

Inapproximability of the Partition Function for the Antiferromagnetic Ising and Hard-Core Models

ANDREAS GALANIS^{1†}, DANIEL ŠTEFANKOVIČ^{2‡}
and ERIC VIGODA^{1§}

¹ School of Computer Science, Georgia Institute of Technology, Atlanta, GA 30332, USA
(email: andreas.galanis@cs.ox.ac.uk, vigoda@cc.gatech.edu)

² Department of Computer Science, University of Rochester, Rochester, NY 14627, USA
(email: stefanko@cs.rochester.edu)

Recent inapproximability results of Sly (2010), together with an approximation algorithm presented by Weitz (2006), establish a beautiful picture of the computational complexity of approximating the partition function of the hard-core model. Let $\lambda_c(\mathbb{T}_\Delta)$ denote the critical activity for the hard-model on the infinite Δ -regular tree. Weitz presented an FPTAS for the partition function when $\lambda < \lambda_c(\mathbb{T}_\Delta)$ for graphs with constant maximum degree Δ . In contrast, Sly showed that for all $\Delta \geq 3$, there exists $\varepsilon_\Delta > 0$ such that (unless $\text{RP} = \text{NP}$) there is no FPRAS for approximating the partition function on graphs of maximum degree Δ for activities λ satisfying $\lambda_c(\mathbb{T}_\Delta) < \lambda < \lambda_c(\mathbb{T}_\Delta) + \varepsilon_\Delta$.

We prove that a similar phenomenon holds for the antiferromagnetic Ising model. Sinclair, Srivastava and Thurley (2014) extended Weitz's approach to the antiferromagnetic Ising model, yielding an FPTAS for the partition function for all graphs of constant maximum degree Δ when the parameters of the model lie in the uniqueness region of the infinite Δ -regular tree. We prove the complementary result for the antiferromagnetic Ising model without external field, namely, that unless $\text{RP} = \text{NP}$, for all $\Delta \geq 3$, there is no FPRAS for approximating the partition function on graphs of maximum degree Δ when the inverse temperature lies in the non-uniqueness region of the infinite tree \mathbb{T}_Δ . Our proof works by relating certain second moment calculations for random Δ -regular bipartite graphs to the tree recursions used to establish the critical points on the infinite tree.

1. Introduction

A remarkable computational transition has recently been established for the complexity of approximating the partition function of the hard-core model. Amazingly, this computational transition coincides exactly with the statistical physics phase transition on infinite Δ -regular trees. In this work we prove that a similar phenomenon holds for a much more general class of two-spin models.

Our general setup is two-spin systems on an input graph $G = (V, E)$ with maximum degree Δ . We follow the setup of several related previous works [12, 26, 18]. Configurations of the system are assignments $\sigma : V \rightarrow \{-1, +1\}$. For a given configuration $\sigma \in \{-1, +1\}^V$, let $n_-(\sigma)$ denote the number of vertices assigned -1 , let $m^-(\sigma)$ be the number of edges with both endpoints assigned -1 and $m^+(\sigma)$ the number of edges with both endpoints assigned $+1$.

There are three non-negative parameters B_1, B_2 and λ , where B_1 is the edge activity for $(-, -)$ edges, B_2 is the edge activity for $(+, +)$ edges, and λ is the vertex activity (or external field). A configuration $\sigma \in \{-1, +1\}^V$ has weight

$$w_G(\sigma) = \lambda^{n_-(\sigma)} B_1^{m^-(\sigma)} B_2^{m^+(\sigma)}.$$

The Gibbs distribution $\mu_G(\cdot)$ is over the set $\{-1, +1\}^V$, where $\mu_G(\sigma) = w_G(\sigma)/Z$ and where $Z = Z_G(B_1, B_2, \lambda)$ is a normalizing factor, known as the partition function, defined as

$$Z := \sum_{\sigma \in \{-1, +1\}^V} w_G(\sigma) = \sum_{\sigma \in \{-1, +1\}^V} \lambda^{n_-(\sigma)} B_1^{m^-(\sigma)} B_2^{m^+(\sigma)}.$$

Let us point out a few examples of two-spin models that are of particular interest. Setting $B_1 = 0$ and $B_2 = 1$, the only configurations with positive weight in the Gibbs distribution induce independent sets of G . This case is known as the *hard-core model* with fugacity λ . Setting $B_1 = B_2 = B$, one recovers the Ising model. The case $\lambda = 1$ is the Ising model without external field. When $B < 1$ it is the antiferromagnetic Ising model, and when $B \geq 1$ it is the ferromagnetic Ising model. In general, when $B_1 B_2 < 1$ the model is called *antiferromagnetic*, and when $B_1 B_2 \geq 1$ it is ferromagnetic.

The Ising and hard-core models (and, more generally, spin models) have been studied thoroughly in various contexts, especially in statistical physics. Their interpretation as idealized models of microscopic interaction within a body have nurtured research on understanding their macroscopic behaviour on appropriate underlying graph structures, such as the infinite grid or the infinite triangular lattice. The intriguing questions arising have made an impact on other fields as well, most notably probability and computer science. For illuminating accounts of such directions, we refer the reader to [21] and [29].

In this paper, we focus on the computational complexity of approximating the partition function in two-spin models. A long series of works, which we only touch upon later in this introduction, have identified the connection of the problem with decay of correlation properties in Gibbs measures of the infinite Δ -regular tree. In statistical physics terms, this is known as the phase transition on the Bethe lattice. To briefly sketch this important concept, let \mathbb{T}_Δ denote the infinite Δ -regular tree. When the parameters of the two-spin model lie in the so-called uniqueness region of \mathbb{T}_Δ , for finite complete trees of height h , the influence on the root from fixing a configuration on the leaves dies off in the limit

as the height goes to infinity. In sharp contrast, in the non-uniqueness region there exist configurations of the leaves for which their influence on the root persists in the limit. For convenience, we often refer to the uniqueness region, omitting the reference to \mathbb{T}_Δ .

The uniqueness/non-uniqueness regions can be determined rather easily on \mathbb{T}_Δ (for other infinite graphs this is far from easy; see [3] and [25] for recent advances in the two-dimensional grid). For the hard-core model, Kelly [17] showed that non-uniqueness holds if and only if

$$\lambda > \lambda_c(\mathbb{T}_\Delta) := \frac{(\Delta - 1)^{\Delta-1}}{(\Delta - 2)^\Delta}.$$

For the antiferromagnetic Ising model without external field, non-uniqueness holds if and only if

$$B < B_c(\mathbb{T}_\Delta) := \frac{\Delta - 2}{\Delta}$$

(see *e.g.* [26]). For general antiferromagnetic two-spin models with soft constraints (*i.e.*, $B_1 B_2 > 0$), non-uniqueness holds if and only if $\sqrt{B_1 B_2} < (\Delta - 2)/\Delta$ and $\lambda \in (\lambda_1, \lambda_2)$ for some critical values $\lambda_1(\Delta, B_1, B_2)$, $\lambda_2(\Delta, B_1, B_2)$ (see *e.g.* [18]).

Before stating our results, it will be instructive to review some known results from the literature. When $\lambda = 1$, the partition function is the number of independent sets in G , and this quantity is #P-complete to compute exactly [30], even when $\Delta = 3$ [13]. For the Ising model, once again, exact computation of the partition function for graphs of maximum degree Δ is #P-complete, even for the ferromagnetic model [16]. Hence, the focus is on the computational complexity of approximating the partition function.

This is where the phase transition on \mathbb{T}_Δ comes into play. Specifically, for the hard-core model, a deep connection has recently been established between uniqueness/non-uniqueness and the complexity of approximating the partition function. Namely, for graphs of constant maximum degree Δ , Weitz [31] presented a beautiful FPTAS for approximating the partition function when $\lambda < \lambda_c(\mathbb{T}_\Delta)$. On the other hand, Sly [27] presented a clever reduction which proves that for every $\Delta \geq 3$ there exists $\varepsilon_\Delta > 0$ such that it is NP-hard (unless $\text{RP} = \text{NP}$) to approximate the partition function for graphs of maximum degree Δ for any λ where $\lambda_c(\mathbb{T}_\Delta) < \lambda < \lambda_c(\mathbb{T}_\Delta) + \varepsilon_\Delta$.

It was believed that Sly's inapproximability result should hold for all $\lambda > \lambda_c(\mathbb{T}_\Delta)$. However, Sly's work utilized results of Mossel, Weitz and Wormald [24] which showed that for a random Δ -regular bipartite graph, an independent set chosen from the Gibbs distribution is 'unbalanced' with high probability for λ under the same condition as in Sly's result.

Mossel, Weitz and Wormald [24] used a technically complicated second moment argument that is characterized in [27] as a 'technical tour-de-force'. In [8] we extended the results of [24] to all $\lambda > \lambda_c(\mathbb{T}_\Delta)$ for $\Delta = 3$ and $\Delta \geq 6$. However, that work built upon [24] and had an even more complicated analysis. In this work, as a by-product of the approach we devise for coping with the Ising model, we can prove inapproximability for the hard-core model for all $\lambda > \lambda_c(\mathbb{T}_\Delta)$ for $\Delta = 3, 4, 5$, and our proof is simpler than in [8]. This settles the picture for the hard-core model, and also proves Conjecture 1.2 of [24].

Theorem 1.1. *Unless $\text{NP} = \text{RP}$, for the hard-core model, for all $\Delta \geq 3$, for all λ in the non-uniqueness region of the infinite tree \mathbb{T}_Δ , there does not exist an FPRAS for the partition function at activity λ for graphs of maximum degree at most Δ .*

Our main focus in this paper is to address whether the above phenomenon for the hard-core model occurs for other models. In particular, our goal is to address for two-spin antiferromagnetic models whether the computational complexity of approximating the partition function for graphs with maximum degree Δ undergoes a transition that coincides exactly with the uniqueness/non-uniqueness phase transition on the infinite tree \mathbb{T}_Δ .

Let us first review existing results in the literature. For the ferromagnetic Ising model, Jerrum and Sinclair [16] presented an FPRAS for all graphs, for all $B \geq 1$ where $B = B_1 = B_2$. For the antiferromagnetic Ising model, for general graphs there does not exist an FPRAS for $B < 1$ [16], unless $\text{NP} = \text{RP}$. For constant Δ , Sinclair, Srivastava and Thurley [26] extend Weitz's approach to the antiferromagnetic Ising model (see also [33, 25, 18] for other results on the Ising model), yielding in the uniqueness region an FPTAS for the partition function of graphs with maximum degree Δ .

We prove for the antiferromagnetic Ising model without external field that for all B in the non-uniqueness region of the infinite tree there does not exist an FPRAS for the antiferromagnetic Ising model.

Theorem 1.2. *Unless $\text{NP} = \text{RP}$, for the antiferromagnetic Ising model without external field, for all $\Delta \geq 3$, for all B in the non-uniqueness region of the infinite tree \mathbb{T}_Δ , there does not exist an FPRAS for the partition function at inverse temperature B for graphs of maximum degree at most Δ .*

The hard-core model and Ising model are the two most well-studied examples of two-spin systems. For ferromagnetic two-spin systems with no external field, that is, $B_1 B_2 > 1$ and $\lambda = 1$, Goldberg, Jerrum and Paterson [12] presented an FPRAS for the partition function for any graph. For any antiferromagnetic two-spin model, for constant Δ , an FPTAS for the partition function was obtained by Sinclair, Srivastava and Thurley [26] for Δ -regular graphs in the uniqueness region of the infinite tree \mathbb{T}_Δ , and by Li, Lu and Yin [18] for graphs of maximum degree Δ in the intersection of the uniqueness regions for infinite trees \mathbb{T}_d for $d \leq \Delta$. For antiferromagnetic two-spin models we obtain a complementary inapproximability result in the non-uniqueness region. Our results do not reach the uniqueness/non-uniqueness threshold in general.

Theorem 1.3. *Unless $\text{NP} = \text{RP}$, for all $\Delta \geq 3$, for all B_1, B_2, λ that lie in the non-uniqueness region of the infinite tree \mathbb{T}_Δ and*

$$\sqrt{B_1 B_2} \geq \frac{\sqrt{\Delta - 1} - 1}{\sqrt{\Delta - 1} + 1},$$

there does not exist an FPRAS for the partition function at parameters B_1, B_2, λ for graphs of maximum degree at most Δ .

1.1. Independent results of Sly and Sun

In an independent and simultaneous work, Sly and Sun [28] obtained closely related results. They prove inapproximability of the partition function in the case of the hard-core model and the antiferromagnetic Ising model. Their result for the Ising model also covers the case of an external field, which then extends to general two-spin antiferromagnetic models [26, 28].

The main technical result in our proof is a second moment argument to analyse the partition function for the Ising model on random Δ -regular bipartite graphs, as outlined in the following section (Section 2). As a consequence we can estimate the partition function within any arbitrarily small polynomial factor, which allows us to use the reduction in [27]. In contrast, Sly and Sun's approach builds upon the recent interpolation scheme of Dembo, Montanari and Sun [6] to analyse the logarithm of the partition function, which yields estimates of the partition function within an arbitrarily small exponential factor. To get their NP-hardness results, they use a modification of the approach appearing in [27] which allows us to use a constant-sized gadget.

1.2. Further work

Since the original publication of this work on arXiv [9] there has been further progress on this topic. In [10] we present a general approach for analysing the second moment of spin systems on random regular bipartite graphs using induced matrix norms. As a by-product, we obtain hardness results for all two-spin antiferromagnetic systems in the tree non-uniqueness region as in [28], and in addition hardness results for approximately counting k -colourings for even $k < \Delta$ and for the antiferromagnetic Potts model in the conjectured tree non-uniqueness region.

We point out that the present high-level approach to analysing the second moment is different to that in [10]. Here we give an analysis of the critical points of the second moment in the case of the antiferromagnetic Ising model and the hard-core model; this analysis is partly possible because the spins take binary values. As a consequence, in certain cases we obtain stronger results than those needed for the desired hardness results (see for example the discussion in Section 5.4). In contrast, in [10] the approach is targeted to the maximizers of the second moment (which is sufficient for the desired hardness results) and connecting them to the maximizers of the first moment.

We should also remark that, in [10], the gadget, which is a random regular bipartite graph, has fixed size as in [28]. In contrast, the gadget studied here, which is Sly's original gadget in [27], allows us to quantify certain properties in terms of its size (see Lemma 6.2), which is desirable in approximation-preserving reductions. For example, the results in this paper combined with those in [10] were used in [5] to prove #BIS-hardness on bipartite graphs for two-spin antiferromagnetic systems in the tree non-uniqueness region.

2. Proof outline

The main element of the proof of Theorem 1.3 (and similarly the proofs of Theorems 1.1 and 1.2) is to analyse random Δ -regular bipartite graphs to show a certain bimodality as in [24] for the hard-core model. From there the same reduction of [27] applies with some

non-trivial modifications in the analysis. Hence, in some sense our main technical result is to show that in random Δ -regular bipartite graphs, under the conditions of Theorem 1.3, a configuration selected from the Gibbs distribution is ‘unbalanced’ with high probability.

To formally state our results, let us introduce some notation. Let $\mathcal{G}(n, \Delta)$ denote the probability distribution over bipartite graphs with $n + n$ vertices formed by taking the union of Δ random perfect matchings. We will denote the two sides of the bipartition of the graphs as V_1, V_2 . Strictly speaking, this distribution is over bipartite multigraphs. However, since our results hold asymptotically almost surely (a.a.s.) over $\mathcal{G}(n, \Delta)$, as noted in [24], by contiguity arguments they also hold a.a.s. for the uniform distribution over bipartite Δ -regular graphs. For a complete account of contiguity, we refer the reader to [15, Chapter 9].

For a graph $G \sim \mathcal{G}(n, \Delta)$ and a given configuration $\sigma \in \{-1, +1\}^V$, let $N_-(\sigma)$ denote the set of vertices assigned -1 and let $N_+(\sigma)$ be the set of vertices assigned $+1$. For $\alpha, \beta \geq 0$, let

$$\Sigma^{\alpha, \beta} = \{\sigma \in \{-1, +1\}^V \mid |N_-(\sigma) \cap V_1| = \alpha n, |N_-(\sigma) \cap V_2| = \beta n\},$$

that is, α (resp. β) is the fraction of the vertices in V_1 (resp. V_2) whose spin is -1 . Let Σ^{Bal} denote the set of all balanced configurations, and for $\rho > 0$, let Σ^ρ be the set of ρn -unbalanced configurations. Formally,

$$\Sigma^{\text{Bal}} = \bigcup_{\alpha} \Sigma^{\alpha, \alpha}, \quad \Sigma^\rho = \bigcup_{|\alpha - \beta| \geq \rho} \Sigma^{\alpha, \beta}.$$

The following theorem, which is proved later in this section, establishes that in the Gibbs distribution of a random Δ -regular bipartite graph, balanced configurations have exponentially small measure.

Theorem 2.1. *Under the hypotheses of Theorem 1.3, there exist constants $a > 1$ and $\rho > 0$, such that asymptotically almost surely, for a graph G sampled from $\mathcal{G}(n, \Delta)$, the Gibbs distribution μ_G satisfies*

$$\mu_G(\Sigma^{\text{Bal}}) \leq a^{-n} \cdot \mu_G(\Sigma^\rho).$$

Therefore, the Glauber dynamics is torpidly mixing.

While Theorem 2.1 is interesting in its own right, our inapproximability results require a far more precise quantification of the bimodality than that given in Theorem 2.1. Nevertheless, it is instructive to outline the proof of Theorem 2.1, since it will allow us to introduce the key ingredients that are needed for the hardness results.

Namely, to establish Theorem 2.1, we use a second moment approach as in [24]. For $G \sim \mathcal{G}(n, \Delta)$ we analyse the random variable $Z_G^{\alpha, \beta}$, where

$$Z_G^{\alpha, \beta} = \sum_{\sigma \in \Sigma^{\alpha, \beta}} w_G(\sigma),$$

for some well-suited values of α, β . In particular, the α, β will be selected to maximize the first moment of $Z_G^{\alpha, \beta}$. One of the key ingredients in obtaining Theorem 2.1 is that the

maximum of the first moment occurs for $\alpha \neq \beta$. More generally, this is true for general antiferromagnetic two-spin systems in the non-uniqueness region of the infinite tree \mathbb{T}_Δ (see the upcoming Lemma 2.2). The proof of this fact is based on an explicit connection with three specific Gibbs measures in the infinite tree \mathbb{T}_Δ which are invariant under parity-preserving transformations. We next introduce these relevant Gibbs measures, deferring a more technical discussion to Section 3.

Let B_1, B_2, λ specify an antiferromagnetic two-spin system. There exists a unique translation-invariant Gibbs measure μ^* on \mathbb{T}_Δ , known as the free measure, whose marginal probability for the root ρ being assigned spin -1 will be denoted by $p^* := \mu^*(\sigma_\rho = -1)$ (the exponent denotes the boundary condition; $*$ refers to the free boundary). In the non-uniqueness region, there also exist two semi-translation-invariant measures μ^+ and μ^- , corresponding to the ‘even’ and ‘odd’ boundary conditions, respectively. These measures can be obtained by conditioning the leaves on level 2ℓ (resp. $2\ell + 1$) of the tree to have spin -1 and taking the weak limit as $\ell \rightarrow \infty$. We will denote the marginals for the root being assigned the spin -1 by $p^+ := \mu^+(\sigma_\rho)$, $p^- := \mu^-(\sigma_\rho)$. In the non-uniqueness region of \mathbb{T}_Δ , it holds that $\mu^* \neq \mu^\pm$; in fact, it holds that $p^- < p^* < p^+$.

The following lemma illustrates the relevance of p^-, p^*, p^+ in our arguments.

Lemma 2.2. *Let $\Delta \geq 3$. For the distribution $\mathcal{G}(n, \Delta)$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\mathcal{G}} [Z_G^{\alpha, \beta}]$$

is maximized for:

- (1) $(\alpha, \beta) = (p^*, p^*)$ when B_1, B_2, λ are in the uniqueness region of the infinite tree \mathbb{T}_Δ ,
- (2) $(\alpha, \beta) = (p^\pm, p^\mp)$ when B_1, B_2, λ are in the non-uniqueness region of the infinite tree \mathbb{T}_Δ .

Note that Lemma 2.2 already yields a weak form of a bimodality in the Gibbs distribution on random Δ -regular graphs: in the non-uniqueness region, the configurations with the largest contribution are unbalanced in expectation. Since $Z_G^{\alpha, \beta}$ is typically exponential in n , Markov’s inequality implies an upper bound on the number of balanced configurations, which holds with high probability over the choice $G \sim \mathcal{G}(n, \Delta)$.

However, to get tight results, we need a strong form of this bimodality. To do this, we look more carefully at the random variable $Z_G^{\alpha, \beta}$ for $(\alpha, \beta) = (p^\pm, p^\mp)$. In particular, suppose that we are able to show that $Z_G^{p^\pm, p^\mp}$ is within a polynomial multiplicative factor from its expectation. Lemma 2.2 would then yield that, for $\varepsilon > 0$ and all α', β' satisfying $\|(\alpha', \beta') - (p^\pm, p^\mp)\| \geq \varepsilon$, $Z_G^{\alpha', \beta'}$ is exponentially smaller than $Z_G^{p^\pm, p^\mp}$. Using arguments in [7] and [24], one can then easily deduce Theorem 2.1.

We will formalize the above argument shortly. Prior to that, let us remark that the same idea underlies the reduction used by Sly [27] (which we will use to obtain Theorems 1.1, 1.2 and 1.3). Namely, define the phase of a configuration of G to be the bipartition which has the largest number of vertices assigned -1 . With high probability, a configuration sampled from μ_G will be from the set Σ^{p^\pm, p^\mp} (roughly) and hence the phase will take binary values. Informally, this allows us to view G as a Boolean gadget (the precise construction of the gadget used by Sly [27] is given in Section 6). Consider now an

arbitrary graph H and replace each vertex of H with a (distinct) copy of the graph G . Sly showed how to encode the edges of H (i.e., make connections between the copies of G), so that the final graph has maximum degree Δ and its partition function is dominated by configurations where the phases of the copies of G correspond to a maximum cut of H . The quantification of this scheme involves hard work, but this has already been done in [27] (certain calculations do require rather lengthy modifications to account for general antiferromagnetic two-spin systems; see Section 6 for details).

Let us now return to the technical core of the argument, which requires analysis of the second moment of $Z_G^{\alpha,\beta}$ for $(\alpha, \beta) = (p^\pm, p^\mp)$. By symmetry, we may clearly focus on $(\alpha, \beta) = (p^+, p^-)$. Pick two configurations σ_1, σ_2 from the set Σ^{p^+, p^-} (not necessarily distinct). We need variables γ, δ that capture the overlap of the configurations σ_1, σ_2 . Formally, for $\gamma, \delta \geq 0$, let

$$\Sigma_{\gamma, \delta}^{\alpha, \beta} = \{(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in \Sigma^{\alpha, \beta}, |N_-(\sigma_1) \cap N_-(\sigma_2) \cap V_1| = \gamma n, \\ |N_-(\sigma_1) \cap N_-(\sigma_2) \cap V_2| = \delta n\},$$

and

$$Y_G^{\gamma, \delta} = \sum_{(\sigma_1, \sigma_2) \in \Sigma_{\gamma, \delta}^{\alpha, \beta}} w_G(\sigma_1) w_G(\sigma_2).$$

We will study $Y_G^{\gamma, \delta}$ for $(\alpha, \beta) = (p^+, p^-)$, so this is why we dropped the dependence on α, β . Observe that $\mathbb{E}_G[Y_G^{\gamma, \delta}]$ is the contribution to the second moment $\mathbb{E}_G[(Z_G^{\alpha, \beta})^2]$ coming from pairs of configurations in $\Sigma^{\alpha, \beta}$ with overlap $\gamma n, \delta n$.

