclidean linguistic form.[11] Indeed, each of those constructions proceeds through steps familiar from Euclid, locating the center of the polygon's circumscribed circle (compare *Elements* III.1) and using that point to launch the triangles between center and edge that will allow Hero to demonstrate the proportional relationships between them (or, in the case of the enneagon and hendecagon, the diameters that define the right triangles used for that purpose).[12] All three books of the *Metrica* proceed similarly, beginning with simpler problems in each domain (plane geometry for book 1, solid geometry for book 2, and proportional division for book 3) and working toward more complex problems.

The later compilations of metrological problems adopt some of the same organizational strategies as the *Metrica*. The *Stereometrica*, at least in Heiberg's recension, often builds up stretches of problems based on the same basic geometric form, starting with simpler cases and working up to more complex variations. So, for example, a string of problems in the second book (which we will examine in more detail later) begins with a semicircular arch inscribed in a rectangular wall, goes on to a free-standing arch where the reader needs to consider the relationship between its inner and outer semicircular peripheries, and then combines those two forms of arch into a single construction. Other connections between problems are more complex; the series of problems above continues with a structure where arches made specifically of bricks are combined with rubble into a construction element, then moves on to a shell-shaped form (*konchē*) made of bricks, and eventually to a house whose roof has to be

11 On the formulaic language of Greek geometry, see especially Netz 1999, 9–11, 127–167. Further discussion of the stylistic features of this genre of text may be found at Asper 2001, 75–76.

12 The equilateral triangle is an exception to this pattern as Hero does not circumscribe a circle, but he does make use of another Euclidean mainstay, the "Pythagorean theorem" of *Elements* I.47.

covered in tiles. The internal structures of these later metrological compilations thus hint strongly at a context blending familiarity with the basics of the Euclidean "elements" with a strong emphasis on practical tasks. Vitrac suggests that while the metrological texts associated with Hero's name certainly suggest associations with technical education such as surveyors and architects may have received, they may also have drawn on a tradition of elementary geometrical education, perhaps offering a kind of geometrical analogue to Nicomachus's *Introduction to Arithmetic*.[13]

Yet the caveat above about reference to Heiberg's recension is an important one. The metrological texts associated with Hero's name (and to a lesser extent with Euclid's, as well as other figures like Didymus of Alexandria) reflect a dizzying codicological history of textual blending and reshuffling. The works constructed by modern editors Hultsch and Heiberg as the *Geometrica, De mensuris,* and *Stereometrica* in fact emerge from an array of manuscripts that collect various subsets of problems and tables of metrological conversions under different titles. To be sure, these manuscripts are far from being random assortments of individual problems; in many cases relatively stable clusters of problems and tables are found in multiple manuscripts, in the same order, and often under the same or similar title. Still, on the scale of the whole work there remains an immense amount of variation between the collections, and Hultsch and Heiberg's editions naturally tend to make sense of the varied problem collections they inherited by grouping similar problems together in the edition, even when this means creating a problem collection that does not entirely match any single manuscript.[14] Vitrac indeed views these editions effectively as novel creations by their editors, and so emphasizes the importance of analyzing the contents and entitled collections found in individual manuscripts.[15]

While it is difficult to assign a particular date to the formation of the collections found in this wealth of manuscripts, there are some indicators that in most cases this process took place sometime in the first few centuries CE. The *De mensuris*, for example, was already collected by the 9th century, as it appears in a branch of the Archimedean manuscript tradition whose earliest known witness dates from then.[16] Heiberg observes that the problems from the *Stereomet-*

13 Vitrac 2010.

14 Comparisons of the lists of problems in the texts I discuss here in the editions assembled by Hultsch and Heiberg can be found at Hero 1976, vii–viii.

15 Hero 2014, 430–448; Vitrac 2010.

16 Hero 2014, 435 n. 16. The manuscript in question is known to have been in the possession of Giorgio Valla in the fifteenth century, and went missing by the sixteenth, but not before having yielded several copies, many of which feature the *De mensuris*.

rica on vaults and arches, which will be a topic of particular interest here, seem to correspond to a work attributed to Hero for which Isidorus of Miletus composed a commentary in the 6th century.[17] Corcoran further suggests a tentative dating of the *Stereometrica* to the first half of the 5th century, based on a reference to a particular praetorian prefect's having fixed the weights per volume of commodities like bacon.[18] While other collections appear somewhat more unruly and resistant to dating, then, we can at least say that the work of constructing these compilations of practically-focused metrological problems was well underway in late antiquity. The resulting corpus, loose-limbed though it may be, is a rich store of practical problem-solving techniques, squarely located in the domain of the marketplace and construction site both by the kinds of problems the texts solve and by the metrological conversion tables themselves.

While the surviving Byzantine manuscripts reflect one set of processes of mathematical knowledge-construction dating at least partially to late antiquity, the mathematical papyri reflect another side of those construction processes. The majority of the surviving mathematical papyri draw on contexts of practical problem-solving comparable to those found in the "Heronian" metrological texts, and many seem to have functioned as tools for teaching and learning. Cuomo gives the example of a demotic papyrus from Hermopolis dating to the 3rd century BCE, which features a selection of arithmetical and geometrical problems framed largely as practical problems about measuring land or cloth, as well as some practice problems with common techniques like finding square roots. Cuomo argues that "a teaching context is suggested by direct appeals to the reader and by statements such as this: 'When another [add-fraction-to-them] (problem) is stated to you, it will be successful according to the model.'"[19] So while the problems in these papyri are solved for the specific case of the given sample numbers rather than in the entirely general manner characteristic of Greek geometry, those sample solutions are meant to serve as templates for solving similar problems encountered later.

Imhausen's comprehensive examination of Egyptian mathematical papyri suggests some similar conclusions. Her study of the major demotic mathematical papyri (Cairo, BM 10399, BM 10520, BM 10794, Carlsberg 30, Griffith I E7, and Heidelberg 663) explores both the mathematical techniques they exemplify and the way these problem collections are contextualized against a backdrop of "practical" problems. The scare quotes here reflect the complication that as

17 Hero 1976, xxxi.

18 Corcoran 1995, 380.

19 Cuomo 2001, 72.

Imhausen points out, several problems that appear quite practical at first glance turn out to be what she calls "suprautilitarian." This term refers to problems designed to showcase a mathematical technique rather than an actual guide to the physical processes one would undertake in carrying out the task described in the problem. For example, problem 8 in *P. Cairo* purports to deal with measuring out a piece of cloth of a given area and cutting it, preserving the total area while reducing the height by one cubit and increasing the width.[20] However, as Imhausen notes, the cut-off strip vanishes from the problem almost immediately, as the problem is actually focused on determining how broad the strip added to the width dimension would have to be in order to keep the area the same once a one-cubit strip was removed from the height. Obviously that same one-cubit strip cannot just be stuck back onto the width dimension; the answer to the problem tells you the required width of the addition, but not how to make it from the existing cut-off strip. So the problem is less practical than it might first have appeared, geared rather for the pedagogical exercise of calculating rectangular areas.