In our case, for the second moment approach to succeed we need the largest contribution to the second moment to come from ‘uncorrelated’ pairs of configurations. This is captured by the following condition.

Condition 2.3. For $(\alpha, \beta) = (p^+, p^-)$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_G[Y_G^{\gamma, \delta}]$$

is maximized at $\gamma = \alpha^2, \delta = \beta^2$.

Lemma 2.4. Under the hypotheses of Theorem 1.3, Condition 2.3 holds.

Lemma 2.5. For the antiferromagnetic Ising model with $\lambda = 1$, $\Delta = 3$ and B in the non-uniqueness region of \mathbb{T}_Δ , Condition 2.3 holds.

Lemma 2.6. For the hard-core model, $\Delta = 3, 4, 5$ and λ in the non-uniqueness region of \mathbb{T}_Δ , Condition 2.3 holds.

The innocuous statements of Lemmas 2.4, 2.5, and 2.6 are rather misleading. Namely, proving that uncorrelated pairs of random variables ‘dominate’ the second moment is a standard complication of the second moment approach in a variety of settings; see for example [1, 2, 24]. To highlight the technical difficulties, the second moment typically has

a number of other extra variables that need to be considered and those are usually treated by sophisticated analysis arguments. We escape this paradigm in the following sense: our arguments try to explain the success of the second moment by relating it to the intrinsic nature of the problem at hand (in our context, uniqueness on the infinite tree). To stress the advantages of such a scheme, it is possibly the foundation of a more generic method to translate uniqueness proofs to similar second moment arguments.

In the hard-core model, due to the hard constraints, only one extra variable (besides γ, δ) was needed and it could be expressed explicitly in terms of γ, δ , so that, in fact, one had a two-variable optimization. Even with this seemingly simple setup and despite the variety and complexity of approaches appearing in [24], [27] and [8], certain regions for $\Delta = 4, 5$ remained open. In the general two-spin model the second moment is more complex than in the hard-core model in the sense that the soft constraints cause the number of extra variables to increase to nine. To make matters even worse, one cannot get rid of those extra variables in some easy algebraic way, since such a scheme reduces to a sixth-order polynomial equation in terms of γ, δ . These obstacles are overcome in the proof of Lemma 2.4 by a clean argument: the analysis of the second moment is reduced to certain tree recursions on \mathbb{T}_Δ , allowing for a short inequality argument.

After establishing Condition 2.3, we obtain for $(\alpha, \beta) = (p^\pm, p^\mp)$ that

$$\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha,\beta})^2] \approx \mathbb{E}_{\mathcal{G}}[Y_G^{\alpha^2,\beta^2}].$$

At this point, we are in a position to study the ratio $\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha,\beta})^2]/(\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}])^2$. This is a technical computation and requires an asymptotic tight approximation of sums by appropriate Gaussian integrals. While the asymptotic value of the ratio is a constant, as in [24], the constant is greater than 1.

Lemma 2.7. *When Condition 2.3 is true, for $(\alpha, \beta) = (p^\pm, p^\mp)$, it holds that*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha,\beta})^2]}{(\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}])^2} = (1 - \omega^2)^{-(\Delta-1)/2} (1 - (\Delta - 1)^2 \omega^2)^{-1/2},$$

where ω is given by (3.4).

Remark. The formula given in the lemma is valid for the hard-core model as well ($B_1 = 0, B_2 = 1$), agreeing with the statement of [24, Theorem 3.3].

The fact that the ratio of the second moment to the first moment squared converges to a constant greater than 1 does not allow us to get a.a.s. results using the standard second moment method. This can be dealt with by the so-called small subgraph conditioning method [32, 15]. The consequence of applying the small subgraph conditioning method is the following result (proved in Section 8.1).

Lemma 2.8. *When Condition 2.3 is true, for $(\alpha, \beta) = (p^\pm, p^\mp)$, it holds asymptotically almost surely over the graph $G \sim \mathcal{G}(n, \Delta)$ that*

$$Z_G^{\alpha,\beta} \geq \frac{1}{n} \mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}].$$

We next conclude Theorem 2.1.

Proof of Theorem 2.1. Lemmas 2.2 and 2.8 establish the necessary ingredients to carry out the argument in [24, Proof of Theorem 2.2] (see also [8, Proofs of Theorems 2 and 3]). The torpid mixing of Glauber dynamics follows immediately by a conductance argument, using [7, Claim 2.3]. \square

The outline of the rest of the paper is as follows. In Section 3 we derive certain properties for the infinite tree which are used throughout the remainder of the paper. We then look at the first and second moments in Section 4. In Section 5 we look at the optimization problems for the first and second moments and prove Lemmas 2.2 and 2.4. In Section 6 we give the proof of our inapproximability results (Theorems 1.1, 1.2 and 1.3) based on Sly's reduction for the hard-core model.

In Section 7 we analyse the ratio of the second moment to the first moment squared for random Δ -regular graphs (Lemma 2.7) and for the modified graph used in Sly's reduction. In Section 8 we apply the small subgraph conditioning method to get a.a.s. results (over the choice of the random graph), where we also prove Lemma 2.8. Finally, in Section 9 we give the proofs of some remaining lemmas, including Lemmas 2.5 and 2.6.

3. Extremal measures on the infinite tree for two-spin models

In Section 2 we defined the measures μ^*, μ^\pm on \mathbb{T}_Δ which will be of interest to us. In our context, the phase transition on the infinite $(\Delta - 1)$ -ary tree $\hat{\mathbb{T}}_\Delta$ rooted at ρ will also be of interest (note that $\hat{\mathbb{T}}_\Delta$ differs from \mathbb{T}_Δ only in that the root has degree $\Delta - 1$). The uniqueness/non-uniqueness regions for $\hat{\mathbb{T}}_\Delta$ coincide with those of \mathbb{T}_Δ , albeit with different occupation probabilities for the root in the respective free and fixed boundary measures.

We let $\hat{\mu}^\pm, \hat{\mu}^*$ denote the measures on $\hat{\mathbb{T}}_\Delta$ which are the analogues of the measures μ^\pm, μ^* on \mathbb{T}_Δ , respectively. Recall that we use $+, -$ to denote the even and odd boundary conditions, and $*$ for the free boundary condition. We will let q^\pm, q^* denote the marginal probabilities of the root being assigned -1 in each of these measures on $\hat{\mathbb{T}}_\Delta$, that is, $q^\pm = \hat{\mu}^\pm(\sigma_\rho = -1)$ and $q^* = \hat{\mu}^*(\sigma_\rho = -1)$. Once again, it can easily be proved that $q^- < q^* < q^+$ using (anti)monotonicity arguments.

A well-known and interesting property that the measures μ^\pm (and $\hat{\mu}^\pm$) possess is that of extremality. Namely, in the non-uniqueness region they cannot be written as non-trivial convex combinations of other Gibbs measures. In this particular setting, this follows from (and can be interpreted as) the fact that the even and odd boundaries force the maximal bias on the probability that the root ρ is assigned the spin -1 as the height of the tree goes to infinity. In other words, for every Gibbs measure $\mu \neq \mu^\pm$ on \mathbb{T}_Δ , we have $p^- < \mu(\sigma_\rho = -1) < p^+$. For a more complete account of extremality, we refer the reader to [11].

In this section we state some further properties for the densities q^\pm, q^* and p^\pm, p^* , which we are going to utilize throughout the text. Standard tree recursions for Gibbs measures

(see e.g. [26]) establish that

$$\frac{q^\pm}{1 - q^\pm} = \lambda \left(\frac{B_1 \cdot \frac{q^\mp}{1 - q^\mp} + 1}{\frac{q^\mp}{1 - q^\mp} + B_2} \right)^{\Delta-1}, \quad \frac{q^*}{1 - q^*} = \lambda \left(\frac{B_1 \cdot \frac{q^*}{1 - q^*} + 1}{\frac{q^*}{1 - q^*} + B_2} \right)^{\Delta-1}. \tag{3.1}$$

We define separate expressions for

$$\frac{q^\pm}{1 - q^\pm}, \quad \frac{q^*}{1 - q^*},$$

that is,

$$Q^+ = \frac{q^+}{1 - q^+}, \quad Q^- = \frac{q^-}{1 - q^-}, \quad Q^* = \frac{q^*}{1 - q^*}. \tag{3.2}$$

The following lemma summarizes the properties of the occupation densities q^+, q^-, q^* which we are going to utilize. Its rather folklore proof is a straightforward analysis of appropriate tree recursions, and is omitted.

Lemma 3.1. *For $\Delta \geq 3$ and $B_1, B_2, \lambda > 0$, consider the system of equations*

$$x = \lambda \left(\frac{B_1 y + 1}{y + B_2} \right)^{\Delta-1}, \quad y = \lambda \left(\frac{B_1 x + 1}{x + B_2} \right)^{\Delta-1}, \tag{3.3}$$

with $x, y \geq 0$.

- (1) *In the uniqueness region of $\hat{\mathbb{T}}_\Delta$, the only solution to (3.3) is (Q^*, Q^*) .*
- (2) *In the non-uniqueness region of $\hat{\mathbb{T}}_\Delta$, (3.3) has exactly three solutions which are given by (Q^\pm, Q^\mp) and (Q^*, Q^*) . It holds that $Q^- < Q^* < Q^+$.*

Proof. See, for example, [20, Section 6.2]. □

We also define the quantities ω, ω^* , which will emerge throughout the paper (and will be motivated shortly):

$$\omega := \frac{(1 - B_1 B_2)^2 Q^+ Q^-}{(B_2 + Q^-)(B_2 + Q^+)(1 + B_1 Q^-)(1 + B_2 Q^+)}, \quad \omega^* := \frac{(1 - B_1 B_2)^2 (Q^*)^2}{(B_2 + Q^*)^2 (1 + B_1 Q^*)^2}. \tag{3.4}$$

Note that ω^* is obtained from ω after identifying Q^\pm with Q^* . The following lemma for ω, ω^* will be crucial for proving some optimality conditions required to establish Lemma 2.2 and Condition 2.3. Its proof is given in Section 9.1.

Lemma 3.2. *Let $\Delta \geq 3$. For (B_1, B_2, λ) in the non-uniqueness region of $\hat{\mathbb{T}}_\Delta$, it holds that*

$$(\Delta - 1)^2 \omega < 1 < (\Delta - 1)^2 \omega^*.$$

Let us explain the seemingly mysterious inequality in Lemma 3.2. While not obvious, it is related to the two-step recurrence on the infinite tree $\hat{\mathbb{T}}_\Delta$. The two-step recurrence is obtained by the composition of two successive one-step recursences. In the non-uniqueness region, the fixed points of the two-step recurrence are exactly q^\pm, q^* (this is basically

Lemma 3.1). The inequality in Lemma 3.2 establishes, after some calculations, that the fixed point q^+ (resp. q^-) is an attractor of the two-step recurrence in $\hat{\mu}^+$ (resp. $\hat{\mu}^-$), while the fixed point q^* is repulsive. More specifically, the derivative of the two-step function at q^\pm is equal to $(\Delta - 1)^2\omega$, and similarly at q^* it is equal to $(\Delta - 1)^2\omega^*$. This property will be used to show that the measures $\hat{\mu}^\pm$ satisfy a strong form of non-reconstruction with exponential decay of correlation. We should also note that the analysis in [26] (see also [18]) establishes the following criterion for uniqueness, which we reformulate for our purposes: uniqueness holds on $\hat{\mathbb{T}}_\Delta$ if and only if $(\Delta - 1)^2\omega^* \leq 1$.

Finally, let us translate the above to obtain a handle on p^\pm, p^* . Using a slight modification of the tree recursions in $\hat{\mathbb{T}}_\Delta$, we obtain that

$$\frac{p^\pm}{1 - p^\pm} = \frac{q^\pm}{1 - q^\pm} \left(\frac{B_1 \cdot \frac{q^\mp}{1 - q^\mp} + 1}{\frac{q^\mp}{1 - q^\mp} + B_2} \right), \quad \frac{p^*}{1 - p^*} = \frac{q^*}{1 - q^*} \left(\frac{B_1 \cdot \frac{q^*}{1 - q^*} + 1}{\frac{q^*}{1 - q^*} + B_2} \right). \tag{3.5}$$

Using (3.2) and (3.5), an easy calculation gives

$$p^\pm = \frac{Q^\pm(1 + B_1Q^\mp)}{B_2 + Q^+ + Q^- + B_1Q^+Q^-}. \tag{3.6}$$

Equation (3.6) will prove to be very useful in simplifying seemingly intricate calculations.

For the case of the Ising model with zero external field ($B_1 = B_2 = B, \lambda = 1$), it holds that $p^+ + p^- = q^+ + q^- = 1, p^* = q^* = 1/2$. These properties may be proved easily using the symmetries of the model. Namely, observe that if (x, y) satisfy (3.3) then so do $(1/y, 1/x)$. By items 1 and 2 of Lemma 3.1, this implies that $q^+ + q^- = 1$ and $q^* = 1/2$. By (3.5), it also follows that $p^+ + p^- = 1, p^* = 1/2$.

4. Moment analysis

In this section we give expressions for $\mathbb{E}_G[Z_G^{\alpha,\beta}], \mathbb{E}_G[Y_G^{\gamma,\delta}]$ and then study the asymptotics of

$$\frac{1}{n} \log \mathbb{E}_G[Z_G^{\alpha,\beta}], \quad \frac{1}{n} \log \mathbb{E}_G[Y_G^{\gamma,\delta}].$$

4.1. First moment

The first moment of $Z_G^{\alpha,\beta}$ is given by

$$\mathbb{E}_G[Z_G^{\alpha,\beta}] = \lambda^{n(\alpha+\beta)} \binom{n}{\alpha n} \binom{n}{\beta n} \left(\sum_{\delta \leq \alpha, \delta \leq \beta} \frac{\binom{\alpha n}{\delta n} \binom{(1-\alpha)n}{(\beta-\delta)n}}{\binom{n}{\beta n}} B_1^{\delta n} B_2^{(1-\alpha-\beta+\delta)n} \right)^\Delta. \tag{4.1}$$

Ignoring the contribution of the external field λ , the terms outside the sum count $|\Sigma_G^{\alpha,\beta}|$. The sum gives the expected contribution to $w_G(\sigma)$ (for $\sigma \in \Sigma_G^{\alpha,\beta}$) from a *single* matching, where δn is the number of edges between $N_-(\sigma) \cap V_1$ and $N_-(\sigma) \cap V_2$ in a single matching. The weight of σ for a single matching depends only on δn and is equal to $B_1^{\delta n} B_2^{(1-\alpha-\beta+\delta)n}$. The fraction in the sum is the probability that there are exactly δn edges from $N_-(\sigma) \cap V_1$ to $N_-(\sigma) \cap V_2$. Since the matchings are chosen independently, the exponentiation to the power Δ gives the expected contribution of $\sigma \in \Sigma_G^{\alpha,\beta}$ to the partition function for Δ

matchings. Finally, the λ term accounts for the contribution of the external field for $\sigma \in \Sigma_G^{\alpha,\beta}$.

A different way to arrive at a slightly different formulation for the first moment is as follows. Here, we introduce some redundant variables at the cost of introducing some complexity of the presentation, but underlying a general approach to treat such expressions.

Observe that the sets $N_-(\sigma) \cap V_1, N_+(\sigma) \cap V_1$ form a partition of V_1 and similarly for V_2 . Thus for a perfect matching let $x_{ij}, (1 \leq i, j \leq 2)$ denote the number of edges having their endpoints in partition i of V_1 and in partition j of V_2 . Note that the x_{ij} must satisfy

$$\begin{aligned} x_{11} + x_{12} &= \alpha, & x_{11} + x_{21} &= \beta, \\ x_{21} + x_{22} &= 1 - \alpha, & x_{12} + x_{22} &= 1 - \beta, \\ x_{ij} &\geq 0. \end{aligned} \tag{4.2}$$

The first moment may be written as

$$\mathbb{E}_G [Z_G^{\alpha,\beta}] = \lambda^{(\alpha+\beta)n} \binom{n}{\alpha n} \binom{n}{\beta n} \left(\sum_{\mathbf{X}} \frac{\binom{\alpha n}{x_{11}n} \binom{(1-\alpha)n}{x_{21}n} \binom{\beta n}{x_{11}n} \binom{(1-\beta)n}{x_{21}n}}{\binom{n}{x_{11}n, x_{12}n, x_{21}n, x_{22}n}} B_1^{x_{11}n} B_2^{x_{22}n} \right)^\Delta. \tag{4.3}$$

In the above sum we let \mathbf{X} denote the x_{ij} which satisfy (4.2). Note that the expressions (4.1) and (4.3) are completely equivalent after identifying δ with x_{11} and using (4.2) to express the variables x_{12}, x_{21}, x_{22} in terms of x_{11} .

4.2. Second moment

We continue with the ideas presented in Section 4.1 to get an expression for $\mathbb{E}_G [Y_G^{\gamma,\delta}]$. Recall that for the formulation of the second moment α and β are fixed, and hence omitted from the notation. Now there are four induced parts by σ_1, σ_2 on each V_1 and V_2 . Let $\gamma n, \delta n$ stand for the number of vertices in V_1 and V_2 whose spin is -1 in both σ_1, σ_2 , and let $y_{ij}n$ be the number of edges incident with endpoints in partition i of V_1 and partition j of V_2 in a single matching. For ease of exposition, define

$$\begin{aligned} L_1 &= \gamma, & L_2 = L_3 &= \alpha - \gamma, & L_4 &= 1 - 2\alpha + \gamma, \\ R_1 &= \delta, & R_2 = R_3 &= \beta - \delta, & R_4 &= 1 - 2\beta + \delta. \end{aligned}$$

In the same spirit, with expression (4.3) for the first moment, we have that

$$\begin{aligned} \mathbb{E}_G [Y_G^{\gamma,\delta}] &= \lambda^{2(\alpha+\beta)n} \binom{n}{\alpha n} \binom{n}{\beta n} \binom{\alpha n}{\gamma n} \binom{(1-\alpha)n}{(\alpha-\gamma)n} \binom{\beta n}{\delta n} \binom{(1-\beta)n}{(\beta-\delta)n} \\ &\times \left(\sum_{\mathbf{Y}} \frac{\prod_{i=1}^4 \binom{L_i n}{y_{i1}n, y_{i2}n, y_{i3}n, y_{i4}n} \prod_{j=1}^4 \binom{R_j n}{y_{1j}n, y_{2j}n, y_{3j}n, y_{4j}n}}{\binom{n}{y_{11}n, y_{12}n, \dots, y_{44}n}} \right)^\Delta \\ &\times B_1^{(2y_{11}+y_{12}+y_{13}+y_{21}+y_{22}+y_{31}+y_{33})n} B_2^{(y_{22}+y_{24}+y_{33}+y_{34}+y_{42}+y_{43}+2y_{44})n}. \end{aligned} \tag{4.4}$$

In the above sum, we let \mathbf{Y} denote the y_{ij} satisfying

$$\begin{aligned} \sum_j y_{ij} &= L_i, & \text{for } i = 1, \dots, 4, \\ \sum_i y_{ij} &= R_j, & \text{for } j = 1, \dots, 4, \\ y_{ij} &\geq 0. \end{aligned} \tag{4.5}$$

The interpretation of (4.4) should be clear: ignoring the external field, the first line accounts for $|\Sigma_{\gamma, \delta}^{\alpha, \beta}|$, while a term in the sum accounts for the weight of a matching specified by the cardinalities Y (last line) as well as the probability of such a matching occurring (middle line).

4.3. Asymptotics of the logarithms of the moments

We next turn to finding the asymptotics of $\frac{1}{n} \log \mathbb{E}_{\mathcal{G}} [Z_G^{\alpha, \beta}]$, $\frac{1}{n} \log \mathbb{E}_{\mathcal{G}} [Y_G^{\gamma, \delta}]$. The asymptotic orders of the sums in (4.3) and (4.4) are dominated by their maximum terms. For the sole purpose of having a unifying treatment for soft and hard constrained two-spin models, here and in the rest of the paper we adopt the usual conventions that $\ln 0 \equiv -\infty$ and $0 \ln 0 \equiv 0$. Standard application of Stirling's approximation yields for the first moment

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\mathcal{G}} [Z_G^{\alpha, \beta}] = \max_{\mathbf{X}} \Phi_1(\alpha, \beta, \mathbf{X}), \tag{4.6}$$

where

$$\begin{aligned} \Phi_1(\alpha, \beta, \mathbf{X}) &:= (\alpha + \beta) \ln \lambda + (\Delta - 1) f_1(\alpha, \beta) + \Delta g_1(\mathbf{X}), \\ f_1(\alpha, \beta) &:= \alpha \ln \alpha + (1 - \alpha) \ln(1 - \alpha) + \beta \ln \beta + (1 - \beta) \ln(1 - \beta), \\ g_1(\mathbf{X}) &:= (x_{11} \ln B_1 + x_{22} \ln B_2) - \sum_{i,j} x_{ij} \ln x_{ij}. \end{aligned}$$

For the second moment it yields

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\mathcal{G}} [Y_G^{\gamma, \delta}] = \max_{\mathbf{Y}} \Phi_2(\gamma, \delta, \mathbf{Y}), \tag{4.7}$$

where

$$\begin{aligned} \Phi_2(\gamma, \delta, \mathbf{Y}) &:= 2(\alpha + \beta) \ln \lambda + (\Delta - 1) f_2(\gamma, \delta) + \Delta g_2(\mathbf{Y}), \\ f_2(\gamma, \delta) &:= 2(\alpha - \gamma) \ln(\alpha - \gamma) + \gamma \ln \gamma + (1 - 2\alpha + \gamma) \ln(1 - 2\alpha + \gamma) \\ &\quad + 2(\beta - \delta) \ln(\beta - \delta) + \delta \ln \delta + (1 - 2\beta + \delta) \ln(1 - 2\beta + \delta), \\ g_2(\mathbf{Y}) &:= (2y_{11} + y_{12} + y_{13} + y_{21} + y_{22} + y_{31} + y_{33}) \ln B_1 \\ &\quad + (y_{22} + y_{24} + y_{33} + y_{34} + y_{42} + y_{43} + 2y_{44}) \ln B_2 \\ &\quad - \sum_{i,j} y_{ij} \ln y_{ij}. \end{aligned}$$

Recall that we are going to study the second moment for specific values of α, β , namely $(\alpha, \beta) = (p^\pm, p^\mp)$, so this is why we dropped the dependence of the functions Φ_2, f_2 on α, β . The functions g_1, g_2 are defined on the regions (4.2), (4.5) respectively. For fixed $\alpha, \beta, \gamma, \delta$, they are strictly concave over the convex regions on which they are defined, and hence they have a unique maximum, so this is also the case for the functions Φ_1, Φ_2 . The limits (4.6) and (4.7) can thus be justified using the standard Laplace method (see for example [4, Chapter 4]).

5. Finding the maxima

In this section we use the information obtained in Section 4.3 to establish Lemmas 2.2 and 2.4. The proofs for Lemmas 2.5 and 2.6 are given in Sections 9.5 and 9.6 respectively, see also Section 5.4 for an overview. These proofs are based on the approach given in this section.

5.1. Optimizing entropy distributions

By the limits (4.6) and (4.7), to establish Lemmas 2.2 and 2.4 (and 2.5, 2.6 as well), it suffices to study the maxima of the functions

$$\phi_1(\alpha, \beta) := \max_{\mathbf{X}} \Phi_1(\alpha, \beta, \mathbf{X}) \quad \text{and} \quad \phi_2(\gamma, \delta) := \max_{\mathbf{Y}} \Phi_2(\gamma, \delta, \mathbf{Y}). \quad (5.1)$$

A small abstraction allows us to treat the two functions in a unifying way.

Namely, observe that ϕ_1, ϕ_2 have the form $\phi(u, v) = \max_{\mathbf{Z}} \Phi(u, v, \mathbf{Z})$, where Φ can be decomposed as $\Phi(u, v, \mathbf{Z}) = f(u, v) + g(\mathbf{Z})$, for some functions f, g , and \mathbf{Z} stands for mn non-negative variables Z_{ij} , $i = 1, \dots, m$, $j = 1, \dots, n$.¹ Thus in the context of ϕ_1 (resp. ϕ_2), u, v, \mathbf{Z} correspond to $\alpha, \beta, \mathbf{X}$ (resp. $\gamma, \delta, \mathbf{Y}$). Moreover, \mathbf{Z} satisfies some prescribed marginals, that is,

$$\sum_j Z_{ij} = \alpha_i(u), \quad \sum_i Z_{ij} = \beta_j(v), \quad Z_{ij} \geq 0, \quad (5.2)$$

for some functions $\alpha_i(u)$ and $\beta_j(v)$, which satisfy $\sum_i \alpha_i(u) = \sum_j \beta_j(v) = 1$. In this section we will only consider u, v such that $\alpha_i(u) \neq 0 \neq \beta_j(v)$ for all i, j , since the remaining cases correspond to boundary conditions in our setup and these are treated more easily in the specific instance of the maximization (see the relevant Lemmas 5.4 and 5.7). Finally, the function $g(\mathbf{Z})$ is an entropy-like function and has the form

$$g(\mathbf{Z}) = \sum_i \sum_j (Z_{ij} \ln M_{ij} - Z_{ij} \ln Z_{ij}), \quad (5.3)$$

where $M \in \mathbb{R}^{m \times n}$ is a matrix with non-negative entries.

We will use another interpretation of the function $g(\mathbf{Z})$, which will be handy when we perform integrations: let \mathbf{Z}^f denote the variables Z_{ij} for $i = 1, \dots, m-1$, $j = 1, \dots, n-1$. Using the equations in (5.2), we may write $g(\mathbf{Z})$ as a function of the variables \mathbf{Z}^f . Note that the region (5.2) must be adjusted to account for the non-negativity of the substituted variables. Other than this, the ‘new’ function has exactly the same behaviour as $g(\mathbf{Z})$ but in a full-dimensional space. Thus we will also think of $g(\mathbf{Z})$ as a function of the variables \mathbf{Z}^f where the Z_{ij} with $i = m$ or $j = n$ are just shorthand for the expressions of these variables when substituted by the equations in (5.2). We will refer to this reformulation as the *full-dimensional representation* of $g(\mathbf{Z})$. We define similarly the full-dimensional representation of $\Phi(u, v, \mathbf{Z})$.

¹ In all the applications of the arguments in this section, we will have $m = n$. The slightly more general setting considered here (which allows $m \neq n$) is not crucial, but the arguments extend to this setting without any additional effort.

It is immediate to see that $g(\mathbf{Z})$ is strictly concave in the convex region (5.2), hence it attains its maximum over the region at a unique point. A tedious but straightforward optimization of $g(\mathbf{Z})$, using Lagrange multipliers, yields the following result.

Lemma 5.1. Fix u, v such that $\alpha_i(u) \neq 0 \neq \beta_j(v)$ for all i, j . Let $\mathbf{Z}^* = \arg \max_{\mathbf{Z}} g(\mathbf{Z})$, the maximum taken over the region (5.2). Then

$$Z_{ij}^* = M_{ij} R_i C_j, \quad (5.4)$$

where R_i and C_j are positive real numbers, which satisfy

$$R_i \sum_j M_{ij} C_j = \alpha_i \quad \text{and} \quad C_j \sum_i M_{ij} R_i = \beta_j. \quad (5.5)$$

Moreover, the full-dimensional representation of $g(\mathbf{Z})$ decays quadratically in a sufficiently small ball around \mathbf{Z}^* .

The proof of Lemma 5.1 is given in Section 9.2. It is relevant to point out that while the values of Z_{ij}^* are unique in the region (5.2), the values of R_i, C_j which satisfy (5.5) are not unique since a positive solution $\{\{R_i\}_i, \{C_j\}_j\}$ of (5.5) gives rise to a positive solution $\{\{tR_i\}_i, \{C_j/t\}_j\}$ for arbitrary $t > 0$. Modulo this mapping, the uniqueness of Z_{ij}^* easily implies that the positive R_i, C_j which satisfy (5.5) are otherwise unique.

Plugging (5.4) in $g(\mathbf{Z})$, we obtain

$$\max_{\mathbf{Z}} g(\mathbf{Z}) = -\sum_i \sum_j M_{ij} R_i C_j \ln(R_i C_j) := g^*(u, v). \quad (5.6)$$

In our setup, it will be hard in general to solve for the R_i, C_j , so that we need to study $\partial g^*/\partial u, \partial g^*/\partial v$ using implicit differentiation. The following lemma will be very useful.

Lemma 5.2. For $g^* = \max_{\mathbf{Z}} g(\mathbf{Z})$, we have

$$\frac{\partial g^*}{\partial u} = -\sum_i \ln(R_i) \frac{\partial \alpha_i}{\partial u} \quad \text{and} \quad \frac{\partial g^*}{\partial v} = -\sum_j \ln(C_j) \frac{\partial \beta_j}{\partial v}.$$

In particular,

$$\frac{\partial(f + g^*)}{\partial u} = \frac{\partial f}{\partial u} - \sum_i \ln(R_i) \frac{\partial \alpha_i}{\partial u} \quad \text{and} \quad \frac{\partial(f + g^*)}{\partial v} = \frac{\partial f}{\partial v} - \sum_j \ln(C_j) \frac{\partial \beta_j}{\partial v},$$

where the R_i, C_j are positive solutions of (5.5).

The proof of Lemma 5.2 is given in Section 9.2.

5.2. The logarithm of the first moment – maximum

In this section we prove Lemma 2.2. As observed in Sections 4.3 and 5.1, it suffices to study the maxima of $\phi_1(\alpha, \beta)$. Lemma 2.2 is implied by the following lemmas.

Lemma 5.3. *The only critical points of $\phi_1(\alpha, \beta)$ are as follows:*

- (1) $(\alpha, \beta) = (p^\pm, p^\mp)$ and $(\alpha, \beta) = (p^*, p^*)$ when B_1, B_2, λ lie in the non-uniqueness region of \mathbb{T}_Δ ,
- (2) $(\alpha, \beta) = (p^*, p^*)$ when B_1, B_2, λ lie in the uniqueness region of \mathbb{T}_Δ .

Lemma 5.4. *The function $\phi_1(\alpha, \beta)$ does not have any local maximum on the boundary of the region*

$$\{(\alpha, \beta) \mid 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1\}.$$

Lemma 5.5. *Let $\mathbf{X}^* = \arg \max_{\mathbf{X}} \Phi_1(p^+, p^-, \mathbf{X})$ and $\mathbf{X}^o = \arg \max_{\mathbf{X}} \Phi_1(p^*, p^*, \mathbf{X})$.*

- (1) *When B_1, B_2, λ lie in the non-uniqueness region of \mathbb{T}_Δ , the function $\Phi_1(\alpha, \beta, \mathbf{X})$ has a local maximum at the point (p^+, p^-, \mathbf{X}^*) and a saddle point at (p^*, p^*, \mathbf{X}^o) .*
- (2) *When B_1, B_2, λ lie in the uniqueness region of \mathbb{T}_Δ , the function $\Phi_1(\alpha, \beta, \mathbf{X})$ has a local maximum at (p^*, p^*, \mathbf{X}^o) .*

Proof of Lemma 2.2. Recall from (4.6) and (5.1) that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\mathcal{G}} [Z_G^{\alpha, \beta}] = \phi_1(\alpha, \beta) = \max_{\mathbf{X}} \Phi_1(\alpha, \beta, \mathbf{X}).$$

By Lemma 5.4, the function $\phi_1(\alpha, \beta)$ does not attain its maximum on the boundary and hence its (global) maximum is achieved at a critical point. Lemma 5.3 gives the critical points in the uniqueness/non-uniqueness region and Lemma 5.5 classifies which are local maxima. The result follows. □

We defer the proofs of Lemmas 5.4 and 5.5 to Section 9.3. We give the more interesting proof of Lemma 5.3.

Proof of Lemma 5.3. We apply Lemma 5.2 with

$$\begin{aligned} \mathbf{Z} \leftarrow \mathbf{X}, \quad u \leftarrow \alpha, \quad v \leftarrow \beta, \quad \alpha_1 \leftarrow \alpha, \quad \alpha_2 \leftarrow 1 - \alpha, \quad \beta_1 \leftarrow \beta, \quad \beta_2 \leftarrow 1 - \beta, \\ f \leftarrow (\alpha + \beta) \ln \lambda + (\Delta - 1)f_1, \quad g \leftarrow \Delta g_1. \end{aligned}$$

The matrix M is given by

$$M = \begin{pmatrix} B_1 & 1 \\ 1 & B_2 \end{pmatrix}.$$

Lemma 5.2 gives that the critical points of $\max_{\mathbf{X}} \Phi_1(\alpha, \beta, \mathbf{X})$ must satisfy

$$\begin{aligned} \frac{\partial \phi_1}{\partial \alpha} &= \ln \left(\lambda \left(\frac{\alpha}{1 - \alpha} \right)^{\Delta - 1} \left(\frac{R_2}{R_1} \right)^\Delta \right) = 0, \\ \frac{\partial \phi_1}{\partial \beta} &= \ln \left(\lambda \left(\frac{\beta}{1 - \beta} \right)^{\Delta - 1} \left(\frac{C_2}{C_1} \right)^\Delta \right) = 0, \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} R_1(B_1C_1 + C_2) &= \alpha, & C_1(B_1R_1 + R_2) &= \beta, \\ R_2(C_1 + B_2C_2) &= 1 - \alpha, & C_2(R_1 + B_2R_2) &= 1 - \beta. \end{aligned} \quad (5.8)$$

Let $r = R_1/R_2$ and $c = C_1/C_2$. Using (5.7) and (5.8), we obtain

$$\begin{aligned} r \frac{B_1c + 1}{c + B_2} &= \frac{\alpha}{1 - \alpha} = r^{1+1/(\Delta-1)}(1/\lambda)^{1/(\Delta-1)} \implies r = \lambda \left(\frac{B_1c + 1}{c + B_2} \right)^{\Delta-1}, \\ c \frac{B_1r + 1}{r + B_2} &= \frac{\beta}{1 - \beta} = c^{1+1/(\Delta-1)}(1/\lambda)^{1/(\Delta-1)} \implies c = \lambda \left(\frac{B_1r + 1}{r + B_2} \right)^{\Delta-1}. \end{aligned}$$

The right-hand side equations are exactly the equations (3.3) and in light of Lemma 3.1 it must be the case that in the non-uniqueness region either that

$$r = \frac{q^\pm}{1 - q^\pm}, \quad c = \frac{q^\mp}{1 - q^\mp} \quad \text{or} \quad r = c = \frac{q^*}{1 - q^*}.$$

By (3.6), we hence obtain that the critical points for the first moment in the non-uniqueness region are given by $(\alpha, \beta) = (p^\pm, p^\mp)$ or $(\alpha, \beta) = (p^*, p^*)$. Similarly for the uniqueness region. \square

5.3. The logarithm of the second moment – maximum

In this section we prove Lemma 2.4. As established in Sections 4.3 and 5.1, it suffices to study the maxima of $\phi_2(\gamma, \delta)$. Lemma 2.4 is immediately implied by the following lemmas.

Lemma 5.6. *Under the hypotheses of Theorem 1.3, for $(\alpha, \beta) = (p^\pm, p^\mp)$, the only critical points of $\phi_2(\gamma, \delta)$ satisfy $\gamma = \alpha^2$, $\delta = \beta^2$.*

Lemma 5.7. *For $(\alpha, \beta) = (p^\pm, p^\mp)$, the function $\phi_2(\gamma, \delta)$ does not have any local maximum on the boundary of the region*

$$\{(\gamma, \delta) \mid \gamma \geq 0, \alpha - \gamma \geq 0, 1 - 2\alpha + \gamma \geq 0, \delta \geq 0, \beta - \delta \geq 0, 1 - 2\beta + \delta \geq 0\}.$$

Lemma 5.8. *For $(\alpha, \beta) = (p^+, p^-)$, let*

$$\mathbf{Y}^* = \arg \max_{\mathbf{Y}} \Phi_2(\alpha^2, \beta^2, \mathbf{Y}).$$

The function $\Phi_2(\gamma, \delta, \mathbf{Y})$ has a local maximum at $(\alpha^2, \beta^2, \mathbf{Y}^)$.*

Proof of Lemma 2.4. Just combine Lemmas 5.6, 5.7 and 5.8 as in the proof of Lemma 2.2 in Section 5.2. \square

The proofs of Lemmas 5.7 and 5.8 are valid for all B_1, B_2, λ , so to prove Lemmas 2.5 and 2.6 one only needs to obtain the analogues of Lemma 5.6 in the respective settings of B_1, B_2 . The proofs of these analogues are given in Sections 9.5 and 9.6; see also Section 5.4

for a proof overview. We defer the proofs of Lemmas 5.7 and 5.8 to Section 9.4 and give here the more interesting proof of Lemma 5.6.

Proof of Lemma 5.6. Once more we apply Lemma 5.2 with

$$\begin{aligned} \mathbf{Z} \leftarrow \mathbf{Y}, \quad u \leftarrow \gamma, \quad v \leftarrow \delta, \quad \alpha_1 \leftarrow \gamma, \quad \alpha_2 = \alpha_3 \leftarrow \alpha - \gamma, \quad \alpha_4 \leftarrow 1 - 2\alpha + \gamma, \\ \beta_1 \leftarrow \gamma, \quad \beta_2 = \beta_3 \leftarrow \beta - \delta, \quad \beta_4 \leftarrow 1 - 2\beta + \delta, \quad f \leftarrow (\Delta - 1)f_2, \quad g \leftarrow \Delta g_2. \end{aligned}$$

The matrix M is given by

$$M = \begin{pmatrix} B_1^2 & B_1 & B_1 & 1 \\ B_1 & B_1 B_2 & 1 & B_2 \\ B_1 & 1 & B_1 B_2 & B_2 \\ 1 & B_2 & B_2 & B_2^2 \end{pmatrix}.$$

We obtain that the critical points of $\max_{\mathbf{Y}} \Phi_2(\gamma, \delta, \mathbf{Y})$ must satisfy

$$\frac{\partial \phi_2}{\partial \gamma} = \ln \left(\left(\frac{\gamma(1 - 2\alpha + \gamma)}{(\alpha - \gamma)^2} \right)^{\Delta-1} \left(\frac{R_2 R_3}{R_1 R_4} \right)^\Delta \right) = 0, \tag{5.9}$$

$$\frac{\partial \phi_2}{\partial \delta} = \ln \left(\left(\frac{\delta(1 - 2\beta + \delta)}{(\beta - \delta)^2} \right)^{\Delta-1} \left(\frac{C_2 C_3}{C_1 C_4} \right)^\Delta \right) = 0, \tag{5.10}$$

where the R_i, C_j satisfy the following equations (\circ denotes the Hadamard product):

$$\begin{pmatrix} R_1 & C_1 \\ R_2 & C_2 \\ R_3 & C_3 \\ R_4 & C_4 \end{pmatrix} \circ M \begin{pmatrix} C_1 & R_1 \\ C_2 & R_2 \\ C_3 & R_3 \\ C_4 & R_4 \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ \alpha - \gamma & \beta - \delta \\ \alpha - \gamma & \beta - \delta \\ 1 - 2\alpha + \gamma & 1 - 2\beta + \delta \end{pmatrix}. \tag{5.11}$$

We will write out an explicit form of the equations after establishing the following claim.

Claim 5.9. *It holds that $R_2 = R_3$ and $C_2 = C_3$.*

Proof. Observe that the matrix M remains invariant upon interchanging its second and third rows and second and third columns, that is,

$$\begin{aligned} (R_1, R_2, R_3, R_4, C_1, C_2, C_3, C_4) \text{ satisfy (5.11)} \\ \iff (R_1, R_3, R_2, R_4, C_1, C_3, C_2, C_4) \text{ satisfy (5.11)}. \end{aligned}$$

Since $g_2(\mathbf{Y})$ is strictly concave, this yields the claim. □

The rest of the proof will work as follows: as in the first moment, we first do some manipulations with the equations so that we arrive at a nice form. This form coincides with an inequality version of the tree equations (3.3) for the ferromagnetic Ising model without external field, which can be analysed easily.

Using Claim 5.9, equalities (5.11) give

$$\begin{aligned} R_1 &= \frac{\gamma}{B_1^2 C_1 + 2B_1 C_2 + C_4}, & C_1 &= \frac{\delta}{B_1^2 R_1 + 2B_1 R_2 + R_4}, \\ R_2 &= \frac{\alpha - \gamma}{B_1 C_1 + (B_1 B_2 + 1)C_2 + B_2 C_4}, & C_2 &= \frac{\beta - \delta}{B_1 R_1 + (B_1 B_2 + 1)R_2 + B_2 R_4}, \\ R_4 &= \frac{1 - 2\alpha + \gamma}{C_1 + 2B_2 C_2 + B_2^2 C_4}, & C_4 &= \frac{1 - 2\beta + \delta}{R_1 + 2B_2 R_2 + B_2^2 R_4}. \end{aligned} \quad (5.12)$$

Also, again using Claim 5.9, the equalities in (5.9) and (5.10) give

$$\frac{\partial \phi_2}{\partial \gamma} = \ln \left(\left(\frac{\gamma(1 - 2\alpha + \gamma)}{(\alpha - \gamma)^2} \right)^{\Delta-1} \left(\frac{R_2^2}{R_1 R_4} \right)^{\Delta} \right) = 0, \quad (5.13)$$

$$\frac{\partial \phi_2}{\partial \delta} = \ln \left(\left(\frac{\delta(1 - 2\beta + \delta)}{(\beta - \delta)^2} \right)^{\Delta-1} \left(\frac{C_2^2}{C_1 C_4} \right)^{\Delta} \right) = 0. \quad (5.14)$$

We now set

$$r_1 = R_1/R_2, \quad r_4 = R_4/R_2, \quad c_1 = C_1/C_2, \quad c_4 = C_4/C_2.$$

After dividing the appropriate pairs of equations in (5.12), we obtain

$$\begin{aligned} r_1 &= \frac{\gamma}{\alpha - \gamma} \cdot \frac{B_1 c_1 + (B_1 B_2 + 1) + B_2 c_4}{B_1^2 c_1 + 2B_1 + c_4}, \\ r_4 &= \frac{1 - 2\alpha + \gamma}{\alpha - \gamma} \cdot \frac{B_1 c_1 + (B_1 B_2 + 1) + B_2 c_4}{c_1 + 2B_2 + B_2^2 c_4}, \end{aligned} \quad (5.15)$$

$$\begin{aligned} c_1 &= \frac{\delta}{\beta - \delta} \cdot \frac{B_1 r_1 + (B_1 B_2 + 1) + B_2 r_4}{B_1^2 r_1 + 2B_1 + r_4}, \\ c_4 &= \frac{1 - 2\beta + \delta}{\beta - \delta} \cdot \frac{B_1 r_1 + (B_1 B_2 + 1) + B_2 r_4}{r_1 + 2B_2 + B_2^2 r_4}. \end{aligned} \quad (5.16)$$

Equations (5.13) and (5.14) become

$$(r_1 r_4)^{\Delta} = \left(\frac{\gamma(1 - 2\alpha + \gamma)}{(\alpha - \gamma)^2} \right)^{\Delta-1}, \quad (c_1 c_4)^{\Delta} = \left(\frac{\delta(1 - 2\beta + \delta)}{(\beta - \delta)^2} \right)^{\Delta-1}. \quad (5.17)$$

Using the identity

$$\begin{aligned} &(B_1^2 c_1 + 2B_1 + c_4)(c_1 + 2B_2 + B_2^2 c_4) \\ &= (B_1 c_1 + (B_1 B_2 + 1) + B_2 c_4)^2 + (1 - B_1 B_2)^2 (c_1 c_4 - 1), \end{aligned}$$

and multiplying the equations in (5.15), we obtain

$$r_1 r_4 = \frac{\gamma(1 - 2\alpha + \gamma)}{(\alpha - \gamma)^2} \cdot \frac{(B_1 c_1 + (B_1 B_2 + 1) + B_2 c_4)^2}{(B_1 c_1 + (B_1 B_2 + 1) + B_2 c_4)^2 + (1 - B_1 B_2)^2 (c_1 c_4 - 1)}. \quad (5.18)$$

It will be convenient at this point to work with $d = \Delta - 1$. We plug the first equation in (5.17) into (5.18), yielding

$$\begin{aligned} (r_1 r_4)^{1/d} - 1 &= \frac{(1 - B_1 B_2)^2 (c_1 c_4 - 1)}{(B_1 c_1 + (B_1 B_2 + 1) + B_2 c_4)^2}, \\ (c_1 c_4)^{1/d} - 1 &= \frac{(1 - B_1 B_2)^2 (r_1 r_4 - 1)}{(B_1 r_1 + (B_1 B_2 + 1) + B_2 r_4)^2}. \end{aligned} \tag{5.19}$$

The second equality in (5.19) follows by a completely symmetric argument for c_1, c_4 . Observe that (5.19) implies that one of the following three cases can hold:

- I $r_1 r_4 = c_1 c_4 = 1$,
- II $r_1 r_4 > 1, c_1 c_4 > 1$,
- III $r_1 r_4 < 1, c_1 c_4 < 1$.

Case I reduces to $\gamma = \alpha^2, \delta = \beta^2$ by (5.17). Thus we may focus on cases II and III. We further restrict our attention to case II, with case III being completely analogous. From AM–GM, we have the inequalities

$$B_1 c_1 + B_2 c_4 \geq 2\sqrt{B_1 B_2 c_1 c_4}, \quad B_1 r_1 + B_2 r_4 \geq 2\sqrt{B_1 B_2 r_1 r_4},$$

so (5.19) gives

$$\begin{aligned} (r_1 r_4)^{1/d} - 1 &\leq \frac{(1 - B_1 B_2)^2 (c_1 c_4 - 1)}{(B_1 B_2 + 1 + 2\sqrt{B_1 B_2 c_1 c_4})^2}, \\ (c_1 c_4)^{1/d} - 1 &\leq \frac{(1 - B_1 B_2)^2 (r_1 r_4 - 1)}{(B_1 B_2 + 1 + 2\sqrt{B_1 B_2 r_1 r_4})^2}. \end{aligned}$$

It is straightforward to verify the identity

$$(B_1 B_2 + 1 + 2z\sqrt{B_1 B_2})^2 + (1 - B_1 B_2)^2 (z^2 - 1) = ((1 + B_1 B_2)z + 2\sqrt{B_1 B_2})^2.$$

Set $x = \sqrt{r_1 r_4}, y = \sqrt{c_1 c_4}$. Using the identity with $z = x, y$ and taking square roots, we obtain

$$x^{1/d} \leq \frac{(1 + B_1 B_2)y + 2\sqrt{B_1 B_2}}{2\sqrt{B_1 B_2}y + B_1 B_2 + 1}, \quad y^{1/d} \leq \frac{(1 + B_1 B_2)x + 2\sqrt{B_1 B_2}}{2\sqrt{B_1 B_2}x + B_1 B_2 + 1}. \tag{5.20}$$

Using the substitution $B' = (1 + B_1 B_2)/2\sqrt{B_1 B_2}$, this can be rewritten in the form

$$x^{1/d} \leq \frac{B'y + 1}{y + B'}, \quad y^{1/d} \leq \frac{B'x + 1}{x + B'}. \tag{5.21}$$

The astute reader will immediately realize the analogy of (5.21) with the tree equations (3.3) for the Ising model without external field. Moreover, since $B' \geq 1$ we are in the case of the *ferromagnetic* Ising model. For the ferromagnetic Ising model on the infinite tree \mathbb{T}_{d+1} , we have uniqueness when $1 < B \leq (d + 1)/(d - 1)$. Hence intuitively the lemma holds when $B' \leq (d + 1)/(d - 1)$. However, in (5.21) we have an inequality version of the tree recursions (3.3) and therefore a bit more work is required.