Among the likely Greek "pedagogical" papyri is *P. Mich.* [inv.] 4966, written on one side of a papyrus dated to the second century CE.[21] This document features a table of fractions (all with prime numbers as denominators) expressed as the sum of unit fractions, combined with a series of practical problems: arithmetical calculations framed as being about quantities of wheat, problems asking the reader to convert different amounts of money to copper or silver drachmai, calculations on areas of land, and so on. Smyly conjectured that the Akhmim mathematical papyrus (*P. Cair.* [inv.] 10758) dated to the seventh century CE and edited by Baillet was a school exercise book, as it consists of a set of division tables and a collection of "disconnected problems, with no method in their arrangement" whose solutions often include conceptual and methodological errors, to say nothing of frequent errors in Greek.[22]

P. Math. combines model business contracts, tables of metrological conversions, and a very diverse group of mathematical problems including calculations of areas of land, volume calculations on excavations and buildings, arithmetical problems about the total wages of different classes of workers working for various amounts of time, and several other kinds of problems. These are very often framed as "model" problems with a formulaic note ("this way for similar problems"), similar to the formula Cuomo observed in the Hermopolis papyrus.

20 Imhausen 2016, 193.

21 Boyaval 1976.

22 Baillet 1892; Smyly 1920.

Bagnall and Jones note that many of the *P. Math.* problems appear to reflect a formulaic “dialogue” of question and answer between teacher and student.[23] This “dialogue” was far from fluent in *P. Math.*, however; they note numerous orthographical and mathematical errors, leading them to conclude that “the codex belonged to a student in a school devoted to training business agents and similar professionals.”[24] These pedagogical mathematical papyri seem to have aimed at teaching a quite different skill set from the “scribal” education more typically associated with grammarians’ schools. Instead of grammar and orthography, *P. Math.* reflects an education focused on algorithms for making calculations and unit conversions, models for calculating daily wages, and drafting contracts.

A key skill set in this educational model was fluency with visual “information technologies” ranging from geometrical diagrams to tables of numbers meant to aid in calculation with fractions, multiplication, division, and so forth (indeed, some of the mathematical papyri consist solely of such tables).[25] The papyri that seem to have served a principally pedagogical function are particularly interesting because they include diagrams that played a role in the learning process. A fragmentary metrological text in a papyrus dated to the second century CE (*P. Corn.* inv. 69) features diagrams of two trapezoidal figures, one dissected into several polygons, whose sides are labeled with numbers given in the problems.[26]

P. Math. includes diagrams for most problems with a spatial component, usually labeled with numbers corresponding to quantities given in the problem, and often the result as well. The images in *P. Math.* are of several kinds; some are “diagrams” in the sense of spatially representing objects or quantities, while several of the problems are additionally separated from one another by decorative borders and drawings of palm fronds and ankhs. Diagrams of the first type will be of most interest here, and they too take several forms. Some depict bird’s-eye views of geometrical forms representing problem topics like the dimensions of fields, granaries, or holes in the ground. Others are slightly more complex representations of three-dimensional objects. A few tabulate step-by-step results of an arithmetic process or conceptual elements in another kind of problem-solving process. Finally, a few are simply baffling, like a curvilinear

23 Bagnall/Jones 2019, 23.
24 Bagnall/Jones 2019, 55.
25 On the broader history of tables of information in the Roman world (and in particular their relative rarity in most contexts), see Riggsby 2019, 42–82.
26 Taisbak/Bülow-Jacobsen 2003.

shape surmounted by a scribbled line that seems to represent a vaulted granary in problem *o1* (fig. 4).

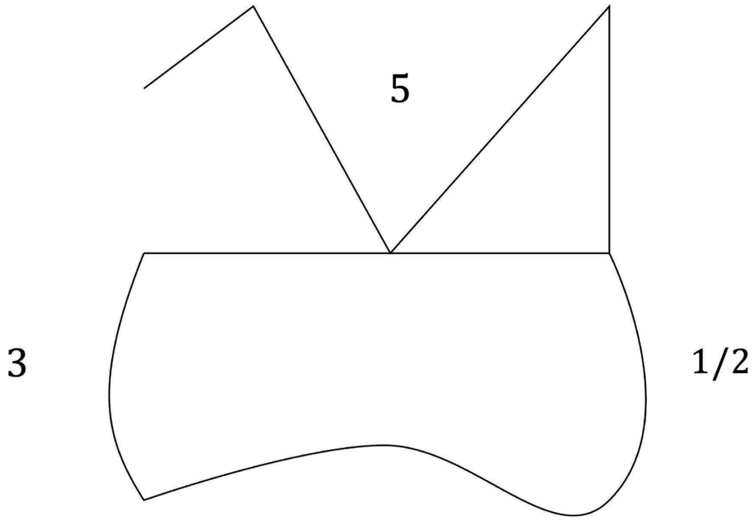

Fig. 4: "Granary" diagram from *P. Math.*, problem 01 (after Bagnall/Jones 2019).

Some medieval manuscripts also include informal diagrams added by readers at later stages of the composition process, but the papyri are particularly rich sources of these diagrams, which largely appear to have been produced spontaneously as part of the solving process, rather than being copied from formal exemplars. This latter feature in particular makes the diagrams in the papyri witnesses to a live learning process that is otherwise difficult to capture. Indeed, the importance of diagrams to Greek mathematics can hardly be overstated.[27] Geometrical proofs are linked at every stage to a letter-labeled diagram depicting every component referenced in the proof. In cases where formal geometrical proof is no longer the principal objective, diagrams take on a variety of other roles and forms. When technical texts in other genres like mechanics or surveying call upon them to represent objects in the world rather than purely geomet-

27 The foundational work on diagrams in Greek geometry is Netz 1999. Further studies that focus on the manuscript traditions of these diagrams include Saito 2012; 2018.

rical ones, they may adapt the systems of spatial representation and letter labeling to new ends.

Several different skills are involved in producing a mathematical diagram. Some amount of scribal skill and draftsmanship, possibly involving the use of compass and straightedge, is necessary to produce a clearly drawn diagram. Provided the diagram is not simply copied from an exemplar, competence in the "graphical languages" that might come into play is also required. For a table, this might mean an ability to distinguish headers from data in individual cells, and to keep the rows and columns properly aligned; for a geometrical diagram, this might include an understanding of the relative placement of letter-labeled points in the diagram. However, as Netz, Carman, and others have pointed out, manuscript diagrams are "underdetermined" with respect to the problem statement and do not reliably reflect relative lengths of line segments or arcs (so a triangle specified as isosceles might be represented as scalene), or even preserve easily assumed features such as line segments bounding a polygon (e.g., polygons in some manuscripts are represented as bounded by arcs or spiked triangles instead of straight lines).[28]

In addition to the skills required to produce *some* diagram of a polygon, a circle, and so forth, another set of skills serves to produce a "correct," or at any rate heuristically useful, diagram for the problem at hand. Van Garderen, Scheuermann, and Poch enumerate a set of "strands of diagram proficiency" for modern mathematics students. These include a conceptual understanding of how to use a diagram to solve a given problem, the procedural skill to generate an accurate diagrammatic representation of the situation in the problem, and the strategic ability to engage the diagram as a problem-solving tool.[29] Additionally, van Garderen et al. identify proficiencies in students' ability to explain how the diagram was used to solve the problem, and their belief in their ability to use the diagram appropriately. These latter two are obviously impossible to extract from the surviving ancient evidence, but clues to the first three can be found, and can yield some insight into the process of mathematical learning at play in the papyrus.