In our setting, we obtain that (5.21) cannot hold when $x > 1, y > 1$. To see this, multiply the inequalities in (5.21) to obtain

$$x^{1/d} y^{1/d} \leq \frac{B'x + 1}{x + B'} \cdot \frac{B'y + 1}{y + B'}. \tag{5.22}$$

Note that (5.22) is reversed in case III. The following lemma implies that (5.22) cannot hold in case II, and similarly for the reverse inequality (5.22) in case III.

Lemma 5.10. *Let $d \geq 2$. When*

$$B' \leq 1/B_c(\mathbb{T}_{d+1}) = \frac{d+1}{d-1},$$

for $z > 1$ it holds that

$$z^{1/d} > \frac{B'z + 1}{z + B'}.$$

The inequality is reversed for $z < 1$.

We give the proof of Lemma 5.10 after observing that

$$B' \leq \frac{d+1}{d-1}$$

reduces to

$$\sqrt{B_1 B_2} \geq \frac{\sqrt{d} - 1}{\sqrt{d} + 1}.$$

This concludes the proof of Lemma 5.6. □

Proof of Lemma 5.10. The case $z < 1$ reduces to the case $z > 1$ using the inversion $z \leftarrow 1/z$. Thus we focus on proving the lemma for $z > 1$. Let $f(w) := w^{d+1} - B'w^d + B'w - 1$. The lemma reduces to proving $f(w) > 0$ when $w > 1$ (under the substitution $w = z^{1/d}$). We have

$$\begin{aligned} f'(w) &= (d+1)w^d - dB'w^{d-1} + B', \\ f''(w) &= dw^{d-2}((d+1)w - (d-1)B'). \end{aligned}$$

The assumptions imply that $(d+1) - (d-1)B' \geq 0$. Hence, when $w > 1$, we have $f''(w) > f''(1)$. Note that $f''(1) \geq 0$ only if $(d+1) - (d-1)B' \geq 0$. Thus, $f''(w)$ is strictly positive when $w > 1$ and hence $f'(w)$ is strictly increasing. It follows that for $w > 1$, we have $f'(w) > f'(1)$ and

$$f'(1) = (d+1) - (d-1)B' \geq 0.$$

Thus, f is strictly increasing for $w > 1$, yielding $f(w) > f(1) = 0$. □

5.4. Some remarks

Here we give the idea behind the arguments establishing Lemmas 2.5 and 2.6. As observed in Section 5.3, these lemmas boil down to proving that, for $(\alpha, \beta) = (p^+, p^-)$, the only critical point of $\phi_2(\gamma, \delta)$ is at $(\gamma, \delta) = (\alpha^2, \beta^2)$ under the respective hypotheses.

Our point of departure is once again the equations in (5.19), although we proceed with a more thorough analysis. Looking at the argument for the case

$$\frac{\sqrt{d} - 1}{\sqrt{d} + 1} \leq \sqrt{B_1 B_2},$$

the point of the proof which is subject to tighter analysis is the use of the AM–GM inequalities. These are weak if the ratios $r_1/r_4, c_1/c_4$ are much bigger than B_2/B_1 . In Section 9.5, we turn this weakness in our favour. Namely, for the antiferromagnetic Ising model without external field, in the region

$$0 < B < \frac{\sqrt{d} - 1}{\sqrt{d} + 1},$$

the densities p^\pm are heavily biased towards 1 and 0 respectively, and this reflects at the values of r_1, r_4, c_1, c_4 . This observation can be turned into a simple proof of Lemma 2.5. For the proof of Lemma 2.6, it is easier to get an explicit handle on the values p^\pm (and thus on the bias of the r_i, c_j), which can be used to analyse the equations (5.19) (see Section 9.6).

Further, let us make some small minor observations. The reader may have noticed that the proof of Lemma 5.6 applies in a more general setup than that stated in Condition 2.3. Namely, in the region

$$\sqrt{B_1 B_2} \geq \frac{\sqrt{d} - 1}{\sqrt{d} + 1},$$

Lemma 5.6 holds for all values of α, β such that $1 > \alpha, \beta > 0$ and not just for $(\alpha, \beta) = (p^\pm, p^\mp)$ (this range of B_1, B_2 covers the whole uniqueness region even with external field). It is thus conceivable that when $\sqrt{B_1 B_2} \geq (\sqrt{d} - 1)/(\sqrt{d} + 1)$, the bound on the partition function $Z_G^{\alpha, \beta}$ in Lemma 2.8 holds for $1 > \alpha, \beta > 0$ (we do not attempt to show a more general version of Lemma 2.8 in this paper since we are only interested in the values of (α, β) which maximize $\mathbb{E}_G[Z_G^{\alpha, \beta}]$).

6. NP-hardness results

In this section we give the proofs for our NP-hardness results. We will prove the following theorem, which allows us to prove Theorems 1.1, 1.2 and 1.3.

Theorem 6.1. *Let $\Delta \geq 3$. Assume that (B_1, B_2, λ) lie in the non-uniqueness region of the infinite tree \mathbb{T}_Δ . Moreover, assume that Condition 2.3 holds. Then, unless $NP = RP$, there is no FPRAS for the partition function of the two-spin model with parameters B_1, B_2, λ in graphs with maximum degree Δ .*

With Theorem 6.1 at hand, it is straightforward to prove Theorems 1.1, 1.2 and 1.3.

Proofs of Theorems 1.1, 1.2, 1.3. Theorem 1.3 follows immediately from Lemma 2.4 and Theorem 6.1. Theorem 1.1 was known to hold for $\Delta = 3$ and $\Delta \geq 6$ ([27], [8]). Lemma 2.6 in combination with Theorem 6.1 proves the cases $\Delta = 4, 5$ as well.

We now prove Theorem 1.2. In the case of the Ising model with no external field, note that Lemma 2.4 and Theorem 6.1 give hardness for $\Delta \geq 3$ when

$$\frac{\sqrt{d} - 1}{\sqrt{d} + 1} \leq B < \frac{d - 1}{d + 1}, \quad \text{where } d = \Delta - 1. \tag{6.1}$$

The theorem will be proved once we show that this can be extended to the region

$$0 < B < \frac{\sqrt{d}-1}{\sqrt{d}+1}.$$

Observe that hardness for B, Δ gives hardness for $B, \Delta + 1$. It is easy to check that for $d \geq 2$ it holds that

$$\frac{\sqrt{d+1}-1}{\sqrt{d+1}+1} < \frac{d-1}{d+1} < \frac{d}{d+2},$$

so the intervals corresponding to Δ and $\Delta + 1$ are overlapping for $\Delta \geq 3$. Thus it suffices to check NP-hardness for $d = 2$ and $0 < B < (\sqrt{d}-1)/(\sqrt{d}+1)$. This follows from Lemma 2.5 and Theorem 6.1, thus completing the proof. \square

We thus focus on proving Theorem 6.1. Sly's reduction [27], while presented for the particular case of the hard-core model, can be used to show hardness for general antiferromagnetic two-spin models. The only part of Sly's reduction that needs modification is in establishing the properties of the gadget he uses. Let us first describe the gadget used in [27].

The construction of the gadget has two parameters $0 < \theta, \psi < 1/8$. Let

$$k := (\Delta - 1)^{\lfloor \theta \log_{\Delta-1} n \rfloor} \quad \text{and} \quad \ell := 2 \left\lfloor \frac{\psi}{2} \log_{\Delta-1} n \right\rfloor$$

and let $m' := k(\Delta - 1)^\ell$. Note that $m' = o(n^{1/4})$. The gadget is constructed in two steps. First, a random bipartite graph \bar{G} with $n + m'$ vertices on each side is constructed. The two sides of the graph will be labelled with $+, -$. For $s \in \{+, -\}$, let the vertices on the s -side be $W_s \cup U_s$, where $|W_s| = n$ and $|U_s| = m'$. The edges of the graph are the union of $\Delta - 1$ uniformly random perfect matchings between $W_+ \cup U_+$ and $W_- \cup U_-$, together with a uniformly random perfect matching between W_+ and W_- . Thus, in the resulting random graph \bar{G} , the vertices in $W := W_+ \cup W_-$ have degree Δ and the vertices in $U := U_+ \cup U_-$ have degree $\Delta - 1$. We denote the graph distribution defined by this construction as \bar{G} .

The second part of the construction appends complete trees of depth ℓ to \bar{G} in the following manner. For $s \in \{+, -\}$, partition the vertices of U_s into k groups of $(\Delta - 1)^\ell$ vertices. For each group we create a new $(\Delta - 1)$ -ary tree of even depth ℓ (where ℓ is an even integer as specified earlier). The leaves of this tree are the group of vertices in U_s ; the other vertices of the tree are new. After the addition of these $2k$ trees to \bar{G} the vertices of U_s have degree Δ ; in fact, all vertices in this new graph have degree Δ except for the roots of the trees which have degree $\Delta - 1$. Let R_+ (respectively, R_-) denote the roots of those trees whose leaves are a subset of U_+ (U_-). Let $R := R_+ \cup R_-$. Informally, the addition of the trees makes it easier to prove stronger concentration properties for the spins of the vertices with degree $\Delta - 1$ (which are now the roots of the trees).

Analogously to [27], for a configuration σ on H , the phase $Y(\sigma)$ of σ is defined to be $+$ if the number of vertices assigned -1 in W_+ is greater than the number of vertices assigned -1 in W_- , otherwise $Y(\sigma) = -$. In other words, the phase $Y(\sigma)$ of a configuration σ simply points to the set between W_+, W_- with the greatest number of vertices assigned -1 in σ .

The properties of Sly’s gadget are stated in the following lemma (proved in Section 6.3). We use the following notation: for a configuration σ on H , σ_R denotes the restriction of σ to vertices in R . Moreover, for a spin $s \in \{-1, +1\}$, $\sigma^{-1}(s)$ denotes the vertices assigned the spin s .

Lemma 6.2 (analogue of [27, Theorem 2.1]). *Let $\Delta \geq 3$. Assume that (B_1, B_2, λ) lie in the non-uniqueness region of the infinite tree \mathbb{T}_Δ . Moreover, assume that Condition 2.3 holds. There exist constants $\theta(\Delta, B_1, B_2, \lambda), \psi(\Delta, B_1, B_2, \lambda) > 0$ such that for all sufficiently large n the graph H satisfies the following with probability $1 - o_n(1)$.*

(1) *The two phases occur with roughly equal probability, that is, for $i \in \{+, -\}$ we have*

$$\mu_H(Y(\sigma) = i) \geq \frac{1}{n}.$$

(2) *Conditioned on the phase $i \in \{+, -\}$, the spins of vertices in R are approximately independent, that is,*

$$\max_{\eta \in \{-1, +1\}^R} \left| \frac{\mu_H(\sigma_R = \eta \mid Y(\sigma) = i)}{Q_R^i(\eta)} - 1 \right| \leq n^{-2\theta},$$

where Q_R^i is the following product distribution on configurations $\eta : R_+ \cup R_- \rightarrow \{\pm 1\}$:

$$Q_R^i(\eta) = (q^i)^{|\eta^{-1}(-1) \cap R_+|} (1 - q^i)^{|\eta^{-1}(+1) \cap R_+|} (q^{-i})^{|\eta^{-1}(-1) \cap R_-|} (1 - q^{-i})^{|\eta^{-1}(+1) \cap R_-|}.$$

Recall that for $i \in \{+, -\}$, q^i is the probability that the root of the infinite $(\Delta - 1)$ -ary tree $\hat{\mathbb{T}}_\Delta$ is assigned the spin -1 in the extremal measure $\hat{\mu}^i$: see Section 3.

We now describe briefly how these properties of H are established in [27]. This is done in two steps.

- (1) First prove that in the graph \overline{G} , conditioned on the phase $i \in \{+, -\}$, the spins of the vertices in U are asymptotically (jointly) distributed as a product distribution Q_U^i on $\{-1, +1\}^U$. The latter distribution is obtained by replacing R with U in the definition of Q_R^i in Lemma 6.2. Roughly, this can be obtained as follows from what we have seen so far. In Section 2 we showed that for a random Δ -regular bipartite graph, with high probability over the choice of the graph, in the Gibbs distribution the total weight of configurations from the set Σ^{p^+, p^-} dominate exponentially over the rest. An analogous phenomenon occurs for a graph $\overline{G} \sim \overline{\mathcal{G}}$, which allows us to quantify the aforementioned behaviour; see Section 6.1 for more details.
- (2) Translate the results to R . This is accomplished in a crafty way in [27]: the product measures Q_U^i on $\{-1, +1\}^U$ are analogous to the even and odd boundaries in the infinite $(\Delta - 1)$ -ary tree, providing for an elegant translation of the results (see the proof of the second part of Lemma 6.2 for more details). Moreover, the extremality of the measures implies that the roots of the trees, conditioned on the phase, are strongly concentrated around their expected value. The latter part follows after studying non-reconstruction in the extremal measures and proving that the correlation of the spins of the vertices in R to the spins of the vertices in U decays doubly exponentially in ℓ ; see Section 6.2. Crucially, this allows us to bound the distance between the (true)

distribution of the spins of the vertices in R (conditioned on the phase i) and the product distribution Q_R^i by an inverse polynomial function of n .

In Section 6.1 we give the lemmas in [27] which require adjustment in the two-spin model. Their proofs are analogous to those in [27], by comparing the quantities in the graph distribution $\overline{\mathcal{G}}$ with those in the distribution \mathcal{G} . In Section 6.2, we describe the non-reconstruction results that Sly uses. Finally, in Section 6.3, we give the proof of Lemma 6.2 and conclude Theorem 6.1, checking that Sly’s arguments go through in our setting as well.

6.1. The partition function for graphs sampled from $\overline{\mathcal{G}}$

In this section we give an outline of the lemmas which are needed to extend Sly’s reduction for general two-spin models.

Let η be an assignment of spins to U , that is, $\eta \in \{-1, +1\}^U$. For a graph $\overline{G} \sim \overline{\mathcal{G}}$, let $Z_{\overline{G}}^{\alpha, \beta}(\eta)$ be the partition function over configurations which agree with η on U , that is, $Z_{\overline{G}}^{\alpha, \beta}(\eta)$ is the total weight of configurations in $\sigma \in \{-1, +1\}^{V(\overline{G})}$ subject to $|N_-(\sigma) \cap W_+| = \alpha n$, $|N_-(\sigma) \cap W_-| = \beta n$ and $\sigma_U = \eta$.

Let η_1^+ denote the number of vertices in U_+ which are assigned spin $+1$ and let $\eta_1^- = m' - \eta_1^+$ (the number of vertices in U_+ that are assigned spin -1). Define similarly η_2^+, η_2^- for U_- . We have the following analogue of [27, Lemma 3.1].

Lemma 6.3. *For any α, β and all $\eta \in \{-1, +1\}^U$ we have*

$$\frac{\mathbb{E}_{\overline{\mathcal{G}}}[Z_{\overline{G}}^{\alpha, \beta}(\eta)]}{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}]} = (1 + o(1)) \left(\frac{\alpha^{\eta_1^-} (1 - \alpha)^{\eta_1^+} \beta^{\eta_2^-} (1 - \beta)^{\eta_2^+}}{(\alpha - x^*)^{\eta_1^-} (1 - \alpha - \beta + x^*)^{\eta_1^+ - \eta_2^-} (\beta - x^*)^{\eta_2^-}} B_2^{\eta_1^+ - \eta_2^-} \right)^{\Delta - 1} \lambda^{\eta_1^- + \eta_2^-},$$

where x^* is the (unique) non-negative solution of the equation

$$B_1 B_2 (\alpha - x)(\beta - x) = x(1 - \alpha - \beta + x).$$

In particular, when $(\alpha, \beta) = (p^+, p^-)$, it holds that

$$\frac{\mathbb{E}_{\overline{\mathcal{G}}}[Z_{\overline{G}}^{\alpha, \beta}(\eta)]}{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}]} = (1 + o(1)) \frac{C^*}{(1 - q^+)^{m'} (1 - q^-)^{m'}} (q^+)^{\eta_1^-} (1 - q^+)^{\eta_1^+} (q^-)^{\eta_2^-} (1 - q^-)^{\eta_2^+}, \tag{6.2}$$

where

$$C^* = \left(\frac{(B_2 + Q^+)(B_2 + Q^-)}{B_2 + Q^+ + Q^- + B_1 Q^+ Q^-} \right)^{(\Delta - 1)m'}. \tag{6.3}$$

The proof of Lemma 6.3 is given in Section 7.2.1. Seeking high-probability lower bounds on $Z_{\overline{G}}^{\alpha, \beta}(\eta)$, we study the second moment of $Z_{\overline{G}}^{\alpha, \beta}(\eta)$.

Lemma 6.4 (analogue of Lemma 3.5 in [27]). *If Condition 2.3 holds, for $(\alpha, \beta) = (p^+, p^-)$ and all $\eta \in \{-1, +1\}^U$ we have*

$$\frac{\mathbb{E}_{\bar{G}}[(Z_{\bar{G}}^{\alpha, \beta}(\eta))^2]}{\mathbb{E}_{\bar{G}}[Z_{\bar{G}}^{\alpha, \beta}(\eta)]^2} = (1 + o(1))(C^*)^2 \left(\lambda \left(\frac{1 + B_1 Q^-}{B_2 + Q^-} \right)^{\Delta-1} \right)^{2\eta_1^-} \left(\lambda \left(\frac{1 + B_1 Q^+}{B_2 + Q^+} \right)^{\Delta-1} \right)^{2\eta_2^-}, \quad (6.4)$$

where Q^-, Q^+ are given by (3.2) and C^* is given by (6.3).

The proof of Lemma 6.4 is given in Section 7.2.2. Once again, it turns out that the second moment method fails to give high probability bounds, as is captured by the following lemma.

Lemma 6.5. *When Condition 2.3 is true, for $(\alpha, \beta) = (p^+, p^-)$ it holds that*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\bar{G}}[(Z_{\bar{G}}^{\alpha, \beta}(\eta))^2]}{(\mathbb{E}_{\bar{G}}[Z_{\bar{G}}^{\alpha, \beta}(\eta)])^2} = (1 - \omega^2)^{-(\Delta-1)/2} (1 - (\Delta - 1)^2 \omega^2)^{-1/2},$$

where ω is given by (3.4).

The proof of Lemma 6.5 is given in Section 7.2.3. We then apply the small subgraph conditioning method to obtain the following analogue of [27, Lemma 3.9].

Lemma 6.6. *For $\Delta \geq 3$, suppose that for B_1, B_2, λ in the non-uniqueness region, Condition 2.3 is true. Then, for $(\alpha, \beta) = (p^+, p^-)$, asymptotically almost surely over the choice of the graph $\bar{G} \sim \bar{\mathcal{G}}$, for $\eta \in \{-1, +1\}^U$, it holds that*

$$Z_{\bar{G}}^{\alpha, \beta}(\eta) > \frac{1}{\sqrt{n}} \mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^{\alpha, \beta}(\eta)].$$

The proof of Lemma 6.6 is given in Section 8.2.

6.2. The reconstruction problem in the extremal measures

The final ingredient we need is proving that a strong form of non-reconstruction holds in the extremal measures. Let us first introduce some notation.

Let $\hat{T}_{\rho, \ell, \Delta}$ denote the subtree of $\hat{\mathbb{T}}_{\Delta}$ rooted at ρ up to depth ℓ . Let $S_{\rho, \ell}$ be the leaves of $\hat{T}_{\rho, \ell, \Delta}$. Let $X_{\rho, \ell, +}$ denote the marginal probability that the root ρ is occupied in a configuration σ generated by the following process.

- (1) Sample a configuration $\hat{\sigma}$ from the projection of the measure $\hat{\mu}^+$ on $\hat{T}_{\rho, \ell, \Delta}$,
- (2) Conditioning on the configuration $\hat{\sigma}_S$ on $S_{\rho, \ell}$, sample a configuration σ from the Gibbs distribution on $\hat{T}_{\rho, \ell, \Delta}$.

Note that the configuration of the vertices in $S_{\rho, \ell}$ is a random vector, so $X_{\rho, \ell, +}$ is a random variable. Define similarly $X_{\rho, \ell, -}$. We need the following analogue of [27, Lemma 4.2].

Lemma 6.7. *Assume that $\Delta \geq 3$ and B_1, B_2, λ lie in the non-uniqueness region of \mathbb{T}_{Δ} . There exist constants $\zeta_1(\Delta, B_1, B_2, \lambda), \zeta_2(\Delta, B_1, B_2, \lambda) > 0$ such that, for all sufficiently large ℓ ,*

$$\mathbb{P}[|X_{\rho, \ell, \pm} - q^{\pm}| \geq \exp(-\zeta_1 \ell)] \leq \exp(-\exp(\zeta_2 \ell)).$$

Proof. Martinelli, Sinclair and Weitz [19, Proof of Theorem 5.1, equation (21)] prove the required strong concentration for the variables $X_{\rho,\ell,\pm}$ for the Ising model with arbitrary external field (in a form different to that stated here but completely equivalent). Their results can be translated to the general two-spin model after carrying out the reduction described in [26]. Finally, in the case of hard constraints Lemma 6.7 follows by the respective result in [27, Lemma 4.2] (slightly extended in [8, Lemma 30]). Of course, one could also apply the methods in [19] or [27] to directly get the result in general. \square

6.3. Finishing the proofs of the hardness results

We first give the proof of Lemma 6.2; this is a straightforward extension of the proof of [27, Theorem 2.1], so we give a description of the main elements and point to the relevant lemmas which verify that Sly’s proof goes through in our setting as well.

Proof of Lemma 6.2. We first introduce some notation that is used by Sly [27]. Define Σ^\pm to be the configurations $\sigma \in \{-1, +1\}^{V(\bar{G})}$ such that $|N_-(\sigma) \cap W_\pm| \geq |N_-(\sigma) \cap W_\mp|$, that is, a configuration in Σ^+ assigns the spin -1 to more vertices in W_+ than in W_- (and conversely for Σ^-). Similarly, for $\eta \in \{-1, +1\}^U$, define $\Sigma^\pm(\eta)$ to be the configurations σ in $\sigma \in \Sigma^\pm$ such that $\sigma_U = \eta$. Further, for $G \sim \bar{G}$, define

$$Z_{\bar{G}}^\pm(\eta) = \sum_{\sigma \in \Sigma^\pm(\eta)} w_G(\sigma), \quad Z_{\bar{G}}^\pm := \sum_{\sigma \in \Sigma^\pm} w_G(\sigma).$$

Note that

$$Z_{\bar{G}}^\pm = \sum_{\eta \in \{-1, +1\}^U} Z_{\bar{G}}^\pm(\eta).$$

Define similarly Z_G^\pm for a random Δ -regular graph G , that is, for $G \sim \mathcal{G}$, let $Z_G^+ = \sum_{\alpha \geq \beta} Z_G^{\alpha, \beta}$ and $Z_G^- = \sum_{\alpha < \beta} Z_G^{\alpha, \beta}$.

Using that $(\alpha, \beta) = (p^\pm, p^\mp)$ maximizes the logarithm of $\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}]$ (see Lemma 2.2), the first item of Lemma 6.3 together with Markov’s inequality shows that the largest contribution to $\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^\pm(\eta)]$ is given by $\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^{p^\pm, p^\mp}(\eta)]$. This gives

$$\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^\pm(\eta)] = (1 + o(1))(C')^{m'} Q_U^\pm(\eta) \mathbb{E}_{\mathcal{G}}[Z_G^\pm], \tag{6.5}$$

where the last equality uses the conclusion (6.2) of Lemma 6.3 (the constant C' can be inferred from (6.2), though it will not be important for what follows). The derivation of (6.5) is identical to the proof of [27, Lemma 3.3, equation (3.9)]. Summing (6.5) over $\eta \in \{-1, +1\}^U$ yields

$$\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^\pm] = (1 + o(1))(C')^{m'} \mathbb{E}_{\mathcal{G}}[Z_G^\pm]. \tag{6.6}$$

Our results from the small subgraph conditioning method (see Section 8.1), combined with the above observations, yield that $Z_{\bar{G}}^\pm(\eta)$ is close to its expectation, in particular

$$\lim_{n \rightarrow \infty} \max_{\eta \in \{-1, +1\}^U} \Pr_{\bar{\mathcal{G}}} \left(Z_{\bar{G}}^\pm(\eta) < \frac{1}{\sqrt{n}} \mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^\pm(\eta)] \right) = 0. \tag{6.7}$$

see the proof of [27, Theorem 3.10] for details.

At this point, one needs to transfer these results to the graph H . Define Z_H^\pm to be the contribution to the partition function of H from configurations with phase \pm (note that the restriction of such configurations on \bar{G} is a configuration in Σ^\pm). To connect the partition function of H with the partition function of \bar{G} , let T be the graph induced by the edges $E(H) \setminus E(\bar{G})$. In particular, T is a union of disjoint trees and the union of the leaves of the trees is precisely the set U . For $\eta \in \{-1, +1\}^U$, define $Z_T(\eta)$ to be the contribution to the partition function of T from configurations $\sigma \in \{-1, +1\}^T$ such that $\sigma_U = \eta$. These definitions allow us to write

$$Z_H^\pm = \sum_{\eta \in \{-1, +1\}^U} Z_{\bar{G}}^\pm(\eta) Z_T(\eta).$$

It follows by (6.7) and Markov’s inequality that

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr_{\bar{G}} \left(\sum_{\eta \in \{-1, +1\}^U} Z_{\bar{G}}^\pm(\eta) Z_T(\eta) < \frac{1}{2\sqrt{n}} \sum_{\eta \in \{-1, +1\}^U} Z_T(\eta) \mathbb{E}_{\bar{G}}[Z_{\bar{G}}^\pm(\eta)] \right) &= 0, \\ \lim_{n \rightarrow \infty} \Pr_{\bar{G}} \left(\sum_{\eta \in \{-1, +1\}^U} Z_{\bar{G}}^\pm(\eta) Z_T(\eta) > \frac{\sqrt{n}}{3} \sum_{\eta \in \{-1, +1\}^U} Z_T(\eta) \mathbb{E}_{\bar{G}}[Z_{\bar{G}}^\pm(\eta)] \right) &= 0, \end{aligned}$$

see also [27, (4.22), (4.23)]. It follows that $\mu_H(Y(\sigma) = \pm) = Z_H^\pm / Z_H \geq 1/n$ a.a.s. over the choice of the graph \mathcal{G} . This proves the first item in Lemma 6.2.

The proof of the second item in Lemma 6.2 utilizes to a greater extent the connection to the tree and relies crucially on the non-reconstruction results described in Section 6.2. First observe that

$$\mu_H(\sigma_U = \eta \mid Y(\sigma) = \pm) = \frac{Z_{\bar{G}}^\pm(\eta) Z_T(\eta)}{Z_H^\pm} = \frac{Z_{\bar{G}}^\pm(\eta) Z_T(\eta)}{\sum_{\eta' \in \{-1, +1\}^U} Z_{\bar{G}}^\pm(\eta') Z_T(\eta')}. \tag{6.8}$$

Now, provided that

$$Z_{\bar{G}}^\pm(\eta) < \frac{1}{\sqrt{n}} \mathbb{E}_{\bar{G}}[Z_{\bar{G}}^\pm(\eta)]$$

(see (6.7)), the right-hand side of (6.8) is (with large probability) within a multiplicative factor n from

$$v^\pm(\eta) := \frac{Z_T(\eta) \mathbb{E}_{\bar{G}}[Z_{\bar{G}}^\pm(\eta)]}{\sum_{\eta' \in \{-1, +1\}^U} Z_T(\eta') \mathbb{E}_{\bar{G}}[Z_{\bar{G}}^\pm(\eta')]} = \frac{Z_T(\eta) Q_U^\pm(\eta)}{\sum_{\eta' \in \{-1, +1\}^U} Z_T(\eta') Q_U^\pm(\eta')},$$

using (6.6) in the first equality. The crucial idea, captured in the proof of [27, Lemma 4.3], is that the measures v^\pm on configurations $\{-1, +1\}^U$ are the same as the distributions induced by the random processes we described in Section 6.2; see the proof of [27, Lemma 4.3] for more details. This allows us to use Lemma 6.7 together with (6.7) to bound the probability mass of ‘bad’ configurations η on U , that is, configurations on U which exert a large influence on the roots R of the trees; see [27, Proof of Theorem 2.1]. This yields item (2) in Lemma 6.2, concluding the argument. \square

With Lemma 6.2 at hand, Theorem 6.1 can be proved completely analogously to [27, Proof of Theorems 1 and 2].

Proof of Theorem 6.1. Use Lemma 6.2 and the reduction in [27, Section 2.2]. \square

7. Calculating moments

7.1. Random bipartite Δ -regular graphs

We use the information we obtained for the first and second moments to prove Lemma 2.7. Its proof is a technical calculation.

Recall that

$$Q^\pm = \frac{q^\pm}{1 - q^\pm}.$$

To aid the presentation, it will be convenient to define the following expressions, which will appear in the analysis of the asymptotics of the first and second moments:

$$\begin{aligned} E_1 &:= B_1 Q^- Q^+ + Q^- + Q^+ + B_2, \\ E_2 &:= B_1 Q^- Q^+ + B_1 B_2 (Q^+ + Q^-) + B_2, \\ E_3 &:= (1 - B_1 B_2)^2 Q^- Q^+ \\ &\quad + (1 + B_1(Q^+ + Q^-) + B_1^2 Q^- Q^+)(B_2^2 + B_2(Q^- + Q^+) + Q^- Q^+). \end{aligned} \tag{7.1}$$

We are also going to need the explicit values of

$$\mathbf{X}^* = \arg \max_{\mathbf{X}} g_1(\mathbf{X}) \quad \text{and} \quad \mathbf{Y}^* = \arg \max_{\mathbf{Y}} g_2(\mathbf{Y})$$

when $\alpha = p^+$, $\beta = p^-$, $\gamma = \alpha^2$, $\delta = \beta^2$. These are given by

$$\begin{pmatrix} x_{11}^* & x_{12}^* \\ x_{21}^* & x_{22}^* \end{pmatrix} = \frac{1}{E_1} \begin{pmatrix} B_1 Q^- Q^+ & Q^+ \\ Q^- & B_2 \end{pmatrix}, \tag{7.2}$$

$$\begin{pmatrix} y_{11}^* & y_{12}^* & y_{13}^* & y_{14}^* \\ y_{21}^* & y_{22}^* & y_{23}^* & y_{24}^* \\ y_{31}^* & y_{32}^* & y_{33}^* & y_{34}^* \\ y_{41}^* & y_{42}^* & y_{43}^* & y_{44}^* \end{pmatrix} = \begin{pmatrix} x_{11}^* & x_{12}^* \\ x_{21}^* & x_{22}^* \end{pmatrix} \otimes \begin{pmatrix} x_{11}^* & x_{12}^* \\ x_{21}^* & x_{22}^* \end{pmatrix}, \tag{7.3}$$

where the operator \otimes denotes the tensor product. The easiest way to derive the validity of these values is to use the strict concavity of the functions $g_1(\mathbf{X})$, $g_2(\mathbf{Y})$ and just check that the required equalities for the critical points are satisfied.

Lemma 7.1. For $(\alpha, \beta) = (p^+, p^-)$, we have that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}]}{\frac{1}{n} \varrho^{n\Phi_1(\alpha, \beta, \mathbf{X}^*)}} = \frac{1}{2\pi} (\alpha\beta(1 - \alpha)(1 - \beta))^{(\Delta-1)/2} \left(\frac{Q^+ Q^- E_2}{E_1^3} \right)^{-\Delta/2}.$$

Lemma 7.2. For $(\alpha, \beta) = (p^+, p^-)$, we have that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha, \beta})^2]}{\frac{1}{n^2} e^{n\Phi_2(\gamma^*, \delta^*, \mathbf{Y}^*)}} = \frac{1}{4\pi^2} (\alpha\beta(1-\alpha)(1-\beta))^{2(\Delta-1)} \left(\frac{(Q^+Q^-)^4 E_2^3 E_3}{E_1^{13}} \right)^{-\Delta/2} \\ \times \left(\frac{(Q^+Q^-)^2 E_2 E_3}{E_1^7} \right)^{1/2} (1 - (\Delta - 1)^2 \omega^2)^{-1/2},$$

where $\gamma^* = \alpha^2, \delta^* = \beta^2, \mathbf{Y}^*$ is given by (7.3) and ω by (3.4).

The proofs of Lemmas 7.1 and 7.2 are given in Sections 7.1.1 and 7.1.2 respectively. We need to state one more lemma before we can give the proof of Lemma 2.7.

Lemma 7.3. It holds that $2\Phi_1(\alpha, \beta, \mathbf{X}^*) = \Phi_2(\alpha^2, \beta^2, \mathbf{Y}^*)$.

Proof. It clearly suffices to prove

$$f_2(\alpha^2, \beta^2) = 2f_1(\alpha, \beta) \quad \text{and} \quad g_2(\mathbf{Y}^*) = 2g_1(\mathbf{X}^*).$$

Both of these can be verified by straightforward calculations. A neater way to do this, using the uncorrelated property represented by Condition 2.3 more directly, is to use entropy arguments. We illustrate this in the case of f_1 and f_2 , with the same technique extending to g_1, g_2 as well.

Define A, B to be independent Bernoulli variables with success probabilities α, β respectively. It is straightforward to see that $f_1(\alpha, \beta) = -(H(A) + H(B))$, where $H(X)$ is the entropy of the random variable X . Let $A' = (A_1, A_2)$, where A_1, A_2 are independent copies of the random variable A . Define similarly B' . Using the definition of entropy, it is easy to check that $f_2(\alpha^2, \beta^2) = -(H(A') + H(B'))$. But clearly $H(A') = H(A_1) + H(A_2) = 2H(A)$ and similarly for B, B' , thus verifying that $f_2(\alpha^2, \beta^2) = 2f_1(\alpha, \beta)$.

Using (7.3), one can easily extend the previous argument to show that $g_2(\mathbf{Y}^*) = 2g_1(\mathbf{X}^*)$. We should emphasize that the proof of Lemma 7.3 does not rely on the fact that Q^+, Q^- are a function of the critical probabilities on the infinite tree, that is, we do not need Lemma 3.1, but rather the representation given in (3.6) and the uncorrelated property represented by Condition 2.3. □

Proof of Lemma 2.7. Let

$$L := \lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha, \beta})^2]}{(\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}])^2}.$$

Employ Lemmas 7.1, 7.2 and 7.3 to obtain

$$L = (1 - (\Delta - 1)^2 \omega^2)^{-1/2} (\alpha\beta(1-\alpha)(1-\beta))^{\Delta-1} \left(\frac{(Q^+Q^-)^2 E_2 E_3}{E_1^7} \right)^{1/2} R^{-\Delta/2},$$

where

$$R = \left(\frac{(Q^+Q^-)^4 E_2^3 E_3}{E_1^{13}} \right) \cdot \left(\frac{Q^+Q^- E_2}{E_1^3} \right)^{-2} = \frac{(Q^+Q^-)^2 E_2 E_3}{E_1^7}.$$

Thus

$$\begin{aligned} L &= (1 - (\Delta - 1)^2 \omega^2)^{-1/2} (\alpha\beta(1 - \alpha)(1 - \beta))^{\Delta-1} \left(\frac{(Q^+ Q^-)^2 E_2 E_3}{E_1^7} \right)^{-(\Delta-1)/2} \\ &= (1 - (\Delta - 1)^2 \omega^2)^{-1/2} \left(\frac{1}{\alpha^2 \beta^2 (1 - \alpha)^2 (1 - \beta)^2} \cdot \frac{(Q^+ Q^-)^2 E_2 E_3}{E_1^7} \right)^{-(\Delta-1)/2}. \end{aligned}$$

By (3.6), we have

$$\alpha = \frac{Q^+(1 + B_1 Q^-)}{E_1} \quad \text{and} \quad \beta = \frac{Q^-(1 + B_1 Q^+)}{E_1},$$

so that

$$L = (1 - (\Delta - 1)^2 \omega^4)^{-1/2} \left(\frac{E_1 E_2 E_3}{(B_2 + Q^-)^2 (B_2 + Q^+)^2 (1 + B_1 Q^-)^2 (1 + B_1 Q^+)^2} \right)^{-(\Delta-1)/2}.$$

The lemma follows after observing the easily proved identity

$$\frac{E_1 E_2 E_3}{(B_2 + Q^-)^2 (B_2 + Q^+)^2 (1 + B_1 Q^-)^2 (1 + B_1 Q^+)^2} = 1 - \omega^2. \quad \square$$

7.1.1. The asymptotics of the first moment. In this section we prove Lemma 7.1, that is, we compute the asymptotics of $\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}]$ for $(\alpha, \beta) = (p^+, p^-)$; see the relevant Sections 4.1 and 4.3.

Proof of Lemma 7.1. We will use the full-dimensional representation (see Section 5.1) of $g_1(\mathbf{X})$: while we are still going to refer to the values x_{ij}^* , in what follows the only free variable is going to be x_{11} . By the approximation

$$\binom{bn}{an} = (1 + o(1)) \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{b}{a(b-a)}} e^{n(b \ln b - a \ln a - (a-b) \ln(a-b))},$$

we obtain

$$\begin{aligned} \frac{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}]}{\frac{1}{n} e^{n\Phi_1(\alpha,\beta,\mathbf{X}^*)}} &= (1 + o(1)) \frac{1}{2\pi} \cdot (\alpha\beta(1 - \alpha)(1 - \beta))^{(\Delta-1)/2} \\ &\quad \times \left(\sum_{x_{11}} \left(\prod_{i,j} x_{ij} \right)^{-1/2} \frac{1}{\sqrt{2\pi n}} \cdot e^{n(g_1(x_{11}) - g_1(x_{11}^*))} \right)^{\Delta}. \end{aligned}$$

Note that in the above sum terms with $|x_{11} - x_{11}^*| \leq O(1)$ dominate exponentially over the rest, and in a sufficiently small ball around x_{11}^* the function decays quadratically, as established by Lemma 5.1. Thus, standard integration arguments (for precise details see [15, Chapter 9]) yield

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}]}{\frac{1}{n} e^{n\Phi_1(\alpha,\beta,\mathbf{X}^*)}} &= \frac{1}{2\pi} (\alpha\beta(1 - \alpha)(1 - \beta))^{(\Delta-1)/2} \left(\prod_{i,j} x_{ij}^* \right)^{-\Delta/2} \\ &\quad \times \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{\partial^2 g_1}{\partial x_{11}^2}(x_{11}^*) \cdot \frac{(x_{11} - x_{11}^*)^2}{2} \right) dx_{11} \right)^{\Delta}. \end{aligned}$$

Let

$$H := \frac{\partial^2 g_1}{\partial x_{11}^2}(x_{11}^*).$$

Since g_1 is strictly concave and it achieves its maximum at x_{11}^* , we have $H < 0$. It follows that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{\partial^2 g_1}{\partial x_{11}^2}(x_{11}^*) \cdot \frac{(x_{11} - x_{11}^*)^2}{2}\right) dx_{11} = (-H)^{-1/2},$$

and consequently

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}]^2}{\frac{1}{n} e^{n\Phi_1(\alpha, \beta, \mathbf{X}^*)}} = \frac{1}{2\pi} (\alpha\beta(1-\alpha)(1-\beta))^{(\Delta-1)/2} \left((-H) \prod_{i,j} x_{ij}^* \right)^{-\Delta/2}.$$

We have

$$\frac{\partial^2 g_1}{\partial x_{11}^2} = -\left(\frac{1}{x_{11}} + \frac{1}{\alpha - x_{11}} + \frac{1}{\beta - x_{11}} + \frac{1}{1 - \alpha - \beta + x_{11}} \right).$$

Substituting the values of α, β from (3.6) and the values of the x_{ij}^* from (7.2), we obtain

$$-H = \frac{E_1 E_2}{B_1 B_2 Q^- Q^+}, \quad \prod_{i,j} x_{ij}^* = \frac{B_1 B_2 (Q^+ Q^-)^2}{E_1^4}, \quad \text{so that } (-H) \prod_{i,j} x_{ij}^* = \frac{Q^+ Q^- E_2}{E_1^3},$$

and the result follows. □

7.1.2. The asymptotics of the second moment. In this section we prove Lemma 7.2, that is, we compute the asymptotics of $\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha, \beta})^2]$ for $(\alpha, \beta) = (p^+, p^-)$. We have

$$\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha, \beta})^2] = \sum_{\gamma, \delta} \mathbb{E}_{\mathcal{G}}[Y_G^{\gamma, \delta}],$$

where $\mathbb{E}_{\mathcal{G}}[Y_G^{\gamma, \delta}]$ was calculated in Section 4.2 (see also the relevant Section 4.3).

Proof of Lemma 7.2. As in the proof of Lemma 7.1, in the following we are going to refer to the full-dimensional representations (see Section 5.1) of g_2, Φ_2 , that is, our free variables are going to be $\{y_{ij}\}$ with $1 \leq i, j \leq 3$, so the rest of the y_{ij} are just shorthand for their respective values (in terms of the free variables). We have that

$$\begin{aligned} \frac{\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha, \beta})^2]}{\frac{1}{n^2} e^{n\Phi_2(\gamma^*, \delta^*, \mathbf{Y}^*)}} &= (1 + o(1)) \frac{1}{4\pi^2} \\ &\times \sum_{\gamma, \delta} \left(\frac{1}{\sqrt{2\pi n}} \right)^2 (\gamma\delta(1-2\alpha+\gamma)(1-2\beta+\delta)(\alpha-\gamma)^2(\beta-\delta)^2)^{(\Delta-1)/2} \\ &\times \left(\sum_{\mathbf{Y}} \left(\frac{1}{\sqrt{2\pi n}} \right)^9 \left(\prod_{i=1}^4 \prod_{j=1}^4 \frac{1}{\sqrt{y_{ij}}} \right) e^{n(\Phi_2(\gamma, \delta, \mathbf{Y}) - \Phi_2(\gamma^*, \delta^*, \mathbf{Y}^*)) / \Delta} \right)^{\Delta}. \end{aligned}$$

The function $\Phi_2(\gamma, \delta, \mathbf{Y})$ has a unique maximum at $(\gamma^*, \delta^*, \mathbf{Y}^*)$. Thus, in the above sum, terms with $\|(\gamma, \delta, \mathbf{Y}) - (\gamma^*, \delta^*, \mathbf{Y}^*)\| \leq O(1)$ dominate exponentially over the rest. Moreover, in the region $\|(\gamma, \delta, \mathbf{Y}) - (\gamma^*, \delta^*, \mathbf{Y}^*)\| \leq O(1)$ the function Φ_2 decays quadratically.

For convenience, let

$$L := \lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha, \beta})^2]}{\frac{1}{n^2} e^{n\Phi_2(\gamma^*, \delta^*, \mathbf{Y}^*)}}.$$

As in the proof of Lemma 7.1, we obtain

$$\begin{aligned} L &= \left(\frac{1}{\sqrt{2\pi}} \right)^{15} (\gamma^* \delta^* (1 - 2\alpha + \gamma^*) (1 - 2\beta + \delta^*) (\alpha - \gamma^*)^2 (\beta - \delta^*)^2)^{(\Delta-1)/2} \\ &\quad \times \left(\prod_{i,j=1}^4 y_{ij}^* \right)^{-\Delta/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{(\sqrt{2\pi})^9} \int_{-\infty}^{\infty} e^{\frac{1}{2}(\gamma, \delta, \mathbf{Y}) \cdot \mathbf{H}(\gamma, \delta, \mathbf{Y})^T} d\mathbf{Y} \right)^{\Delta} d\gamma d\delta \\ &= \frac{1}{4\pi^2} (\alpha\beta(1-\alpha)(1-\beta))^{2(\Delta-1)} \left(\prod_{i,j=1}^4 y_{ij}^* \right)^{-\Delta/2} \\ &\quad \times \left(\frac{1}{\sqrt{2\pi}} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{(\sqrt{2\pi})^9} \int_{-\infty}^{\infty} e^{\frac{1}{2}(\gamma, \delta, \mathbf{Y}) \cdot \mathbf{H}(\gamma, \delta, \mathbf{Y})^T} d\mathbf{Y} \right)^{\Delta} d\gamma d\delta, \end{aligned} \tag{7.4}$$

where \mathbf{H} denotes the Hessian matrix of Φ_2 evaluated at $(\gamma^*, \delta^*, \mathbf{Y}^*)$ scaled by $1/\Delta$ and the operator \cdot stands for matrix multiplication.