Cases where students of mathematics are not told what a diagram should look like are particularly revealing in this sense. Van Garderen, Scheuermann, and Jackson record the results of several experiments where students of different ability levels were asked to solve word problems with diagrams, but not told

28 On these features see Carman 2018; Netz 2020, 512, 521.

29 Van Garderen/Scheuermann/Poch 2014, 137.

what type of diagram to use.[30] The example below (fig. 5) shows two very different approaches, along with interviews that illuminate the reasoning process the students used. The second student's carefully counted divisions of the sandwich, and the accompanying strategy of making further subdivisions and counting those up, are worlds apart from the iconic depiction of several stick-figure students next to an assortment of sandwiches drawn by the first student, who ultimately resorts to guesswork when that graphical strategy fails.

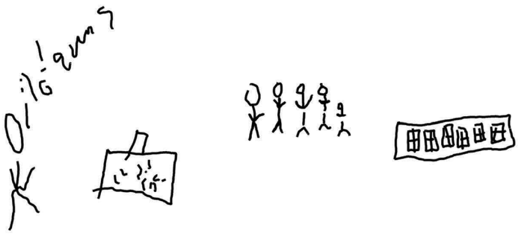

Fig. 5a: Figures 5a and 5b: Two different approaches to diagrammatic problem-solving, author's drawing after van Garderen *et al.* 2013.

Interview a:
I: All right, how did you get that answer? Tell me about that.
S: I don't know.
I: Where did the 10 come from?
S: People.
I: How did you use this picture then to help solve it? Tell me about that.
S: I don't know.
I: Tell me about what's this and what's this.
S: Those are the students and those are the sandwiches.
I: OK, and then what were you counting to get to 10?
S: I just guessed on the 10.

30 Van Garderen/Scheuermann/Jackson 2013.

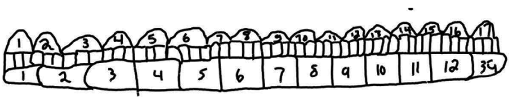

Fig. 5b: Figures 5a and 5b: Two different approaches to diagrammatic problem-solving; author's drawing, after van Garderen *et al.* 2013.

Interview b:
I: How did you solve this one?
S: I drew the sandwich. Then I divided it up into 12 ¾ feet long. I drew little lines above it and counted all the little fourths, all the way up to 17 sets of 3's.

The practically oriented problems found in *P. Math.* and other metrological texts are in many ways close analogues to the kinds of problems students were asked to solve in studies like van Garderen's. They engage the diagram in a very different way from geometrical texts, where the construction of the diagram is typically explained in the course of the proof. The metrological problems do not specify the drawing process in this way. Some are phrased in such a way that the appropriate diagram is obvious, like problem *f6* from *P. Math.*: "A right-angled (triangle) whose hypotenuse is 17. To find the other sides." Others allow for more latitude in selecting a diagram, like *o2*, which specifies the length of each side of a quadrilateral plot of land but does not call it a quadrilateral, accompanied by a diagram that is just a horizontal line with the measurements marked above, below, and on either side. Such a diagram might be viewed as incorrect for the analogous geometrical problem carried out on an abstract quadrilateral, but the simplified line diagram includes all the information needed to carry out the calculation. As it happens, the example dimensions given to carry out the calculation turn out to refer to a quadrilateral which is actually impossible to construct, a problem that could have been illuminated by a more faithful diagram. However, the same kind of breakdown between the problem's sample numbers and the geometrical object depicted also occurs elsewhere in the papyrus even where the diagrams are more robust.

Papyri like *P. Math* and problem collections like the metrological problems associated with Hero's name can thus be construed as valuable witnesses to how education in "practical mathematics" might have been constructed in late antiquity. Yet on their own the ancient texts leave many gaps in our understanding of how mathematical concepts might have been inculcated and practiced, and how this education might have worked on the learner's side. In addressing these mysteries we can call not only on scholarship on ancient pedagogy, where mathematics education is not particularly strongly represented, but also on investigations of modern mathematics pedagogy exploring how students grapple with a growing corpus of mathematical concepts, some more successfully than others.

A particularly lucid and influential study of mathematics learning is Davis's *Learning Mathematics*, which largely focuses on errors as evidence for how students learn mathematics. Davis argues that student errors often follow distinctive and regular patterns of their own, and explains many of these common errors as the result of selecting the wrong conceptual "frame" from the collection of frames students acquire in the course of their mathematics education. Davis uses the term "frame" flexibly to refer to different types of "knowledge representation structures."[31] While his particular approach takes an information-processing view of how the mind handles those structures, the principles of mathematics learning he invokes are flexible enough to suit other cognitive models as well, such as more embodiment or enaction-focused approaches.[32]

A second source of comparisons that will prove particularly useful here is Lave's *Cognition in Practice*, a groundbreaking study of how non-mathematicians perform mathematical tasks in everyday environments like the grocery store. Lave found that her experimental subjects typically performed quite poorly on a written test of their ability to make calculations. However, when they were observed doing everyday tasks like grocery shopping and meal preparation demanding those very same calculations, they performed with a very high degree of confidence and accuracy. Lave concludes that there is an important distinction between contextualized "math-in-practice" and "math conceived as a system of propositions and relations (a 'knowledge domain')."[33] In what follows, I will focus on three main lines of investigation: the ways mathematics learners seem to acquire mathematical concepts and apply them to new problems (appropriately or inappropriately), the relationship between abstract mathematical concepts and what Lave calls "cognition in practice," and the uses of diagrams in mathematics learning.

31 Davis 1984, 107.

32 On the role of embodiment and gesture in mathematics education, see for example Alibali/Nathan 2012; de Freitas/Sinclair 2012.

33 Lave 1988, 97.

3 Problems with Problem-Solving

Papyri like *P. Math.* contain a rich variety of types of errors. Some are simple scribal errors or departures from Greek orthographical conventions, which these papyri of course have in common with papyri of every genre. Some are the numerical equivalent of scribal errors: mistakes in the modified alphabetic system used to represent numbers in Greek mathematics. Other types of errors are more interesting, since conceptual errors offer another window into the student's learning process. The solver of *P. Math.*, while highly competent in some respects like unit conversions and arithmetical calculations, stumbles into a range of conceptual errors. These include failures to match up a problem with a diagram that illustrates features of the structure under investigation in a sensible way, inappropriate selection of algorithms for calculation, and confusion about the elements of a geometrical object. Other "errors" are not mathematical mistakes in and of themselves but rather common-sense breakdowns in choices of dimensions, yielding improbably tiny vineyards or granaries, or worse, structures that turn out to be impossible given the specified dimensions, e.g., of outer and inner perimeters and wall thickness. Even though these are not exactly errors, they do seem to be "precursors" to errors in the sense that they often lead to mistakes in constructing diagrams and performing calculations. That is, a breakdown between the solver's mental conception of the problem and a real-world object that can actually be pictured does seem to lead him into errors that he otherwise might not make.