We thus focus on computing the integral in (7.4). This will be achieved by two successive Gaussian integrations. First, we split the exponent of the quantity inside the integral as follows:

$$\frac{1}{2}(\gamma, \delta, \mathbf{Y}) \cdot \mathbf{H} \cdot (\gamma, \delta, \mathbf{Y})^T = \frac{1}{2}(\gamma, \delta) \cdot \mathbf{H}_{\gamma, \delta} \cdot (\gamma, \delta)^T - \frac{1}{2}\mathbf{Y} \cdot (-\mathbf{H}_{\mathbf{Y}}) \cdot \mathbf{Y}^T + \sum_{i=1}^3 \sum_{j=1}^3 T_{ij} y_{ij},$$

where $\mathbf{H}_{\gamma, \delta}$ denotes the principal minor of \mathbf{H} corresponding to γ, δ , $\mathbf{H}_{\mathbf{Y}}$ denotes the principal minor of \mathbf{H} corresponding to \mathbf{Y} , and

$$T_{ij} = \frac{1}{\Delta} \left(\gamma \cdot \frac{\partial^2 \Phi_2}{\partial \gamma \partial y_{ij}} + \delta \cdot \frac{\partial^2 \Phi_2}{\partial \delta \partial y_{ij}} \right).$$

It follows that

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} e^{\frac{1}{2}(\gamma, \delta, \mathbf{Y}) \cdot \mathbf{H}(\gamma, \delta, \mathbf{Y})^T} d\mathbf{Y} \right)^{\Delta} \\ &= e^{\frac{\Delta}{2}(\gamma, \delta) \cdot \mathbf{H}_{\gamma, \delta} \cdot (\gamma, \delta)^T} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}\mathbf{Y} \cdot (-\mathbf{H}_{\mathbf{Y}}) \cdot \mathbf{Y}^T + \sum_{i=1}^3 \sum_{j=1}^3 T_{ij} y_{ij}} d\mathbf{Y} \right)^{\Delta}. \end{aligned}$$

Let \mathbf{T} denote the row vector with entries T_{ij} . Note that $\mathbf{H}_{\mathbf{Y}}$ is the Hessian of $g_2(\mathbf{Y})$ evaluated at \mathbf{Y}^* . Since $g_2(\mathbf{Y})$ is concave and g_2 achieves its maximum at \mathbf{Y}^* , we have that

\mathbf{H}_Y is negative definite. Hence,

$$\begin{aligned} & \left(\frac{1}{(\sqrt{2\pi})^9} \int_{-\infty}^{\infty} e^{\frac{1}{2}(\gamma, \delta, \mathbf{Y}) \cdot \mathbf{H}(\gamma, \delta, \mathbf{Y})^T} d\mathbf{Y} \right)^\Delta \\ &= \frac{1}{(\text{Det}(-\mathbf{H}_Y))^{\Delta/2}} e^{\frac{\Delta}{2}(\mathbf{T} \cdot (-\mathbf{H}_Y)^{-1} \cdot \mathbf{T}^T + (\gamma, \delta) \cdot \mathbf{H}_{\gamma, \delta} \cdot (\gamma, \delta)^T)}. \end{aligned}$$

We are thus left with the task of computing the integral

$$\left(\frac{1}{\sqrt{2\pi}} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{\Delta}{2}(\mathbf{T} \cdot (-\mathbf{H}_Y)^{-1} \cdot \mathbf{T}^T + (\gamma, \delta) \cdot \mathbf{H}_{\gamma, \delta} \cdot (\gamma, \delta)^T)}. \tag{7.5}$$

It can easily be seen that the expression in the exponent is a quadratic polynomial in γ, δ , so that we may again perform Gaussian integration. We have

$$\frac{\Delta}{2}(\mathbf{T} \cdot (-\mathbf{H}_Y)^{-1} \cdot \mathbf{T}^T + (\gamma, \delta) \cdot \mathbf{H}_{\gamma, \delta} \cdot (\gamma, \delta)^T) = D\gamma^2 + E\gamma\delta + F\delta^2, \tag{7.6}$$

for some complicated D, E, F , which can be determined explicitly by (7.6), once we perform the substitutions. Hence,

$$\left(\frac{1}{\sqrt{2\pi}} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{\Delta}{2}(\mathbf{T} \cdot (-\mathbf{H}_Y)^{-1} \cdot \mathbf{T}^T + (\gamma, \delta) \cdot \mathbf{H}_{\gamma, \delta} \cdot (\gamma, \delta)^T)} d\gamma d\delta = \frac{1}{(4DF - E^2)^{1/2}}, \tag{7.7}$$

provided that $4DF - E^2 > 0$, so the integration is meaningful. We will check this condition later.

Combining equations (7.4), (7.5), (7.7), we obtain

$$\begin{aligned} L &= \frac{1}{4\pi^2} (\alpha\beta(1-\alpha)(1-\beta))^{2(\Delta-1)} \left(\text{Det}(-\mathbf{H}_Y) \prod_{i,j=1}^4 y_{ij}^* \right)^{-\Delta/2} (4DF - E^2)^{-1/2} \\ &= \frac{1}{4\pi^2} (\alpha\beta(1-\alpha)(1-\beta))^{2(\Delta-1)} \cdot R, \end{aligned} \tag{7.8}$$

where

$$R := \left(\text{Det}(-\mathbf{H}_Y) \prod_{i,j=1}^4 y_{ij}^* \right)^{-\Delta/2} (4DF - E^2)^{-1/2}.$$

At this point we use Maple to perform the remaining calculations. The solution in the form of (7.3) is particularly handy, since it rationalizes the relevant expressions. We have

$$\begin{aligned} \text{Det}(-\mathbf{H}_Y) &= \frac{E_1^{19} E_2^3 E_3}{(B_1 B_2)^8 (Q^+ Q^-)^{12}}, \quad \prod_{i,j=1}^4 y_{ij}^* = \frac{(B_1 B_2)^8 (Q^+ Q^-)^{16}}{E_1^{32}}, \\ 4DF - E^2 &= \frac{E_1^7}{(Q^+ Q^-)^2 E_2 E_3} \cdot (1 - (\Delta - 1)^2 \omega^2). \end{aligned}$$

Note that by Lemma 3.2, we have $4DF - E^2 > 0$, so that the integration in (7.7) is indeed meaningful. After plugging the above values into R and substituting back in (7.8), the lemma follows. \square

7.2. Gadget

In this subsection we prove Lemmas 6.3 and 6.4. We will need the following bound.

Lemma 7.4 (general form of Lemma 3.2 in [27]). *Let $b_1, \dots, b_\ell \geq 0$ and y_1, \dots, y_ℓ be integers. Let $a = b_1 + \dots + b_\ell$ and $x = y_1 + \dots + y_\ell$. Assume $y_i^2 \leq b_i$ for each $i \in \{1, \dots, \ell\}$. Then*

$$\frac{\binom{a+x}{b_1+y_1, b_2+y_2, \dots, b_\ell+y_\ell}}{\binom{a}{b_1, b_2, \dots, b_\ell}} = \left(1 + O\left(\sum_{i=1}^{\ell} y_i^2/b_i\right)\right) \frac{a^x}{b_1^{y_1} \dots b_\ell^{y_\ell}}.$$

Proof. We have

$$\begin{aligned} \frac{\binom{a+x}{b_1+y_1, b_2+y_2, \dots, b_\ell+y_\ell}}{\binom{a}{b_1, b_2, \dots, b_\ell}} &= \frac{(a+x)!}{a!} \prod_{i=1}^{\ell} \frac{b_i!}{(b_i+y_i)!} \\ &= \frac{a^x}{b_1^{y_1} \dots b_\ell^{y_\ell}} \left(\prod_{j=1}^x (1+j/a)\right) / \prod_{i=1}^{\ell} \left(\prod_{j=1}^{y_i} (1+j/b_i)\right) \\ &= \frac{a^x}{b_1^{y_1} \dots b_\ell^{y_\ell}} \left(1 + O\left(\frac{x^2}{a} + \frac{y_1^2}{b_1} + \dots + \frac{y_\ell^2}{b_\ell}\right)\right). \end{aligned}$$

The lemma now follows from the fact that $x^2/a \leq y_1^2/b_1 + \dots + y_\ell^2/b_\ell$ (which follows from the Cauchy–Schwarz inequality). \square

7.2.1. First moment.

Proof of Lemma 6.3. By (4.1), recall that

$$\mathbb{E}_{\mathcal{G}} [Z_G^{\alpha, \beta}] = \lambda^{(\alpha+\beta)n} \binom{n}{\alpha n} \binom{n}{\beta n} \left(\sum_{x \leq \min\{\alpha, \beta\}} \kappa_G^{\alpha, \beta, x}\right)^\Delta,$$

where

$$\kappa_G^{\alpha, \beta, x} = \frac{\binom{\alpha n}{x n} \binom{(1-\alpha)n}{(\beta-x)n}}{\binom{n}{\beta n}} B_1^{x n} B_2^{(1-\alpha-\beta+x)n}.$$

Similarly, we have

$$\begin{aligned} \mathbb{E}_{\bar{\mathcal{G}}} [Z_{\bar{\mathcal{G}}}^{\alpha, \beta}(\eta)] &= \lambda^{(\alpha+\beta)n + \eta_1^- + \eta_2^-} \binom{n}{\alpha n} \binom{n}{\beta n} \\ &\quad \times \left(\sum_{x \leq \min\{\alpha + \eta_1^-/n, \beta + \eta_2^-/n\}} \kappa_{\bar{\mathcal{G}}}^{\alpha, \beta, x}(\eta)\right)^{\Delta-1} \left(\sum_{x \leq \min\{\alpha, \beta\}} \kappa_G^{\alpha, \beta, x}\right), \end{aligned}$$

where

$$\kappa_{\bar{\mathcal{G}}}^{\alpha, \beta, x}(\eta) = \frac{\binom{\alpha n + \eta_1^-}{x n} \binom{(1-\alpha)n + \eta_1^+}{(\beta-x)n + \eta_2^-}}{\binom{n + m'}{\beta n + \eta_2^-}} B_1^{x n} B_2^{(1-\alpha-\beta+x)n + \eta_1^+ - \eta_2^-}.$$

From Lemma 7.4 and the fact that $\eta_1^+, \eta_1^-, \eta_2^+, \eta_2^- \leq n^{1/4}$, we have the following estimate on the ratio of $\kappa_G^{\alpha, \beta, x}(\eta)$ and $\kappa_G^{\alpha, \beta, x}$:

$$\frac{\kappa_G^{\alpha, \beta, x}(\eta)}{\kappa_G^{\alpha, \beta, x}} = (1 + o(1)) \frac{\alpha^{\eta_1^-} (1 - \alpha)^{\eta_1^+} \beta^{\eta_2^-} (1 - \beta)^{\eta_2^+}}{(\alpha - x)^{\eta_1^-} (1 - \alpha - \beta + x)^{\eta_1^+ - \eta_2^-} (\beta - x)^{\eta_2^-}} B_2^{\eta_1^+ - \eta_2^-}. \tag{7.9}$$

It is easy to see that for any α, β , the logarithm of the function $\kappa_G^{\alpha, \beta, x}$ scaled by n is exactly the full-dimensional representation of the function $g_1(\mathbf{X})$ (see Section 4.3). Since $g_1(\mathbf{X})$ is strictly concave with respect to \mathbf{X} , it has a unique maximum. With these observations, it can easily be calculated that the value of $x = x_{11}$ for which $g_1(\mathbf{X})$ achieves its maximum is given by the unique positive solution of

$$B_1 B_2 (\alpha - x^*) (\beta - x^*) = x^* (1 - \alpha - \beta + x^*).$$

By Lemma 5.1, the logarithm of $\kappa_G^{\alpha, \beta, x}$ decays quadratically in the distance from x^* . Let

$$\mathcal{A} = \{x : |x - x^*| \leq n^{-1/4}\}.$$

The relative contribution of terms outside \mathcal{A} to $\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}]$ is $\exp(-\Omega(n^{1/2}))$ and hence can be omitted. Similarly, using (7.9), the relative contribution of terms outside \mathcal{A} to $\mathbb{E}_{\overline{\mathcal{G}}}[Z_{\overline{\mathcal{G}}}^{\alpha, \beta}(\eta)]$ is $\exp(-\Omega(n^{1/2}))$ and hence can also be omitted.

For $x \in \mathcal{A}$ we have

$$\frac{\kappa_G^{\alpha, \beta, x}(\eta)}{\kappa_G^{\alpha, \beta, x}} = (1 + o(1)) \frac{\alpha^{\eta_1^-} (1 - \alpha)^{\eta_1^+} \beta^{\eta_2^-} (1 - \beta)^{\eta_2^+}}{(\alpha - x^*)^{\eta_1^-} (1 - \alpha - \beta + x^*)^{\eta_1^+ - \eta_2^-} (\beta - x^*)^{\eta_2^-}} B_2^{\eta_1^+ - \eta_2^-}.$$

Thus,

$$\begin{aligned} & \frac{\mathbb{E}_{\overline{\mathcal{G}}}[Z_{\overline{\mathcal{G}}}^{\alpha, \beta}(\eta)]}{\mathbb{E}_{\mathcal{G}}[Z_{\mathcal{G}}^{\alpha, \beta}]} \\ &= (1 + o(1)) \left(\frac{\alpha^{\eta_1^-} (1 - \alpha)^{\eta_1^+} \beta^{\eta_2^-} (1 - \beta)^{\eta_2^+}}{(\alpha - x^*)^{\eta_1^-} (1 - \alpha - \beta + x^*)^{\eta_1^+ - \eta_2^-} (\beta - x^*)^{\eta_2^-}} B_2^{\eta_1^+ - \eta_2^-} \right)^{\Delta - 1} \lambda^{\eta_1^- + \eta_2^-}, \end{aligned}$$

which proves the first part of the lemma.

When $(\alpha, \beta) = (p^+, p^-)$, we express α, β and $x^* = x_{11}^*$ in terms of Q^+, Q^- using equations (3.6) and (7.2). We obtain

$$\begin{aligned} & \frac{\alpha^{\eta_1^-} (1 - \alpha)^{\eta_1^+} \beta^{\eta_2^-} (1 - \beta)^{\eta_2^+}}{(\alpha - x^*)^{\eta_1^-} (1 - \alpha - \beta + x^*)^{\eta_1^+ - \eta_2^-} (\beta - x^*)^{\eta_2^-}} B_2^{\eta_1^+ - \eta_2^-} \\ &= \left(\frac{B_2(1 - \alpha)(1 - \beta)}{1 - \alpha - \beta + x^*} \right)^{m'} \left(\frac{\alpha(1 - \alpha - \beta + x^*)}{B_2(\alpha - x^*)(1 - \alpha)} \right)^{\eta_1^-} \left(\frac{\beta(1 - \alpha - \beta + x^*)}{B_2(\beta - x^*)(1 - \beta)} \right)^{\eta_2^-} \\ &= \left(\frac{(B_2 + Q^+)(B_2 + Q^-)}{B_2 + Q^+ + Q^- + B_1 Q^+ Q^-} \right)^{m'} \left(\frac{1 + B_1 Q^-}{B_2 + Q^-} \right)^{\eta_1^-} \left(\frac{1 + B_1 Q^+}{B_2 + Q^+} \right)^{\eta_2^-}, \end{aligned}$$

and hence

$$\frac{\mathbb{E}_{\overline{\mathcal{G}}}[Z_{\overline{\mathcal{G}}}^{\alpha, \beta}(\eta)]}{\mathbb{E}_{\mathcal{G}}[Z_{\mathcal{G}}^{\alpha, \beta}]} = (1 + o(1)) C^* \left(\lambda \left(\frac{1 + B_1 Q^-}{B_2 + Q^-} \right)^{\Delta - 1} \right)^{\eta_1^-} \left(\lambda \left(\frac{1 + B_1 Q^+}{B_2 + Q^+} \right)^{\Delta - 1} \right)^{\eta_2^-}, \tag{7.10}$$

where C^* is given by (6.3). Using the fact that Q^+, Q^- satisfy (3.3), we have

$$Q^+ = \lambda \left(\frac{1 + B_1 Q^-}{B_2 + Q^-} \right)^{\Delta-1} \quad \text{and} \quad Q^- = \lambda \left(\frac{1 + B_1 Q^+}{B_2 + Q^+} \right)^{\Delta-1}.$$

Hence, we obtain

$$\begin{aligned} \frac{\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^{\alpha,\beta}(\eta)]}{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}]} &= (1 + o(1))C^*(1 + Q^+)^{m'}(1 + Q^-)^{m'} \\ &\quad \times \left(\frac{Q^+}{1 + Q^+} \right)^{\eta_1^-} \left(\frac{1}{1 + Q^+} \right)^{\eta_1^+} \left(\frac{Q^-}{1 + Q^-} \right)^{\eta_2^-} \left(\frac{1}{1 + Q^-} \right)^{\eta_2^+}. \end{aligned}$$

The second part of the lemma follows after observing that

$$q^\pm = \frac{Q^\pm}{1 + Q^\pm}, \quad 1 - q^\pm = \frac{1}{1 + Q^\pm}. \quad \square$$

7.2.2. Second moment.

Proof of Lemma 6.4. By (4.4), we have

$$\begin{aligned} \mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha,\beta})^2] &= \lambda^{2(\alpha+\beta)n} \binom{n}{\alpha n} \binom{n}{\beta n} \\ &\quad \times \sum_{\gamma,\delta} \binom{\alpha n}{\gamma n} \binom{(1-\alpha)n}{(\alpha-\gamma)n} \binom{\beta n}{\delta n} \binom{(1-\beta)n}{(\beta-\gamma)n} \left(\sum_{\mathbf{Y}} \kappa_G^{\gamma,\delta,\mathbf{Y}} \right)^\Delta, \end{aligned} \quad (7.11)$$

where

$$\begin{aligned} \kappa_G^{\gamma,\delta,\mathbf{Y}} &= \frac{\prod_{i=1}^4 \binom{L_i n}{y_{i1}n, y_{i2}n, y_{i3}n, y_{i4}n} \prod_{j=1}^4 \binom{R_j n}{y_{1j}n, y_{2j}n, y_{3j}n, y_{4j}n}}{\binom{n}{y_{11}n, y_{12}n, \dots, y_{44}n}} \\ &\quad \times B_1^{(2y_{11}+y_{12}+y_{13}+y_{21}+y_{22}+y_{31}+y_{33})n} B_2^{(y_{22}+y_{24}+y_{33}+y_{34}+y_{42}+y_{43}+2y_{44})n}, \end{aligned}$$

$$\begin{aligned} L_1 &= \gamma, & L_2 &= L_3 = \alpha - \gamma, & L_4 &= 1 - 2\alpha + \gamma, \\ R_1 &= \delta, & R_2 &= R_3 = \beta - \delta, & R_4 &= 1 - 2\beta + \delta, \end{aligned}$$

and \mathbf{Y} stands for the non-negative variables y_{ij} , $1 \leq i, j \leq 4$ such that

$$\sum_j y_{ij} = L_i \quad \text{and} \quad \sum_i y_{ij} = R_j.$$

Similarly, we have that

$$\begin{aligned} \mathbb{E}_{\bar{\mathcal{G}}}[(Z_{\bar{G}}^{\alpha,\beta}(\eta))^2] &= \lambda^{2(\alpha+\beta)n+2(\eta_1^-+\eta_2^-)} \binom{n}{\alpha n} \binom{n}{\beta n} \\ &\quad \times \sum_{\gamma,\delta} \binom{\alpha n}{\gamma n} \binom{(1-\alpha)n}{(\alpha-\gamma)n} \binom{\beta n}{\delta n} \binom{(1-\beta)n}{(\beta-\gamma)n} \\ &\quad \times \left(\sum_{\hat{\mathbf{Y}}} \kappa_{\bar{G}}^{\gamma,\delta,\hat{\mathbf{Y}}}(\eta) \right)^{\Delta-1} \left(\sum_{\mathbf{Y}} \kappa_G^{\gamma,\delta,\mathbf{Y}} \right), \end{aligned} \quad (7.12)$$

where

$$\kappa_{\bar{G}}^{\gamma, \delta, \hat{\mathbf{Y}}}(\eta) = \frac{\prod_{i=1}^4 \binom{\hat{L}_i n}{\hat{y}_{i1} n, \hat{y}_{i2} n, \hat{y}_{i3} n, \hat{y}_{i4} n} \prod_{j=1}^4 \binom{\hat{R}_j n}{\hat{y}_{1j} n, \hat{y}_{2j} n, \hat{y}_{3j} n, \hat{y}_{4j} n}}{\binom{n+m'}{\hat{y}_{11} n, \hat{y}_{12} n, \dots, \hat{y}_{44} n}} \times B_1^{(2\hat{y}_{11} + \hat{y}_{12} + \hat{y}_{13} + \hat{y}_{21} + \hat{y}_{22} + \hat{y}_{31} + \hat{y}_{33})n} B_2^{(\hat{y}_{22} + \hat{y}_{24} + \hat{y}_{33} + \hat{y}_{34} + \hat{y}_{42} + \hat{y}_{43} + 2\hat{y}_{44})n},$$

$$\begin{aligned} \hat{L}_1 &= \gamma + \eta_1^-, & \hat{L}_2 &= \hat{L}_3 = \alpha - \gamma, & \hat{L}_4 &= 1 - 2\alpha + \gamma + \eta_1^+, \\ \hat{R}_1 &= \delta + \eta_2^-, & \hat{R}_2 &= \hat{R}_3 = \beta - \delta, & \hat{R}_4 &= 1 - 2\beta + \delta + \eta_2^+, \end{aligned}$$

and $\hat{\mathbf{Y}}$ stands for the non-negative variables \hat{y}_{ij} , $1 \leq i, j \leq 4$ such that

$$\sum_j \hat{y}_{ij} = \hat{L}_i \quad \text{and} \quad \sum_i \hat{y}_{ij} = \hat{R}_j.$$

For a given \mathbf{Y} , consider $\hat{\mathbf{Y}}$ such that $y_{ij} = \hat{y}_{ij}$ for all $i, j \in \{1, 2, 3, 4\}$, except $\hat{y}_{14} = y_{14} + \eta_1^-$, $\hat{y}_{41} = y_{41} + \eta_2^-$, and $\hat{y}_{44} = y_{44} + m' - \eta_1^- - \eta_2^-$. This is always possible, except at the boundary, but as we will see shortly, these cases can be safely ignored. By Lemma 7.4, we have

$$\frac{\kappa_{\bar{G}}^{\gamma, \delta, \hat{\mathbf{Y}}}(\eta)}{\kappa_{\bar{G}}^{\gamma, \delta, \mathbf{Y}}} = (1 + o(1)) \frac{\gamma^{\eta_1^-} (1 - 2\alpha + \gamma)^{\eta_1^+} \delta^{\eta_2^-} (1 - 2\beta + \delta)^{\eta_2^+}}{y_{14}^{\eta_1^-} y_{41}^{\eta_2^-} y_{44}^{m' - \eta_1^- - \eta_2^-}} B_2^{2m' - 2\eta_1^- - 2\eta_2^-}. \tag{7.13}$$

Let

$$\mathcal{A} = \{(\gamma, \delta, \mathbf{Y}) : \|(\gamma, \delta, \mathbf{Y}) - (\gamma^*, \delta^*, \mathbf{Y}^*)\|_2 \leq n^{-1/4}\},$$

where $\gamma^* = \alpha^2$, $\delta^* = \beta^2$ and \mathbf{Y}^* is given by (7.3). By Condition 2.3 and Lemma 5.1, the relative contribution of terms outside \mathcal{A} to $\mathbb{E}_{\bar{G}}[(Z_G^{\alpha, \beta})^2]$ is $\exp(-\Omega(n^{1/2}))$ and hence can be omitted. Similarly, using (7.13), the relative contribution of terms outside \mathcal{A} to

$$\mathbb{E}_{\bar{G}}[(Z_G^{\alpha, \beta}(\eta))^2]$$

is $\exp(-\Omega(n^{1/2}))$ and hence can also be omitted. For $(\gamma, \delta, \mathbf{Y}) \in \mathcal{A}$ we have

$$\frac{\kappa_{\bar{G}}^{\gamma, \delta, \hat{\mathbf{Y}}}(\eta)}{\kappa_{\bar{G}}^{\gamma, \delta, \mathbf{Y}}} = (1 + o(1)) \frac{(\gamma^*)^{\eta_1^-} (1 - 2\alpha + \gamma^*)^{\eta_1^+} (\delta^*)^{\eta_2^-} (1 - 2\beta + \delta^*)^{\eta_2^+}}{(y_{14}^*)^{\eta_1^-} (y_{41}^*)^{\eta_2^-} (y_{44}^*)^{m' - \eta_1^- - \eta_2^-}} B_2^{2m' - 2\eta_1^- - 2\eta_2^-}.$$

Using equations (3.6) and (7.3), we express α, β and \mathbf{Y} in terms of Q^+, Q^- . For $(\gamma, \delta, \mathbf{Y}) \in \mathcal{A}$, we obtain that

$$\begin{aligned} \frac{\kappa_{\bar{G}}^{\gamma, \delta, \hat{\mathbf{Y}}}(\eta)}{\kappa_{\bar{G}}^{\gamma, \delta, \mathbf{Y}}} &= (1 + o(1)) \left(\frac{B_2^2 (1 - 2\alpha + \gamma^*) (1 - 2\beta + \delta^*)}{y_{44}^*} \right)^{m'} \\ &\quad \times \left(\frac{\gamma^* y_{44}^*}{B_2^2 y_{14}^* (1 - 2\alpha + \gamma^*)} \right)^{\eta_1^-} \left(\frac{\delta^* y_{44}^*}{B_2^2 y_{41}^* (1 - 2\beta + \delta^*)} \right)^{\eta_2^-} \\ &= (1 + o(1)) \left(\frac{(B_2 + Q^+) (B_2 + Q^-)}{B_2 + Q^+ + Q^- + B_1 Q^+ Q^-} \right)^{2m'} \left(\frac{1 + B_1 Q^-}{B_2 + Q^-} \right)^{2\eta_1^-} \left(\frac{1 + B_1 Q^+}{B_2 + Q^+} \right)^{2\eta_2^-}. \end{aligned}$$

Plugging the last equation into (7.11) and (7.12), we obtain the lemma. □

7.2.3. Proof of Lemma 6.5.

Proof of Lemma 6.5. By equations (6.4) and (7.10), we have

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha,\beta}(\eta))^2]}{(\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}(\eta)])^2} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}_{\mathcal{G}}[(Z_G^{\alpha,\beta})^2]}{(\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha,\beta}])^2}.$$

The lemma follows, after using Lemma 2.7. □

8. Asymptotically almost surely results

8.1. Overview

In this section we apply the small subgraph conditioning method to obtain Lemmas 2.8 and 6.6, see Section 2 for a brief discussion. Closely related results have appeared in [24, 27] for the hard-core model, but here we need to account for the extra parameters B_1, B_2 . The proofs are similar but the technical calculations are a bit more complicated due to the extra parameters.