Case study 1: Faults with Vaults

A particularly interesting conceptual error plagues two problems in *P. Math* with the same basic aim: to calculate the volume of a granary shaped as a rectangular building surmounted by a vaulted (*kamarōtos*) roof. In neither case is the form of the vault specified, though the default form (at least in mathematical teaching problems in the metrological collections) is the relatively mathematically simple case where the vault is a section of a circle. However, this is the least of the troubles the *P. Math.* solver encounters. In the first of these problems (*n4*), the solver multiplies the granary's length times its breadth, which yields the floor area. So far so good, but then he multiplies the depth by a dimension he calls the "vault (*kamara*)," and finally multiplies the two products by one another. The resulting product of the four dimensions is then converted from solid-cubit volume to grain measure in artabas using the standard conversion figure. But of course, the four-dimensional product of lengths in cubits is no longer in solid cubits: Bagnall and Jones delightfully suggest "hypercubits" as a name for this newly coined unit of measure.[34] The numerical result is an impossibly large 364,500 artabas.[35]

Perhaps the solver realized he had gone astray upon revealing this answer, because the next problem is framed just the same, though the dimensions are reduced: the length from 25 cubits in *n4* to 5 cubits in *o1*, the breadth from 15 cubits to 3, the depth from 16 cubits to 2, and the "vault" from 18 to 2. The problem-solving process is close to identical, but a little more deliberate: instead of multiplying the two pairs of numbers and then finding their product, the author first multiplies the length (misnamed "breadth") by the depth, then the result by the breadth, and finally that result by the "vault." Of course the result is once again nonsense, even after this second attempt with smaller (and thus perhaps more tractable) numbers: this tiny granary is calculated to hold 60 solid cubits, or about 9 cubic meters. This repetition of the initial error is common in problem-solvers even today; Davis notes from a study of student mathematical errors carried out by Erlwanger that "the malfunction occurs, as it were, at the same location in the cognitive machinery. *In nearly every case, a super-procedure selects the wrong sub-procedure*."[36] As in Erlwanger's study, the solver of *P. Math.* is stuck in the same faulty routine the second time he attempts the problem.

The solver's inappropriate introduction of the "vault" dimension thus renders the problem completely intractable. What was so appealing about that framework for solution that the solver attempted it not once, but twice? Bagnall and Jones consider that the solver may have had in mind the formula for calculating the area of a half-oval using the formula $A = 3w/4h$, but since the ¾ coefficient (or its unit-fraction equivalent) doesn't appear here, he certainly did not get far if that was his intent. To better understand where the solver of *P. Math* went wrong, we might search the metrological corpus for models of correct solutions. Hero's *Metrica* does discuss the measurement of vaults, but only briefly. *Metrica* II.12 proposes a method for measuring a washtub (or bathtub?) conceptualized as a slice of a spherical shell: a figure consisting of the space between two concentric spheres is sliced by two parallel planes, one defining the top of the tub, and the other the flat surface upon which it rests. Hero intro-

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34 Bagnall/Jones 2019, 149.

35 Converting this result to more familiar units is tricky as the value of the *artabē* could vary, but converting directly from the result in solid cubits, this volume would be around 15,500 cubic meters!

36 Davis 1984, 98. Italics in original.

duces II.13 with a retrospective look back at II.12, saying that now that the reader has encountered strategies for measuring conical, cylindrical, and spherical shapes, he can use the "tub" example as a model for how to perform calculations on vaults having any of those forms. However, he does not go on to calculate the space inside a cylindrical vault (or any other form), but rather proceeds with the process of measuring the torus. He mentions vaults again in II.15, where the topic is a cube containing two cylinders intersecting perpendicularly to yield a form which he says is useful for designing baths with windows or doors on all sides, or "places difficult to roof over with wood." He does not follow up further on this tantalizingly opaque description, however, and Acerbi and Vitrac note that while a comparable figure is mentioned in the preface to Archimedes' *Method*, the solution does not survive.[37]

The connection between vaults and tori at first appears contextual rather than mathematical, since Hero refers explicitly to the use of segments of tori as decorative elements on architectural columns. However, in measuring the volume of the torus (II.13, fig. 6) he appeals to a result relating a torus to a cylinder, which he credits to a lost work *On the torus* by a certain Dionysodorus. The torus in Dionysodorus's result is generated by translating the circle ΒΓΔΕ around the circle formed by looping the line segment ΑΒ to connect to itself. The cylinder he relates to this torus has axis ΗΘ and base radius ΕΘ. Finally, the proportional relationship Dionysidoros discovered is that the circle ΒΓΔΕ has the same ratio to half ΔΕΗΘ as the torus and cylinder defined above have to one another. A neat result indeed, but the *Metrica* then takes a puzzling turn. As is often the case in this text, the numerical "synthesis" where the dimensions and proportions are actually calculated follows the demonstrative "analysis" where they were introduced. The synthesis in this case does involve a cylinder, but the cylinder turns out to be not the one defined by Dionysodorus, but the one produced by unfurling the torus so that ΑΒ becomes a straight line rather than a circle. Vitrac and Acerbi suggest that the metrical procedure here is likely to be a later insertion, though it is not incorrect. Whatever the particular textual history of this proposition, the close association between tori and cylinders is clear.

37 Hero 2014, 293 n. 145. On the different ways this form is treated by Archimedes and the Chinese mathematician Liu Hui, see Netz 2018.

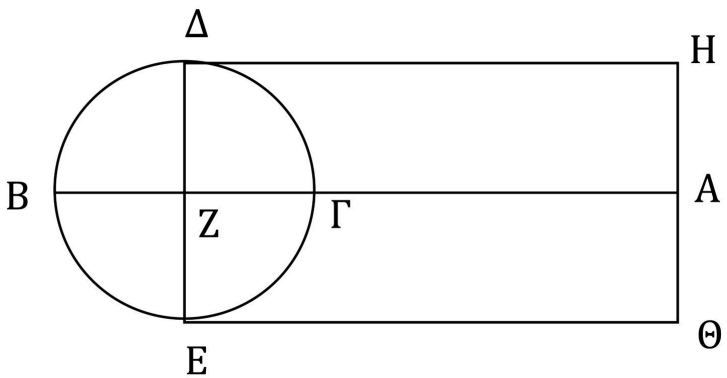

Fig. 6: Schematic depiction of the features of a torus from Hero *Metrica* II.13.

The next section of the *Metrica* (II.14) gets us one final step closer to the connection between the vault and the cylinder. This proposition offers a method for measuring the volume of a segment of a cylinder sliced by a plane through the center of its base, citing a result from Archimedes' *Method* that this segment will be one-sixth the volume of the rectangular prism with cross-section defined by the square circumscribing the cylinder's base and the same height (maybe more easily conceived of as length) as the cylinder. The diagram associated with this problem shows the cylinder mounted on the circumscribing square, in a way that immediately summons the image of a structure with a cylindrical vaulted roof. So in a roundabout way, Hero does relate the torus to the cylinder, a form that could be related to a vault, but the application to problems like the vaulted granary in *P. Math.* is hardly obvious.

However, in the texts of the metrological corpus compiled in the centuries following Hero, problems having to do with the measurement of vaults became more common as the focus on real-world objects grew stronger. Heiberg's edition of the *Stereometrica* includes two clusters of problems related to the calculation of the area or volume enclosed by vaults and arches. The more obviously relevant of these is found in the collection edited by Heiberg as the second book of the *Stereometrica*. Given the cautions mentioned above about the difficulties of creating an edition of the *Stereometrica*, I should begin with a word about how the problems I discuss here fit into that complicated manuscript tradition. All are found in the manuscript denoted S (*Codex Seragliensis* G.I.1 or *Constantinopolitanus Palatii Veteris* 1), which likely dates to the early tenth century, making it the earliest of the metrological manuscripts, and the only one to contain Hero's own *Metrica*.[38] Some are also found in the other major manuscripts that contained material found in Heiberg's edition of the work, and for the most part appear in the same order in all manuscripts.[39] This problem series is thus built on relatively firm foundations by the standards of the metrological corpus.

This cluster (2.28–45) begins by telling the reader how to construct a semicircular arch within a square framework, then how to find the area enclosed by the outer and inner semicircular perimeters of a free-standing arch, then combines the two structures in the form of a square framework enclosing an interior semicircular arch and surmounted by another. The next problem (32) introduces a new method for calculating the area under a semicircular arch, and then proceeds to describe a comparable process for finding the area under a "disproportionate (*apeulogos*)" arch, which is not clearly described in the problem (itself beset with scribal errors). Returning to more readily comprehensible objects, the author next integrates the method from the first part of problem 32 into a problem involving arches made of bricks which border the exterior of a segment of rubble wall. The next few problems continue with the now-established context of architectural construction while introducing a new shape, the "conch (*konchē*)": first constructed of bricks, then covered in mosaic tiles. Next the author returns to the vault. Let us study this problem more closely:

> To measure a vault whose enclosure is less than a semicircle, of which the base of the interior space is 14 feet, and the "front-wedges" [πρωτοσφῆνες; this term seems to refer to the thickness of the wall] on each side 2 feet, whose perpendicular in the interior space is 6 feet, and whose length is 15 feet. Do it like this: add the 14 feet of the interior space and the 6 of the perpendicular; the result is 20. Of this [take] the half; the result is 10. [Multiply] this by the 6; the result is 60. Again, add the 14 feet of the interior space and the "front-wedges" of 2 feet on each side; the result is 18. To these add the 6 of perpendicular of the interior space and the 2 feet; the result is 26. Of these [take] half; the result is 13. [Multiply] these by the whole height/extension (*anatasis*), by 8; the result is 104. Divide this by the 60 feet of the interior space; the remainder of the framework/foundation is 44 feet. Multiply this by the 15 feet of the length; the result is 660 feet. So large is the vault.[40]

We may immediately note how the author here has gone about calculating the cross-section of the vault: the formula $(d + h)/2 \times H$ (in this case, $(18 + 6)/\ 2 \times 8$). Just as in *P. Math.*, an incorrect formula has again been engaged to find this area. However, the erroneous formula here may point the way to the

38 For more details on this manuscript, see Hero 2014, 85–97; Lévy/Vitrac 2018, 190–192.
39 For a detailed discussion of how the problems are ordered in the manuscripts, see Hero 2014, 471–474.
40 [Hero] *Stereometrica* 2.37.

reasoning behind the error in *P. Math*, since the error here is slight — the author has forgotten to include a corrective factor of (1 + 1/21), which was previously introduced in problem 28. The same formula, or variants of it meant to apply to different cross-sections, is used several times on problems of this type in the *Stereometrica*, and it is plausible that it was a common formula for this kind of calculation in late antiquity.[41] The solver of *P. Math.* could have recalled that there was a formula for calculating the volume under a roughly semicircular vault that involved combining three different dimensions of the cross-section, including the height of the space under the vault, and multiplying those by the length to produce a volume. In Davis's terms, however, he inappropriately retrieved the "multiplication" rather than the "addition" frame in the course of carrying out that calculation, hence ending up with a granary occupying four dimensions rather than the more conventional three. The additional errors of failing to divide by half and omitting the (1 + 1/21) corrective here seem trivial by comparison.

Of course it is impossible actually to reconstruct the thought process of the *P. Math.* solver, particularly given the lack of a parallel, correctly solved problem in that text. But, in a sense, that lack is precisely the point: the structure and scale of the *Stereometrica* are such that the author can introduce new geometrical forms gradually, starting from simple shapes and building up to more complex variations and combinations. The reader of manuscript S has by this point been led carefully along a path where each step usually involves a fairly minor variation on what has come before. The same would have been true for readers of other sequences of problems in other versions of the manuscript, which typically group similar problems together, even if the particular groupings change from one manuscript to the next. The steps of the path are by no means flawlessly laid; the *Stereometrica*, including this sequence of chapters, contains a great many erroneous formulas. However, unlike the case of the comparatively short *P. Math.*, the lengthy Byzantine codices of metrological problems afford the opportunity to check a formula against a similar problem, and if a variation occurs (as in this case), the reader is provoked to compare the two and select what appears to be the correct algorithm rather than doubling down on an incorrect formula as in *P. Math.*

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41 Other ancient pedagogical contexts may furnish some comparable examples. Monika Amsler suggested in her comments to this paper that the remarkable stability of the exemplary rhetorical progymnasmata described by Theon, Aphthonius, Nicolaus, and other authors might reflect a similar case. On the pedagogical context and sources of these examples, see Webb 2009, 39–49.

The process of gradually accumulating related problem-solving techniques invites comparisons between recently acquired techniques and new ones, and provides opportunities for self-correction not unlike the problem-solving dialogues between students and teachers (or researchers) common to contemporary studies of mathematical problem-solving. Davis notes that in the studies he analyzes, when a teacher intervened by following up a student's retrieval of an incorrect procedural frame (e.g., "4*4=8," where the "addition" frame is inappropriately retrieved) by asking them the question that would have generated the wrong answer (in this case "what is 4+4?"), the student nearly always immediately corrected the previous wrong answer rather than simply answering the new question. Even though many of the problems in *P. Math.* are framed as though they represented a question-and-answer dialogue between teacher and student, the actual learning process does not seem to have involved a similar one-on-one dialogue where the student could have been alerted to his incorrect problem-solving "frames." While the *Stereometrica* is clearly even further removed from the classroom context of genuine question-and-answer dialogue, its more robust structure, with a greater number of similar problems gathered together, could have facilitated a cognitive process in the reader more like what Davis posits for the contemporary student who corrects her response thanks to an "interesting phenomenon of perception, control or short-term memory" fostered by the dialogue with the teacher.[42]

Besides the greater availability of "checks" on incorrect formulas and problem-solving techniques in the *Stereometrica* compared to *P. Math.*, we should not neglect to mention the value of the *Stereometrica*'s chains of problems related to a particular context, in this case the construction of buildings. *P. Math.* is one of the largest collections of mathematical problems in a papyrus, but even so it is a short text relative to the Heronian metrological works. Within a span of relatively few problems which aim to address a very wide-range of problem-solving techniques, it is simply not possible to follow the *Stereometrica*'s strategy of gathering together a large number of problems focused on vaults and arches and incrementally building complications onto a relatively firm problem-solving foundation. By the time the reader works through a series of problems like Heiberg's *Stereometrica* 2.28–44, she has built up a fairly solid mental picture of the walls, archways, peristyles, and roofs in those problems. Those images may not be elegantly drawn, as is the case for the Vatican manuscript (fig. 7), but simply seeing how the geometrical objects fit into the more complex structures allows the reader to anchor their problem-solving process in their