The following theorem is taken from [24]. The notation $[X]_m$ refers to the m th-order falling factorial of the variable X .

Theorem 8.1. *Let $\lambda_i > 0$ and $\delta_i > -1$ be real numbers for $i = 1, 2, \dots$. Let $r(n) \rightarrow 0$ and suppose that for each n there are random variables $X_i = X_i(n)$, $i = 1, 2, \dots$ and $Y = Y(n)$, all defined on the same probability space $\mathcal{G} = \mathcal{G}_n$ such that X_i is non-negative integer valued, Y is non-negative and $\mathbb{E}[Y] > 0$ (for n sufficiently large). Suppose furthermore that:*

- (1) *for each $k \geq 1$, the variables X_1, \dots, X_k are asymptotically independent Poisson random variables with $\mathbb{E}X_i \rightarrow \lambda_i$,*
- (2) *for every finite sequence m_1, \dots, m_k of non-negative integers,*

$$\frac{\mathbb{E}[Y [X_1]_{m_1} \cdots [X_k]_{m_k}]}{\mathbb{E}[Y]} \rightarrow \prod_{i=1}^k (\lambda_i (1 + \delta_i))^{m_i}, \tag{8.1}$$

- (3) $\sum_i \lambda_i \delta_i^2 < \infty$,
- (4) $\mathbb{E}[Y^2]/(\mathbb{E}[Y])^2 \leq \exp(\sum_i \lambda_i \delta_i^2) + o(1)$ as $n \rightarrow \infty$.

Then $Y > r(n)\mathbb{E}[Y]$ asymptotically almost surely.

Proof. See, for example, [24, Theorem 7.1]. □

Recall that $Z_G^{\alpha,\beta}$ is the measure of configurations in $\Sigma_G^{\alpha,\beta}$ for a random Δ -regular bipartite graph G . Let X_i be the number of cycles of length i in G . Clearly $X_i = 0$ if and only if i is odd. We have the following lemmas.

Lemma 8.2 (Lemma 7.3 in [24]). *Condition 1 of Theorem 8.1 holds for even i with*

$$\lambda_i = \frac{r(\Delta, i)}{i},$$

where $r(\Delta, i)$ is the number of ways one can properly edge colour a cycle of length i with Δ colours.

Proof. The proof of Lemma 8.2 is given in [24, Lemma 7.3]. □

Lemma 8.3. For $(\alpha, \beta) = (p^\pm, p^\mp)$, we have that

$$\frac{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta} X_i]}{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}]} \rightarrow \lambda_i(1 + \delta_i) \quad \text{and} \quad \frac{\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^{\alpha, \beta}(\eta) X_i]}{\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^{\alpha, \beta}(\eta)]} \rightarrow \lambda_i(1 + \delta_i),$$

where

$$\delta_i = \frac{(1 - B_1 B_2)^{i/2} (Q^+ Q^-)^{i/2}}{((1 + B_1 Q^+)(1 + B_1 Q^-)(B_2 + Q^+)(B_2 + Q^-))^{i/2}}.$$

The proof of Lemma 8.3 is given in Section 8.2.

Lemma 8.4. It holds that

$$\exp\left(\sum_{\text{even } i \geq 2} \lambda_i \delta_i^2\right) = (1 - \omega^2)^{-(\Delta-1)/2} (1 - (\Delta - 1)^2 \omega^2)^{-1/2},$$

where ω is given by (3.4).

The proof of Lemma 8.4 is given in Section 8.2.

Lemma 8.5. For $(\alpha, \beta) = (p^\pm, p^\mp)$ and for every finite sequence m_1, \dots, m_k of non-negative integers, it holds that

$$\frac{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}(\eta)[X_2]_{m_1} \cdots [X_{2k}]_{m_k}]}{\mathbb{E}_{\mathcal{G}}[Z_G^{\alpha, \beta}]} \rightarrow \prod_{i=1}^k (\lambda_i(1 + \delta_i))^{m_i}$$

and

$$\frac{\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^{\alpha, \beta}(\eta)[X_2]_{m_1} \cdots [X_{2k}]_{m_k}]}{\mathbb{E}_{\bar{\mathcal{G}}}[Z_{\bar{G}}^{\alpha, \beta}(\eta)]} \rightarrow \prod_{i=1}^k (\lambda_i(1 + \delta_i))^{m_i}.$$

Proof. The proof of Lemma 8.5 is identical to [24, Proof of Lemma 7.5]. □

We are now in a position to prove Lemmas 2.8 and 6.6.

Proofs of Lemmas 2.8 and 6.6. We verify the conditions of Theorem 8.1 in each case: Condition 1 holds (Lemma 8.2), Condition 2 holds (Lemma 8.5), Conditions 3 and 4 hold (Lemma 8.4 and Lemmas 2.7 and 6.5). Lemmas 2.8 and 6.6 follow by picking $r(n) = 1/n$ and $r(n) = 1/\sqrt{n}$ respectively. □

8.2. Random bipartite Δ -regular graphs

In this section we prove Lemma 8.3. We give the proof for the variables $Z_G^{\alpha,\beta}$ and omit the proof for the variables $Z_G^{\alpha,\beta}(\eta)$, which can be carried out the same way as in [27, Lemma 3.8] and along the lines of Section 7.2.1.

Recall that $r(\Delta, i)$ denotes the number of proper Δ -edge colourings of a cycle of length i .

Lemma 8.6. $r(\Delta, i) = (\Delta - 1)^i + (-1)^i(\Delta - 1)$.

Proof. The (simple) proof of Lemma 8.6 is given in [24, Lemma 7.6]. □

Proof of Lemma 8.3. For sets S, T such that $S \subset V_1, T \subset V_2, |S| = \alpha n, |T| = \beta n$, let $Y_{S,T}$ denote the measure of the configuration σ induced by S, T , that is, for a vertex $v \in V(G)$, $\sigma(v) = 1$ if and only if $v \in S \cup T$.

Fix a specific pair of S, T . By symmetry,

$$\frac{\mathbb{E}[Z_G^{\alpha,\beta} X_i]}{\mathbb{E}[Z_G^{\alpha,\beta}]} = \frac{\mathbb{E}[Y_{S,T} X_i]}{\mathbb{E}[Y_{S,T}]} \tag{8.2}$$

We now decompose X_i as follows.

- ξ will denote a proper Δ -edge coloured, rooted and oriented i -cycle ($r(\Delta, i)$ possibilities), in which the vertices are coloured with red (R), blue (B) and white (W) such that no two adjacent vertices in the cycle have both R or B colour, that is, the only assignments which are prohibited for an edge are $(R, R), (B, B)$. The colour of the edges will prescribe which of the Δ perfect matchings an edge of a (potential) cycle will belong to. R denotes vertices belonging to S , B denotes vertices belonging to T , and W the remaining vertices.
- Given ξ , ζ denotes a position that an i -cycle can be in (*i.e.*, the exact vertices it traverses, in order) such that prescription of the vertex colours of ξ is satisfied.
- $\mathbf{1}_{\xi,\zeta}$ is the indicator function whether a cycle specified by ξ, ζ is present in the graph G .

Note that each possible cycle corresponds to exactly $2i$ different configurations ξ (the number of ways to root and orient the cycle). For each of those ξ , the respective sets of configurations ζ are the same. Hence, we may write

$$X_i = \frac{1}{2i} \sum_{\xi} \sum_{\zeta} \mathbf{1}_{\xi,\zeta}.$$

Let $P_1 := \Pr[\mathbf{1}_{\xi,\zeta} = 1]$. It follows that

$$\mathbb{E}[Y_{S,T} X_i] = \frac{1}{2i} \sum_{\xi} \sum_{\zeta} P_1 \cdot \mathbb{E}[Y_{S,T} | \mathbf{1}_{\xi,\zeta} = 1].$$

In light of (8.2), we need to study the ratio $\mathbb{E}[Y_{S,T} | \mathbf{1}_{\xi,\zeta} = 1] / \mathbb{E}[Y_{S,T}]$. At this point, to simplify notation, we may assume that ξ, ζ are fixed.

We have shown in Section 4.1 that

$$\mathbb{E}[Y_{S,T}] = \lambda^{(\alpha+\beta)n} \left(\sum_{\delta} \frac{\binom{\alpha n}{\delta n} \binom{(1-\alpha)n}{(\beta-\delta)n}}{\binom{n}{\beta n}} B_1^{\delta n} B_2^{(1-\alpha-\beta+\delta)n} \right)^{\Delta},$$

where δ denotes the number of edges between S, T in one matching.

To calculate $\mathbb{E}[Y_{S,T} | \mathbf{1}_{\xi, \zeta} = 1]$ we need some notation. For colours $c_1, c_2 \in \{R, B, W\}$, we say that an edge is of type $\{c_1, c_2\}$ if its endpoints have colours c_1, c_2 . Let j_1, j_2 denote the number of red and blue vertices in the colouring prescribed by ξ . For $k = 1, \dots, \Delta$, let $a_1(k)$ denote the number of edges of colour k of type $\{R, B\}$, let $a_2(k)$ be the number of edges of colour k of type $\{W, W\}$, let $d_1(k)$ be the number of edges of colour k of type $\{R, W\}$, and let $d_2(k)$ be the number of edges of colour k of type $\{B, W\}$. Finally, for $j = 1, 2$ let $a_j = \sum_k a_j(k)$ and $d_j = \sum_k d_j(k)$. By considering the sum of the degrees of R vertices, the sum of the degrees of B vertices and the total number of edges of the cycle, we obtain the following equalities:

$$a_1 + d_1 = 2j_1, \quad a_1 + d_2 = 2j_2, \quad a_1 + a_2 + d_1 + d_2 = i. \tag{8.3}$$

A straightforward modification of the arguments in Section 4.1 yields

$$\mathbb{E}[Y_{S,T} | \mathbf{1}_{\xi, \zeta} = 1] = \lambda^{(\alpha+\beta)n} \prod_{k=1}^{\Delta} \left(\sum_{\delta_k} \frac{\binom{\alpha n - a_1(k) - d_1(k)}{\delta_k n} \binom{(1-\alpha)n - a_2(k) - d_2(k)}{(\beta - \delta_k)n - a_1(k) - d_2(k)}}{\binom{n - a_1(k) - a_2(k) - d_1(k) - d_2(k)}{\beta n - a_1(k) - d_2(k)}} B_1^{\delta_k n + a_1(k)} B_2^{(1-\alpha-\beta+\delta_k)n + a_1(k)} \right).$$

By Lemma 7.4, we have

$$\begin{aligned} \frac{\binom{\alpha n - a_1(k) - d_1(k)}{\delta_k n}}{\binom{\alpha n}{\delta_k n}} &\sim \left(\frac{\alpha - \delta_k}{\alpha} \right)^{a_1(k) + d_1(k)}, \\ \frac{\binom{(1-\alpha)n - a_2(k) - d_2(k)}{(\beta - \delta_k)n - a_1(k) - d_2(k)}}{\binom{(1-\alpha)n}{(\beta - \delta_k)n}} &\sim \left(\frac{1 - \alpha - \beta + \delta_k}{1 - \alpha} \right)^{a_2(k) + d_2(k)} \left(\frac{\beta - \delta_k}{1 - \alpha - \beta + \delta_k} \right)^{a_1(k) + d_2(k)}, \\ \frac{\binom{n - a_1(k) - a_2(k) - d_1(k) - d_2(k)}{\beta n - a_1(k) - d_2(k)}}{\binom{n}{\beta n}} &\sim \beta^{a_1(k) + d_2(k)} (1 - \beta)^{a_2(k) + d_1(k)}. \end{aligned}$$

By the same line of arguments as in the proofs of Lemmas 6.3 and 6.4, we obtain that

$$\frac{\mathbb{E}[Y_{S,T} | \mathbf{1}_{\xi, \zeta} = 1]}{\mathbb{E}[Y_{S,T}]} \sim \frac{(\delta^*)^{a_2 - a_1} (\alpha - \delta^*)^{a_1 + d_1} (\beta - \delta^*)^{a_1 + d_2} (B_1 B_2)^{a_1}}{\alpha^{a_1 + d_1} \beta^{a_1 + d_2} (1 - \alpha)^{a_2 + d_2} (1 - \beta)^{a_2 + d_1}}.$$

Clearly $P_1 \sim n^{-i}$ and for given ξ , the number of possible ζ is asymptotic to

$$\alpha^{j_1} \beta^{j_2} (1 - \alpha)^{i/2 - j_1} (1 - \beta)^{i/2 - j_2} n^i.$$

Thus, for the given ξ , we have

$$\frac{\sum_{\zeta} P_1 \mathbb{E}[Y_{S,T} | \mathbf{1}_{\xi, \zeta} = 1]}{\mathbb{E}[Y_{S,T}]} \sim R,$$

where

$$R := \frac{(1 - \alpha - \beta + \delta^*)^{a_2 - a_1} (\alpha - \delta^*)^{a_1 + d_1} (\beta - \delta^*)^{a_1 + d_2} (B_1 B_2)^{a_1}}{\alpha^{a_1 + d_1 - j_1} \beta^{a_1 + d_2 - j_2} (1 - \alpha)^{a_2 + d_2 + j_1 - i/2} (1 - \beta)^{a_2 + d_1 + j_2 - i/2}}.$$

Utilize (8.3) to express everything in terms of i, j_1, j_2, a_2 to get

$$R = \frac{(1 - \alpha - \beta + \delta^*)^{i-2j_1-2j_2} (\alpha - \delta^*)^{2j_1} (\beta - \delta^*)^{2j_2} (B_1 B_2)^{a_2 - i + 2j_1 + 2j_2}}{\alpha^{j_1} \beta^{j_2} (1 - \alpha)^{(i/2) - j_1} (1 - \beta)^{(i/2) - j_2}}.$$

Substituting

$$\alpha = \frac{Q^+(1 + B_1 Q^-)}{B_2 + Q^- + Q^+ + B_1 Q^- Q^+}, \quad \beta = \frac{Q^-(1 + B_1 Q^+)}{B_2 + Q^- + Q^+ + B_1 Q^- Q^+}$$

and

$$\delta^* = \frac{B_1 Q^- Q^+}{B_2 + Q^- + Q^+ + B_1 Q^- Q^+},$$

we obtain

$$R = \frac{B_1^{a_2 - i + 2j_1 + 2j_2} B_2^{a_2} (Q^+)^{j_1} (Q^-)^{j_2}}{(1 + B_1 Q^-)^{j_1} (1 + B_1 Q^+)^{j_2} (B_2 + Q^-)^{i/2 - j_1} (B_2 + Q^+)^{i/2 - j_2}}.$$

We thus obtain

$$\frac{\mathbb{E}[Y_{S,T} | \mathbf{1}_{\xi, \zeta} = 1]}{\mathbb{E}[Y_{S,T}]} \sim x^i y^{j_1} z^{j_2} w^{a_2},$$

where

$$x = \frac{1}{B_1 \sqrt{(B_2 + Q^-)(B_2 + Q^+)}}}, \quad y = \frac{B_1^2 Q^+(B_2 + Q^-)}{1 + B_1 Q^-},$$

$$z = \frac{B_1^2 Q^-(B_2 + Q^+)}{1 + B_1 Q^+}, \quad w = B_1 B_2.$$

Thus, by (8.2), we obtain

$$\frac{\mathbb{E}[Y X_i]}{\mathbb{E}[Y]} = \frac{r(\Delta, i)}{2i} \cdot x^i \sum_{j_1, j_2, a_2} a_{j_1, j_2, a_2} y^{j_1} z^{j_2} w^{a_2}, \quad (8.4)$$

where a_{i, j_1, j_2, a_2} is the number of possible ξ with j_1 red vertices, j_2 blue vertices and a_2 edges of type $\{W, \bar{W}\}$. To analyse such sums, an elegant technique (given in [14]) is to define an appropriate transition matrix. The powers of the matrix count the (multiplicative) weight of walks in the implicit graph. In our setting, the transition matrix is given by

$$\mathbf{A} = \begin{pmatrix} 0 & y & 0 & y \\ z & 0 & z & 0 \\ 0 & 1 & 0 & w \\ 1 & 0 & w & 0 \end{pmatrix}.$$

Note that a rooted oriented cycle with specification ξ, ζ corresponds to a unique closed walk in the implicit graph defined by \mathbf{A} . We have that $\text{Tr}(\mathbf{A}^i) = \sum_{j=1}^4 e_j^i$, where e_j are the eigenvalues of \mathbf{A} ($j = 1, \dots, 4$). We thus obtain

$$\frac{\mathbb{E}[Y X_i]}{\mathbb{E}[Y]} = \frac{r(\Delta, i)}{2i} \sum_{j=1}^4 (x e_j)^i.$$

A computation gives

$$e_1 = -e_2 = B_1(1 - B_1B_2)\sqrt{\frac{Q^+Q^-}{(1 + B_1Q^+)(1 + B_1Q^-)}},$$

$$e_3 = -e_4 = B_1\sqrt{(B_2 + Q^+)(B_2 + Q^-)}.$$

It follows that for even i , we have

$$\frac{\mathbb{E}[YX_i]}{\mathbb{E}[Y]} = \frac{r(\Delta, i)}{i} \left(1 + \left(\frac{(1 - B_1B_2)\sqrt{Q^+Q^-}}{\sqrt{(1 + B_1Q^+)(1 + B_1Q^-)(B_2 + Q^+)(B_2 + Q^-)}} \right)^i \right),$$

thus proving the lemma. □

Proof of Lemma 8.4. Using Lemma 8.6, we have

$$\sum_{\text{even } i \geq 2} \lambda_i \delta_i^2 = \sum_{\text{even } i \geq 2} \frac{r(\Delta, i)}{i} \cdot \omega^i = \sum_{\text{even } i \geq 2} \frac{(\Delta - 1)^i + (\Delta - 1)}{i} \cdot \omega^i.$$

Observe that

$$\sum_{j \geq 1} \frac{x^{2j}}{2j} = -\frac{1}{2} \ln(1 - x^2)$$

for all $|x| < 1$. Clearly this holds for ω and by Lemma 3.2, it also holds for $(\Delta - 1)\omega$. It follows that

$$\sum_{\text{even } i \geq 2} \lambda_i \delta_i^2 = -\frac{1}{2} (\ln(1 - (\Delta - 1)^2 \omega^2) + (\Delta - 1) \ln(1 - \omega^2)),$$

thus proving the lemma. □

9. Remaining proofs

9.1. Proof of Lemma 3.2

In this section we give the proof of Lemma 3.2. We prove only the inequality for ω , since the inequality for ω^* is the condition for non-uniqueness; see the discussion in Section 3. We will in fact establish a slightly stronger inequality, as is captured by the following lemma. To aid the presentation, let us work with $x = Q^+, y = Q^-, d = \Delta - 1$.

Lemma 9.1. *For any $d \geq 2$, it holds that*

$$B_1xy + B_1B_2(x + y) + B_2 > (d - 1)(1 - B_1B_2)\sqrt{xy}.$$

Proof of Lemma 3.2. Let $W := B_1xy + B_1B_2(x + y) + B_2$. Observe that

$$W = (B_1x + 1)(y + B_2) - (1 - B_1B_2)y = (B_1y + 1)(x + B_2) - (1 - B_1B_2)x.$$

Using the two expressions of W and the AM–GM inequality, we obtain

$$\begin{aligned} W^2 &= (B_1x + 1)(y + B_2)(B_1y + 1)(x + B_2) + (1 - B_1B_2)^2xy \\ &\quad - (1 - B_1B_2)(y(B_1x + 1)(B_2 + y) + x(B_1y + 1)(B_2 + x)) \\ &\leq (\sqrt{(B_1x + 1)(B_1y + 1)(B_2 + x)(B_2 + y)} - (1 - B_1B_2)\sqrt{xy})^2. \end{aligned} \quad (9.1)$$

Trivially

$$(B_1x + 1)(B_1y + 1)(B_2 + x)(B_2 + y) > xy > (1 - B_1B_2)^2xy,$$

so (9.1) and Lemma 9.1 give

$$\sqrt{(B_1x + 1)(y + B_2)(B_1y + 1)(x + B_2)} - (1 - B_1B_2)\sqrt{xy} > (d - 1)(1 - B_1B_2).$$

which after trivial manipulations reduces to the desired inequality. \square

Proof of Lemma 9.1. The x, y satisfy

$$x = \lambda \left(\frac{B_1y + 1}{y + B_2} \right)^d \quad \text{and} \quad y = \lambda \left(\frac{B_1x + 1}{x + B_2} \right)^d.$$

It follows that

$$x \left(\frac{B_1x + 1}{x + B_2} \right)^d = y \left(\frac{B_1y + 1}{y + B_2} \right)^d \Rightarrow x((B_1x + 1)(y + B_2))^d = y((B_1y + 1)(x + B_2))^d. \quad (9.2)$$

Let $W := B_1xy + B_1B_2x + B_1B_2y + B_2$. Observe that

$$\begin{aligned} (B_1x + 1)(y + B_2) &= B_1xy + B_1B_2x + y + B_2 = W + (1 - B_1B_2)y, \\ (B_1y + 1)(x + B_2) &= B_1xy + B_1B_2y + x + B_2 = W + (1 - B_1B_2)x. \end{aligned}$$

Hence, (9.2) gives

$$x(W + (1 - B_1B_2)y)^d = y(W + (1 - B_1B_2)x)^d.$$

Expanding using Newton's formula gives

$$x \sum_{k=0}^d \binom{d}{k} W^{d-k} ((1 - B_1B_2)y)^k = y \sum_{k=0}^d \binom{d}{k} W^{d-k} ((1 - B_1B_2)x)^k,$$

which is equivalent to

$$W^d(x - y) = xy \sum_{k=2}^d \binom{d}{k} W^{d-k} (1 - B_1B_2)^k (x^{k-1} - y^{k-1}).$$

Since $x \neq y$, this can be rewritten as

$$W^d = xy \sum_{k=2}^d \binom{d}{k} W^{d-k} (1 - B_1B_2)^k \left(\frac{x^{k-1} - y^{k-1}}{x - y} \right). \quad (9.3)$$

Claim 9.2. For $k \geq 2$ and $x, y > 0$ with $x \neq y$, it holds that

$$\frac{x^{k-1} - y^{k-1}}{x - y} > (k - 1)(xy)^{(k-2)/2}.$$

The simple proof of Claim 9.2 is given at the end. Using Claim 9.2, (9.3) gives

$$W^d > \sum_{k=2}^d \binom{d}{k} (k-1) W^{d-k} (1 - B_1 B_2)^k (xy)^{k/2},$$

or equivalently

$$\underbrace{W^d + \sum_{k=2}^d \binom{d}{k} W^{d-k} (1 - B_1 B_2)^k (xy)^{k/2}}_C > \underbrace{\sum_{k=2}^d \binom{d}{k} k W^{d-k} (1 - B_1 B_2)^k (xy)^{k/2}}_D. \tag{9.4}$$

Using Newton’s formula again and the identity

$$\binom{d}{k} = \frac{d}{k} \binom{d-1}{k-1},$$

we have

$$\begin{aligned} C &= (W + (1 - B_1 B_2)\sqrt{xy})^d - dW^{d-1}(1 - B_1 B_2)\sqrt{xy}, \\ D &= d(1 - B_1 B_2)\sqrt{xy} \sum_{k=2}^d \binom{d-1}{k-1} W^{d-k} (1 - B_1 B_2)^{k-1} (\sqrt{xy})^{k-1} \\ &= d(1 - B_1 B_2)\sqrt{xy}((W + (1 - B_1 B_2)\sqrt{xy})^{d-1} - W^{d-1}). \end{aligned}$$

Thus, (9.4) gives

$$W > (d - 1)(1 - B_1 B_2)\sqrt{xy},$$

which is exactly the inequality we wanted. Finally, we give the proof of Claim 9.2.