42 Davis 1984, 100.

lived experience. By mentally constructing the more complex structures step by step, the reader is less likely to make a grave conceptual error like the extra dimension attributed to the granary by the *P. Math.* solver. As we will see, this kind of mental slippage in picturing the object under study is a recurrent problem in *P. Math.*

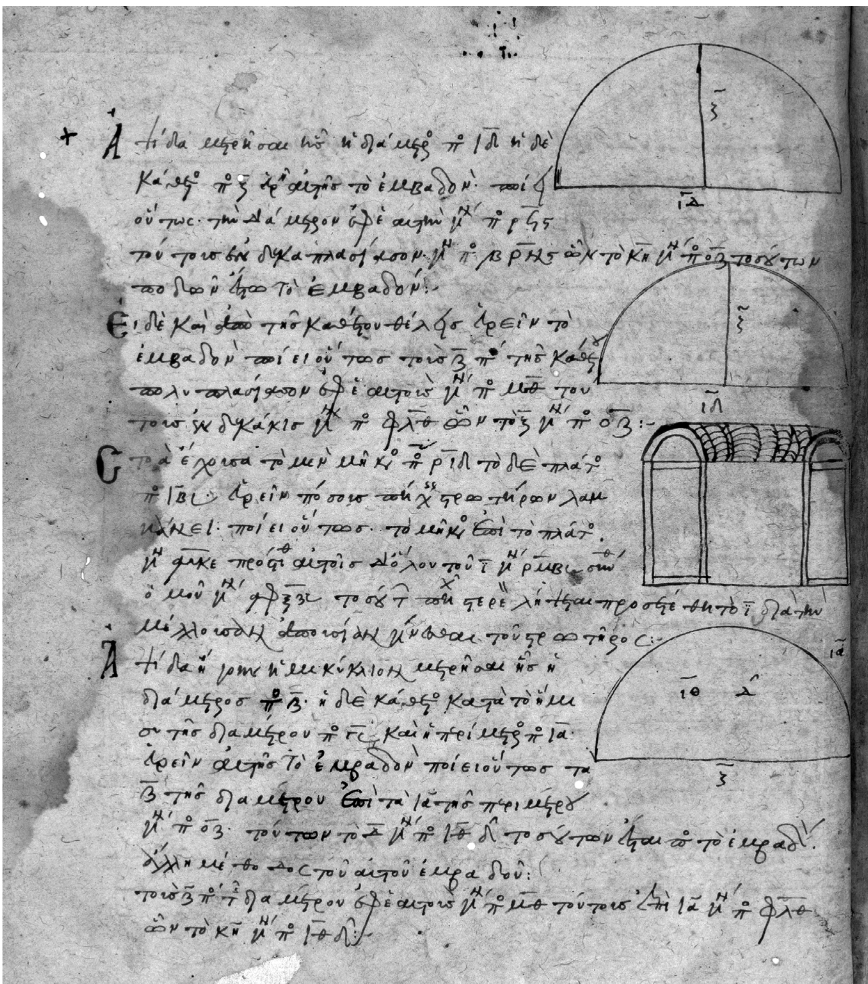

Fig. 7: Stereometrica text and images from Vat. gr. 215, fol. 9r.

Case study 2: Reality Breaks

Some of the "errors" in *P. Math.* are not really mathematical errors at all, but rather involve a breakdown between the numerical values stipulated in the problem and the real-world objects they allegedly correspond to. The vagueness of some of the problem statements can make this assessment difficult, as when problem *b5* calculates the volume of a form called a "quadrangular trapezoid," an odd nomenclature that turns out to refer to an extremely elongated trapezoidal prism. Certainly it is true that a trapezoidal prism (like any polygonal prism) has quadrangular sides, but this is an odd way to frame the shape given that prisms are usually just identified by their cross section. The trouble the solver seems to have giving the shape a name appears to mirror a difficulty in visualizing its dimensions; Bagnall and Jones point out the improbability of a "rod-like object" 48 cubits in length, just 10 cubits wide, and only 5 fingers and 2 fingers on the parallel trapezoidal surfaces. Other forms are improbably small rather than too large; the vaulted granary with a 5x3x2-cubit rectangular base surmounted by a 2-cubit high vault was already mentioned above. Trivial as these errors may seem, they suggest a breakdown in common sense at some point in the process from devising the problem in the first place, to copying it down, to attempting a solution, which makes it difficult for the solver to perform a "reality check" on whether the numbers make sense. In the case of problem *b5* (the "quadrangular trapezoid"), the solver has so much trouble envisioning the object under consideration that he fails to perform the necessary unit conversion from cubits to fingers, a rare mistake for him.

Problem *b3* demands the solver picture a more complicated object: a tower (*porgos*) with "substructures (*krēpidai*)," where the aim is to calculate how many bricks the tower contains. This kind of calculation has a long tradition in both the Mesopotamian and Egyptian traditions.[43] In this case, the tower is described in an unnecessarily confusing way and is quite difficult to picture. First, the tower: we are given the outer and inner perimeters, as well as the thickness of the walls, but not the shape. In fact, the given dimensions (where the thickness of the wall is equal to the difference between the two perimeters) render the tower impossible to construct, whatever its form (in the likely event it was a rectangle, for example, the wall's thickness would be ¼ the difference between the perimeters). Still, the solver chooses an appropriate frame for this calculation, recognizing that the wall's cross-section can be dissected into four trape-

43 On "brick numbers" in Babylonian mathematics, see Friberg 2007, 89, 93–95, 169–174.

zoids and calculating the walls' volume using the algorithm for calculating the volume of a trapezoidal prism.

So far so good, but what about the "substructures"? One might reasonably imagine that the tower would be supported on some kind of rectangular prism, likely hollow on the inside like the tower. The solver seems to have had this idea in mind in stating the problem: "the length of the rectangle 10 [cubits], breadth 8 cubits." However, alert readers will already have noticed that the substructure does not have enough given dimensions, and the solver runs into trouble because of this later in the problem when he goes to calculate the number of bricks it is made of.

Recognizing the missing dimension of thickness, he spontaneously introduces a factor of 2 cubits into his calculations on the substructures, likely importing it from the "tower" section of the problem and assuming incorrectly that they have the same thickness as the tower walls. However, rather than calculating as though he were picturing an 8x10x2 cubit rectangular prism (which would be a rickety support indeed for the tower but at least makes some sense spatially), he lets the plural "substructures" in the problem statement lead him to satisfy the "rectangular prism" schema in a different way. He posits two "substructures" of 8x2 and 10x2 cubits respectively, and then adds together the size of these two-dimensional bases. But since the brick calculations require a three-dimensional structure, the schema is lacking a dimension. The seeds of the solver's catastrophic response to this lack have already been sown in the form of a numerical error at the very start of the 10*8 multiplication: "Likewise also of the rectangle, [the length, 10] times 60. The result is 20." Obviously, 60 is an error for 2, given that the result is 20. But that 2 itself has just been introduced by the solver's need to fill in one missing element in the dimensional schema, and his decision to import it from the "tower" frame. And now at this later point in the problem, he repeats the strategy: he again draws the missing dimension from the "tower" frame, this time multiplying the two bases each by the height of the *tower* (60 cubits), rather than any given measure corresponding to the "substructures." The solver then adds up the volumes of the two tower-height "substructure" sheets, finding the sum slightly larger than the volume of the tower. He then converts the solid cubits to bricks with a factor of 48, yielding the answer 213,120 bricks.

As is often the case in *P. Math.*, the solver's arithmetic is unproblematic, but this tower is built on the shakiest of conceptual foundations. The solver is not disturbed by the fact that the "substructures" represent the bulk of the construction because he has not visualized the tower-substructure complex in enough detail to have a sense for whether this should be the case. The diagram, drawn on the verso of B, emphasizes this confusion: it has suffered considerable damage but appears to have consisted simply of a trapezoid with some horizontal lines across it, marked with “8” at the bottom (the length of one of the “substructures”), “60” at the side (the height of the tower) and a “54” that figures nowhere in the problem, like the trapezoidal shape itself. However, a trapezoid marked with a “54” dimension *did* feature in the problem illustrated at the top of the recto of B. Given the conceptual confusion the solver experienced with the basic task of picturing the tower/substructure complex, it would not be surprising if he borrowed the last stable conceptual image he had of a mathematical object, the trapezoid from problem *b1* (= *a5*), to fill the gap.

As in the case of the vaults described above, the total collapse of the solver’s mental image of the tower/substructure complex seems likely to stem in part from the structure of the text itself. Problem *b3* occurs early in the text, following a series of problems on trapezoids and trapezoidal solids. The only exceptions to this pattern are a unit-fractions exercise and the problem that immediately precedes it, a trivially simple calculation of the area of a square field. It would not be surprising, then, if the solver carried the “trapezoid” conceptual frame from these prior problems into problem *b3*. Moreover, he used the “trapezoid” frame correctly in dissecting the tower’s cross-section into four trapezoids to find its area. It is only when he attempts to determine the volume of the pathologically underdetermined “substructures” that the strength with which this conceptual frame has been lodged in his mind leads him astray, producing the incorrectly visualized “substructures” and the nonsensical diagram. Had problem *b3* been preceded instead by a series of problems on rectangular prisms, the solver might have performed better even if the tower’s structure was still underdetermined in the problem. Much like the case of the vaults, a text like the *Stereometrica* creates a hedge against these lapses in visual comprehension by building up the components of more complex structures more gradually.

Students attempting to solve mathematical problems today often indicate comparable difficulties with common-sense checks on numerical calculations. Davis ascribes several such errors to inappropriate retrieval of mathematical “frames.” One student, asked to divide 6 into 3606, arrived at the incorrect answer of 61.[44] When the interviewer, attempting to spark a self-correction, then asked the student to divide 6 into 366 (which would in fact yield 61), the student was not surprised, but accepted the identity of the answers as correct, since she had learned that “adding zero doesn’t change [the answer]” and “zero means

44 Davis 1984, 199–200.

nothing." The student's inappropriate retrieval of the "addition" frame for the 0 in the division problem, coupled with the semantic drift in the phrase "zero means nothing," overwhelmed what should have been their common-sense reaction that these very different dividends should not yield the same quotient.

Davis explores another kind of "framing" error that speaks more specifically to the breakdown between numerical and real-world objects, centered on problems of the following type.[45] When students (even engineering, physics, or math students) asked to put a phrase like "there are six times as many students as professors" into the form of an equation involving the variables S and P, they overwhelmingly wrote the equation erroneously as "6S=P." The error (called the Rosnick-Clement phenomenon after the psychologists who first studied it) affects people regardless of their level of mathematical skill and experience, and even trumps real-world experience with the objects named in the problem. One might think that the error simply stems from writing the S and P in the order in which "students" and "professors" were encountered in the problem statement, but switching the word order had no perceptible effect on the results. The students, familiar with the fact that students outnumber professors (usually by a factor of considerably more than 6) should have been able to perform the common-sense check that multiplying the number of students by 6 should not yield the number of professors.

The very same error persisted in different formulations of the problem. A student familiar with the recipe for vinaigrette (which calls for more oil than vinegar), who was even given the correct formula 3V = O for the proportions, still managed to talk herself into reversing the proportions, drawing pictures with the reversed proportions, and finally insisting that the formula meant the dressing contained more vinegar than oil. Even a mathematically adept physics major, given a formula describing the proportions of people in England and China, persuaded himself to reverse the meanings of "E" and "C" in the formula rather than renounce his incorrect problem-solving frame. These students are not unlike the *P. Math.* solver, who finds the "trapezoid" frame so firmly lodged in mind after several repetitions that it is difficult to break even when encountering the apparently trapezoid-free tower-complex problem. Like the students Davis describes, the *P. Math.* solver seems to talk himself into a mindset where an absurd fusion of a tower supported by thin rectangular sheets as tall as the tower itself seems like a plausible construction. Davis makes the key point that solvers were not plugging in sample numbers to check that their formulas made

45 Davis 1984, 111–123.

sense, which would have called attention to the breakdown between reality and number here.

How can Davis's characterization of these errors help us better understand what is going on in *P. Math.*? Let us consider a few more troubled mappings from reality onto mathematical problems in the text. Problem *f2* posits three granaries containing various amounts of wheat (200, 300, and 400 artabas respectively). The problem's drama begins when "someone came in and mixed them up. To find 630 artabas. We proceed as follows." But proceed with what, exactly — what is this mysterious "630 artabas" about, and why did someone come and mix up the grain from different granaries? One might surmise that they came not only to mix up the grain, but also to steal it, and took 630 artabas out of the original 900. And indeed that is quite correct, as Bagnall and Jones point out a parallel in *P.Cair. cat* 19758 (problems 47–49), where someone came along, mixed up the grain from each granary, and then stole some grain, so the problem is to determine how much of the stolen wheat originally came from each granary. So in this case, rather than failing to correctly picture a realistic *object*, as in the case of the very long "quadrangular trapezoid" of *b5*, here the solver has failed to imagine a plausible *process* for the grain-theft.

A comparable case occurs at *c2*, where "someone loaded on a boat, from the granary half, and for the taxes one-third, and for the pay of the donkey-driver one twelfth, and there remained on the boat 50 artabas of wheat." Now, as Bagnall and Jones note, the activities described in the problem make sense in the context of *unloading* a boat rather than loading one. The problem continues: we must find the solution to this indeterminate equation by adding the fractions listed, finding that their sum is 1/12 less than a unit, and calculating that since the 50 artabas remaining are 1/12 of the original amount, the original amount on the boat (before unloading) was 600 artabas. Yet the solver frames the solution as "the boat will hold 600 artabas of wheat" — again, suggesting he has a framework of loading rather than unloading in mind. Still, even though the "loading" frame does not make sense with the tax calculation and so forth, the solver seems to have in mind a robust and stable context of activity within that framework, and carries it out correctly. So this case differs from *f2* in that the solver is able to picture the process correctly, implausible though that process may be in a strictly real-world context.

Lave warns that typical studies of mathematical cognition presuppose that all action is preceded by a separate "structuring" step, which leads to a misunderstanding of the relationship between experience and strategic thought:

> The view is consistent with an emphasis on thought distanced from experience as the canonical form of human experience to be investigated, but it is not compatible with the everyday math practices just described, nor with a theory of practice.[46]

Lave's experimental study of ordinary people (referred to in the text as "just plain folks" or "jpfs") involved first giving them a formal mathematical test, and then following them around the grocery store, watching and questioning them as they select, buy, and prepare food. Most of the people in her study performed quite poorly on the math test but extremely well on practical tasks like determining which size jar of mayonnaise was the better buy (averaging 98% accuracy). As she points out, "98% accuracy in the supermarket is practically error-free arithmetic, and belies the image of the hapless jpf failing cognitive challenges in an everyday world."[47] Lave's study found

> not a single significant correlation between frequency of calculation in supermarket, and scores on math test, multiple choice test, or number facts. There is a significant correlation between weight and volume facts (but not length) and frequency of calculation in the supermarket.[48]

The high correlation between shoppers' mastery of facts helping them make weight or volume conversions and their fluency of calculation in the supermarket is of particular interest in the context of the ancient metrological texts. *P. Math.* and other mathematical papyri, as well as texts like the *Stereometrica*, provide the reader with an astonishing array of conversion mechanisms for length, area, volume, and weight. These include universal conversions that could work for any substance as well as more specific conversions. For example, *Stereometrica* 2.54 includes standards set by a praetorian prefect named Modestus for converting fresh or stored barley from *xestai* to cubic feet, bacon from cubic feet to *litrai*, and so on.

And in fact the writer of *P. Math.* functions very fluidly with those conversions, nearly always performing them correctly. His fluency with conversions and other arithmetical tasks suggests an orientation much like Lave's jpfs. He is really very competent within a known framework of mathematical action, but he often struggles to retrieve and apply the correct problem-solving frame for new kinds of problems. These problems are exacerbated by the relative lack of problem-solving supports in the text such as are found in the *Stereometrica*,

46 Lave 1988, 130.
47 Lave 1988, 58.
48 Lave 1988, 57.

notably chains of problems on similar geometric forms, generally built up from simpler to the more complex. Crucially, in the *Stereometrica* these chains are often focused on a common real-world application like buildings (or vessels containing water, ships, theaters full of seats, etc.), encouraging the reader to build up a relatively robust mental picture of the object in question that may help them perform a "reality check" on the results. Lave's study offers an important intervention in conventional studies of mathematical problem-solving as divorced from the "real world":

> I propose to address cognition and culture and their various entailments at different levels of social analysis. Among other things, this requires a broadening of the terms of analysis to reflect the claim that the 'person,' including the person thinking, is constituted in relation with other aspects of the lived-in world.[49]

These texts (*P. Math.* and other papyri focused on practical problem-solving, as well as the "Heronian" metrological texts) make their meaning not merely from recording the arithmetical structures of calculating algorithms, but more profoundly from fitting those algorithms into a concrete and populated world. Crucially, the *Stereometrica* constructs knowledge by arranging problems into clusters that replicate the construction of objects in the world itself: from abstract arches to arches of bricks, to walls, to roofed-over buildings. These clusters would seem to constitute a pedagogical process in their own right, affording the reader a familiar and grounded problem-solving environment more like the grocery store where Lave's subjects thrived than the abstract math test she administered her "hapless jpfs" beforehand.

4 Conclusion

The corpus of Greek metrological texts developed after the *Metrica*, like the *Geometrica* and *Stereometrica* as well as papyri containing related "practical mathematics" problems, owe a great deal to the much older Egyptian and Mesopotamian traditions of arithmetical problems focused on techniques for measuring and manipulating real-world objects. Not only do the techniques and content of the problems differ between the "demonstrative" geometrical and "algorithmic" arithmetic traditions, but so does the very language in which those problems are couched. Greek geometers in general hewed remarkably

49 Lave 1988, 180.

closely to a canonical model of proof-writing. In this model, proofs proceed through a series of formulaic steps couched in equally formulaic language: from enunciation (*protasis*) to conclusion (*sumperasma*), all progress evidently made not by the author's own hand but a shadowy entity adumbrated by the third-person passive imperatives that are such a peculiar verbal marker of these texts. Fowler and Taisbak memorably characterize this mysterious actor as the "Helping Hand," which "is always there first to see that things are done and to keep the operations free from contamination by our mortal fingers."[50] The "Helping Hand" is an effective rhetorical tool, suggesting that the work of mathematical proof happens in a domain far removed from human fallibility. The geometrical proof comes prepackaged and sanitized, as it were, without any indication of the trials and errors the text's author doubtless experienced in its discovery.[51] By contrast, texts in the arithmetical tradition typically frame the mathematical activities they recount as direct instructions to the reader, as an active account of steps being taken by the author, or both.

This difference results from the different generic expectations of the two traditions, to be sure, but it also resonates powerfully with questions of "knowledge construction." Texts in the geometrical tradition, with their impersonal, passive constructions mediated by the "Helping Hand," are worlds away from the first- and second-person constructions of the arithmetical tradition. The difference between a problem framed as being solved through an impersonal and permanently valid demonstrative act on the one hand, and a problem framed as being solved through a person's selecting an algorithm, carrying it out, and inviting the reader to do the same (explicitly or implicitly) on the other, also reframes the meaning of "error." An error in a geometrical demonstration might be seen as a fatal flaw because of the impression of impersonal eternity the generically imposed form of the solution creates. But in a problem framed as a personal adventure in problem-solving, errors and other idiosyncrasies have a value of their own as witnesses to that peculiar personal experience.

When the *P. Math.* solver accidentally imagines a four-dimensional granary, or collapses the area diagram of a field down to just one dimension, those choices open up a window — however hazy — into a living process of "knowledge con-

50 Fowler and Taisbak 1999, 362.

51 This is not to say that Greek geometrical texts are devoid of personality, of course; Netz describes some of the authorial personae developed in Greek geometrical texts at Netz 2013. Still, he describes a textual tradition where "the mathematician's results cannot be otherwise" (225) even if the writing style varies from author to author, which is clearly not the case here. For additional considerations of authorial personae in mathematical commentaries, see Asper 2019.

struction" that works by fits and starts as students of different ability levels grapple with mathematical concepts and their real-world analogues. The same may be said for the compilers of the metrological problems that crystallized into the *Stereometrica* and related texts: their choices to bring a certain set of problems and tables together represent a form of "knowledge construction" in its own right. That construction process roots the invariant principles of mathematics in a gloriously varied world, where the mathematical system's users navigate a complex landscape of culturally determined units of measurement and assemble geometrical forms into concrete constructions, building meaning from the progress from a semicircular arch to a roofed building every bit as much as they build meaning from a growing collection of algorithms for calculation.

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