Proof of Claim 9.2. Since $k \geq 2$, observe that

$$\begin{aligned} \frac{x^{k-1} - y^{k-1}}{x - y} &= x^{k-2} + x^{k-3}y + \dots + y^{k-2} \\ &\geq (k - 1)((xy)^{(k-1)(k-2)/2})^{1/(k-1)} = (k - 1)(xy)^{(k-2)/2}. \end{aligned}$$

The inequality is an application of the AM–GM inequality to x^{k-2}, \dots, y^{k-2} . Equality holds if and only if $x = y$. □

This completes the proof of Lemma 9.1. □

9.2. Proofs for Section 5.1

Proof of Lemma 5.1. We first give the proof when $M_{ij} > 0$ for all i, j . Then it is easy to see that the function $g(\mathbf{Z})$ attains its maximum in the interior of the region (5.2), since

$$\frac{\partial g}{\partial Z_{ij}} = \ln M_{ij} - \ln Z_{ij} - 1 \rightarrow +\infty \text{ as } Z_{ij} \rightarrow 0^+.$$

Since $g(\mathbf{Z})$ is differentiable in the interior of the region (5.2) (by $\alpha_i \neq 0 \neq \beta_j$), we may formulate the maximization of $g(\mathbf{Z})$ using Lagrange multipliers. Let

$$\Lambda(\mathbf{Z}, \lambda, \lambda') = g(\mathbf{Z}) + \lambda_i \left(\sum_j Z_{ij} - \alpha_i \right) + \lambda'_j \left(\sum_i Z_{ij} - \beta_j \right).$$

The optimal \mathbf{Z} must satisfy

$$\frac{\partial \Lambda}{\partial Z_{ij}} = 0 \iff \ln M_{ij} - \ln Z_{ij} - 1 + \lambda_i + \lambda'_j \iff Z_{ij} = \exp(\lambda_i + \lambda'_j - 1) M_{ij}.$$

Setting $R_i = \exp(\lambda_i)$, $C_j = \exp(\lambda'_j - 1)$ one obtains that the stationary points of $g(\mathbf{Z})$ have the form stated in Lemma 5.1. The equalities (5.5) come from the conditions

$$\frac{\partial \Lambda}{\partial \lambda_i} = \frac{\partial \Lambda}{\partial \lambda'_j} = 0.$$

We next treat the case where some entries of M_{ij} are zero. Recall the convention (Section 4.3) that $\ln 0 \equiv -\infty$ and $0 \ln 0 \equiv 0$. To see why these conventions are relevant, note that whenever $M_{ij} = 0$, $g(\mathbf{Z})$ can have a maximum at \mathbf{Z}^* only if $Z_{ij}^* = 0$. Going back to the interpretations of $g(\mathbf{Z})$ in the first and second moments, the reader will immediately notice that these are in complete accordance with the combinatorial meaning of the variables Z_{ij} . Let $P = \{(i, j) : M_{ij} = 0\}$. We may restrict our attention for the maximum of $g(\mathbf{Z})$ in the region

$$\begin{aligned} \sum_j Z_{ij} &= \alpha_i(u), & \text{for } i \in 1, \dots, m, \\ \sum_i Z_{ij} &= \beta_j(v), & \text{for } j \in 1, \dots, n, \\ Z_{ij} &\geq 0, & \text{for } (i, j) \notin P, \\ Z_{ij} &= 0, & \text{for } (i, j) \in P. \end{aligned} \tag{9.5}$$

The proof given above (under the assumptions $M_{ij} > 0$ for all i, j) now goes through for the region (9.5), as can readily be checked. Since $M_{ij} = 0$ for $(i, j) \in P$, the form of \mathbf{Z}^* stated in Lemma 5.1 is consistent with the region (9.5).

Finally, the desired quadratic decay is an immediate consequence of the stationarity of \mathbf{Z}^* , the strict concavity of g and Taylor's expansion. \square

Proof of Lemma 5.2. The lemma is essentially an application of the envelope theorem, but we give a short proof for the sake of completeness. From (5.5) we have

$$\left(\frac{\partial}{\partial u} R_i \right) \sum_{j=1}^m M_{ij} C_j + R_i \sum_{j=1}^m M_{ij} \frac{\partial}{\partial u} C_j = \frac{\partial}{\partial u} \alpha_i, \tag{9.6}$$

$$\left(\frac{\partial}{\partial u} C_j \right) \sum_{i=1}^n M_{ij} R_i + C_j \sum_{i=1}^n M_{ij} \frac{\partial}{\partial u} R_i = 0. \tag{9.7}$$

Then

$$\begin{aligned} \frac{\partial g^*}{\partial u} = & - \sum_{i=1}^n \sum_{j=1}^m M_{ij} \left(\left(\frac{\partial}{\partial u} R_i \right) (C_j \ln(R_i) + C_j \ln(C_j)) \right. \\ & \left. + \left(\frac{\partial}{\partial u} C_j \right) (R_i \ln(R_i) + R_i \ln(C_j)) + \left(\frac{\partial}{\partial u} R_i \right) C_j + R_i \left(\frac{\partial}{\partial u} C_j \right) \right), \end{aligned}$$

which, using (9.6) and (9.7), simplifies to

$$\begin{aligned} \frac{\partial g^*}{\partial u} = & - \sum_{i=1}^n \ln(R_i) \left(\left(\frac{\partial}{\partial u} R_i \right) \sum_{j=1}^m M_{ij} C_j + R_i \left(\sum_{j=1}^m M_{ij} \frac{\partial}{\partial u} C_j \right) \right) \\ & - \sum_{j=1}^m \ln(C_j) \left(\left(\frac{\partial}{\partial u} C_j \right) \sum_{i=1}^n M_{ij} R_i + C_j \left(\sum_{i=1}^n M_{ij} \frac{\partial}{\partial u} R_i \right) \right) \\ & - \sum_{j=1}^m \left(\left(\frac{\partial}{\partial u} C_j \right) \sum_{i=1}^n M_{ij} R_i + C_j \sum_{i=1}^n M_{ij} \frac{\partial}{\partial u} R_i \right) \\ = & - \sum_{i=1}^n \ln(R_i) \frac{\partial}{\partial u} \alpha_i. \end{aligned}$$

□

9.3. Proofs for Section 5.2

Proof of Lemma 5.4. To avoid unnecessary complications, we give the proof for $B_1 B_2 > 0$. The case with $B_1 = 0$ (essentially the hard-core model) is covered in [7, Claim 2.2]. By (5.7) we have

$$\frac{\partial \phi_1}{\partial \alpha} = \ln \left(\lambda \frac{1-\alpha}{\alpha} \left(\frac{\alpha}{1-\alpha} \frac{R_2}{R_1} \right)^\Delta \right), \quad \frac{\partial \phi_1}{\partial \beta} = \ln \left(\lambda \frac{1-\beta}{\beta} \left(\frac{\beta}{1-\beta} \frac{C_2}{C_1} \right)^\Delta \right), \tag{9.8}$$

where the R_i, C_j satisfy (5.8). By (5.8), we have

$$\frac{\alpha}{1-\alpha} = \frac{R_1}{R_2} \cdot \frac{B_1 C_1 + C_2}{C_1 + B_2 C_2}, \quad \frac{\beta}{1-\beta} = \frac{C_1}{C_2} \cdot \frac{B_1 R_1 + R_2}{R_1 + B_2 R_2}.$$

Hence, the derivatives $\partial \phi_1 / \partial \alpha, \partial \phi_1 / \partial \beta$ in (9.8) become

$$\frac{\partial \phi_1}{\partial \alpha} = \ln \left(\lambda \frac{1-\alpha}{\alpha} \left(\frac{B_1 C_1 + C_2}{C_1 + B_2 C_2} \right)^\Delta \right), \quad \frac{\partial \phi_1}{\partial \beta} = \ln \left(\lambda \frac{1-\beta}{\beta} \left(\frac{B_1 R_1 + R_2}{R_1 + B_2 R_2} \right)^\Delta \right). \tag{9.9}$$

We next establish the following claim.

Claim 9.3. *When $B_1 > 0, B_2 > 0$, the function*

$$f(x, y) := \frac{B_1 x + y}{x + B_2 y},$$

defined for $x, y \geq 0$ such that at least one of x, y is non-zero, is upper- and lower-bounded by strictly positive constants depending only on B_1, B_2 .

Proof. Observe that for $t > 0$ it holds that $f(x, y) = f(tx, ty)$, so that we may assume $x + y = 1$. Thus, $B_1 x + y$ and $x + B_2 y$ are convex combinations of $\{1, B_1\}$ and $\{1, B_2\}$

respectively. Since we assumed that $B_1, B_2 > 0$, $f(x, y)$ is bounded by strictly positive constants. □

Observe that Claim 9.3 is applicable to our case since not all of R_1, R_2 or C_1, C_2 can be equal to 0 (by equations (5.8)).

We consider the boundaries with $\alpha(1 - \alpha) \rightarrow 0^+$ and prove that the derivatives with respect to α go the right way. The boundary cases with $\beta(1 - \beta) \rightarrow 0^+$ are completely analogous. We consider two cases.

(1) $\alpha \rightarrow 0^+$. Then $1 - \alpha \rightarrow 1$. By Claim 9.3 and (9.9), we obtain $\partial\phi_1/\partial\alpha \rightarrow +\infty$.

(2) $1 - \alpha \rightarrow 0^+$. Then $\alpha \rightarrow 1$. By Claim 9.3 and (9.9), we obtain $\partial\phi_1/\partial\alpha \rightarrow -\infty$.

This concludes the proof of Lemma 5.4. □

Proof of Lemma 5.5. Let \mathbf{H}_1 denote the Hessian of Φ_1 evaluated at (p^+, p^-, X^*) . We keep only α, β, x_{11} as free variables in Φ_1 (the rest of the x_{ij} are substituted by (4.2)). We check that the principal minors along the diagonal are positive, the first principal minor being the lower right corner of the matrix. This is equivalent to applying Sylvester’s criterion in a rearranged version of $-\mathbf{H}_1$. We use Maple to perform the necessary computations and plug into the resulting formulas the values of p^\pm, X^* as given in (3.6) and (7.2) respectively.

Let P_i denote the determinant of the i th leading principal minor. We obtain that

$$\begin{aligned}
 P_1 &= \frac{\Delta E_1 E_2}{B_1 B_2 Q^- Q^+}, \\
 P_2 &= \frac{\Delta E_1^2 (B_2 + Q^-)(1 + B_1 Q^-)(1 + (\Delta - 1)\omega)}{B_1 B_2 (Q^-)^2 Q^+}, \\
 P_3 &= \frac{\Delta E_1^4 (1 - (\Delta - 1)^2 \omega)}{B_1 B_2 (Q^+ Q^-)^2},
 \end{aligned}$$

where E_1, E_2 and ω are given by (7.1) and (3.4). Trivially $P_1, P_2 > 0$. The fact that $P_3 > 0$ is just Lemma 3.2.

To prove that (p^*, p^*, X^o) is a saddle point, note that the respective expressions for P_1, P_2, P_3 can be obtained by substituting $Q^- \leftarrow Q^*, Q^+ \leftarrow Q^*$, where $Q^* = q^*/(1 - q^*)$. We thus have that $P_3 < 0$ by Lemma 3.2. □

9.4. Proofs for Section 5.3

Proof of Lemma 5.7. To avoid unnecessary complications, we give the proof for $B_1 B_2 > 0$. The case with $B_1 = 0$ (essentially the hard-core model) is covered in [8, Lemma 12].

By (5.13), we have

$$\frac{\partial\phi_2}{\partial\gamma} = \ln \left(\frac{(\alpha - \gamma)^2}{\gamma(1 - 2\alpha + \gamma)} \left(\frac{\gamma(1 - 2\alpha + \gamma)}{(\alpha - \gamma)^2} \frac{R_2^2}{R_1 R_4} \right)^\Delta \right),$$

where the R_i, C_j satisfy (5.12). By (5.12), we have

$$\frac{\gamma(1 - 2\alpha + \gamma)}{(\alpha - \gamma)^2} = \frac{R_1 R_4}{R_2^2} \cdot \frac{(B_1^2 C_1 + 2B_1 C_2 + C_4)(C_1 + 2B_2 C_2 + B_2^2 C_4)}{(B_1 C_1 + (B_1 B_2 + 1)C_2 + B_2 C_4)^2},$$

Hence, the derivative $\partial\phi_2/\partial\gamma$ becomes

$$\frac{\partial\phi_2}{\partial\gamma} = \ln\left(\frac{(\alpha - \gamma)^2}{\gamma(1 - 2\alpha + \gamma)} \left(\frac{(B_1^2C_1 + 2B_1C_2 + C_4)(C_1 + 2B_2C_2 + B_2^2C_4)}{(B_1C_1 + (B_1B_2 + 1)C_2 + B_2C_4)^2}\right)^\Delta\right). \tag{9.10}$$

Analogously, starting from (5.14), we can rewrite $\partial\phi_2/\partial\delta$ as

$$\frac{\partial\phi_2}{\partial\delta} = \ln\left(\frac{(\beta - \delta)^2}{\delta(1 - 2\beta + \delta)} \left(\frac{(B_1^2R_1 + 2B_1R_2 + R_4)(R_1 + 2B_2R_2 + B_2^2R_4)}{(B_1R_1 + (B_1B_2 + 1)R_2 + B_2R_4)^2}\right)^\Delta\right). \tag{9.11}$$

Before proceeding, we need the following claim.

Claim 9.4. *When $B_1 > 0, B_2 > 0$, the function*

$$f(x, y, z) := \frac{(B_1^2x + 2B_1y + z)(x + 2B_2y + B_2^2z)}{(B_1x + (B_1B_2 + 1)y + B_2z)^2},$$

defined for $x, y, z \geq 0$ such that at least one of x, y, z is non-zero, is upper- and lower-bounded by strictly positive constants depending only on B_1, B_2 .

Proof. Observe that for $t > 0$ it holds that $f(x, y, z) = f(tx, ty, tz)$, so we may assume that $x + y + z = 1$. Thus,

$$B_1^2x + 2B_1y + z, \quad x + 2B_2y + B_2^2z \quad \text{and} \quad B_1x + (B_1B_2 + 1)y + B_2z$$

are convex combinations of $\{B_1^2, 2B_1, 1\}, \{1, 2B_2, B_2^2\}$ and $\{B_1, B_1B_2 + 1, B_2\}$ respectively. Since we assumed that $B_1, B_2 > 0$, $f(x, y, z)$ is bounded by strictly positive constants. \square

Observe that Claim 9.4 is applicable to our case since not all of R_1, R_2, R_4 or C_1, C_2, C_4 can be equal to 0 (by equations (5.12)).

We next consider the boundaries with $\gamma(\alpha - \gamma)(1 - 2\alpha + \gamma) \rightarrow 0^+$ and prove that the derivative $\partial\phi_2/\partial\gamma$ goes to infinity the right way, using the expression in (9.10). The boundary cases with $\beta(\beta - \delta)(1 - 2\beta + \delta) \rightarrow 0^+$ can be treated analogously using (9.11).

We consider three cases.

- (1) $\gamma \rightarrow 0^+$. This case may only occur if $1 - 2\alpha \geq 0$. Then $\alpha - \gamma \rightarrow \alpha > 0$ and $1 - 2\alpha + \gamma \rightarrow 1 - 2\alpha \geq 0$. By Claim 9.4 and (9.10), we obtain $\partial\phi_2/\partial\gamma \rightarrow +\infty$.
- (2) $1 - 2\alpha + \gamma \rightarrow 0^+$. This case may only occur if $2\alpha - 1 \geq 0$. Then $\gamma \rightarrow 2\alpha - 1 \geq 0$ and $\alpha - \gamma \rightarrow 1 - \alpha > 0$. By Claim 9.4 and (9.10), we obtain $\partial\phi_2/\partial\gamma \rightarrow +\infty$.
- (3) $\alpha - \gamma \rightarrow 0^+$. Then $\gamma \rightarrow \alpha > 0$ and $1 - 2\alpha + \gamma \rightarrow 1 - \alpha > 0$. By Claim 9.4 and (9.10), we obtain $\partial\phi_2/\partial\gamma \rightarrow -\infty$.

This concludes the proof of Lemma 5.7. \square

Proof of Lemma 5.8. For $(\alpha, \beta) = (p^+, p^-)$, let \mathbf{H}_2 denote the Hessian of Φ_2 evaluated at $(\alpha^2, \beta^2, \mathbf{Y}^*)$. We keep only γ, δ, y_{ij} with $i, j = 1, 2, 3$ as free variables in Φ_2 (the rest of the y_{ij} are substituted by (4.5)).

We use Sylvester’s criterion to check that $-\mathbf{H}_2$ is positive definite. We check that the principal minors are positive, the first principal minor being the lower right corner of

the matrix. This is equivalent to applying Sylvester's criterion in a rearranged version of $-\mathbf{H}_2$. Note that the first nine principal minors are guaranteed to be positive since they correspond to the Hessian of the convex function $-g_2(\mathbf{Y})$ evaluated at its minimum point. Thus we need to check only the remaining two principal minors. We use Maple to perform the necessary computations and we plug into the resulting formulas the values of p^\pm from (3.6) and the \mathbf{Y}^* as given in (7.3).

Denote by P_i the determinant of the i th leading principal minor. We have

$$P_{10} = \frac{\Delta^9 E_1^{22} E_2^2 (B_2 + Q^-)^2 (1 + B_1 Q^-)^2 (1 + (\Delta - 1)\omega^2)}{(B_1 B_2)^8 (Q^-)^{14} (Q^+)^{12}},$$

$$P_{11} = \det(-\mathbf{H}_2) = \frac{\Delta^9 E_1^{26} E_2^2 (1 - (\Delta - 1)^2 \omega^2)}{(B_1 B_2)^8 (Q^-)^{14} (Q^+)^{14}},$$

where E_1, E_2 and ω are given by (7.1) and (3.4). Trivially $P_{10} > 0$. And we also have $P_{11} > 0$ by Lemma 3.2. \square

9.5. Proof of Lemma 2.5

We now prove Condition 2.3 for the Ising model without external field and for $d = 2$, that is, $B_1 = B_2 = B, \lambda = 1$ and $\Delta = 3$. See Section 5.4 for the proof overview.

Our point of departure is (5.19). As in the proof of Lemma 2.4, we may assume that $r_1 r_4, c_1 c_4$ are either both greater than 1 or both less than 1. We are going to argue that (5.19) cannot hold in these cases. For $B_1 = B_2 = B$, multiplying the equations in (5.19) gives

$$\begin{aligned} & (Bc_1 + (B^2 + 1) + Bc_4)^2 (Br_1 + (B^2 + 1) + Br_4)^2 \\ &= \frac{(1 - B^2)^2 (c_1 c_4 - 1)}{(c_1 c_4)^{1/2} - 1} \frac{(1 - B^2)^2 (r_1 r_4 - 1)}{(r_1 r_4)^{1/2} - 1}. \end{aligned}$$

We will prove that for $0 < B < (\sqrt{2} - 1)/(\sqrt{2} + 1)$, when $r_1 r_4 \neq 1$ (and similarly when $c_1 c_4 \neq 1$),

$$(Br_1 + (B^2 + 1) + Br_4)^2 > \frac{(1 - B^2)^2 (r_1 r_4 - 1)}{\sqrt{r_1 r_4} - 1}. \quad (9.12)$$

In the region $0 < B < (\sqrt{2} - 1)/(\sqrt{2} + 1)$ the values of α and β are heavily biased towards 1 and 0 respectively, and this reflects at the values of r_1, r_4, c_1, c_4 . The required bias of the values of α, β is captured by the following lemma.

Lemma 9.5. *Assume*

$$0 < B < \frac{\sqrt{2} - 1}{\sqrt{2} + 1}.$$

For $(\alpha, \beta) = (p^+, p^-)$ it holds that

$$\frac{\alpha}{1 - \alpha} = \frac{1 - \beta}{\beta} > \frac{4}{9} \cdot \frac{1}{B^3}.$$

The proof of Lemma 9.5 is given in Section 9.5.1. We utilize Lemma 9.5 to prove the following result.

Lemma 9.6. *Assume*

$$0 < B < \frac{\sqrt{2} - 1}{\sqrt{2} + 1}.$$

For $(\alpha, \beta) = (p^+, p^-)$ and r_1, r_4, c_1, c_4 satisfying the equations in (5.15), (5.16), it holds that

(1) $r_1 > 1/(3B^2)$ and $c_4 > 1/(3B^2)$.

(2) If in addition $r_1 r_4, c_1 c_4 > 1$, then $r_1 > 4/(9B^2)$ and $c_4 > 4/(9B^2)$.

The proof of Lemma 9.6 is given in Section 9.5.2. We now prove inequality (9.12). Rewrite (9.12) as

$$B^2(r_1 + r_4)^2 + 2B(B^2 + 1)(r_1 + r_4) + (B^2 + 1)^2 > (1 - B^2)^2(1 + \sqrt{r_1 r_4}),$$

or

$$B^2(r_1 + r_4)^2 + 2B(B^2 + 1)(r_1 + r_4) + 4B^2 > (1 - B^2)^2 \sqrt{r_1 r_4}. \tag{9.13}$$

At this point we split the analysis into the cases $r_1 r_4 > 1$ and $r_1 r_4 < 1$, the first being considerably harder.

Claim 9.7. *Inequality (9.13) holds when $r_1 r_4 < 1$.*

Proof. We prove the stricter inequality (since $1 > B > 0$ and $r_1, r_4 > 0$)

$$2Br_1 > \sqrt{r_1 r_4}.$$

By item (1) of Lemma 9.6 and the assumption $r_1 r_4 < 1$, we have

$$2Br_1 > \frac{2}{3} \cdot \frac{1}{B} > 1 > \sqrt{r_1 r_4}. \quad \square$$

Claim 9.8. *Inequality (9.13) holds when $r_1 r_4 > 1$.*

Proof. We prove the stricter inequality (since $1 > B > 0$ and $r_1, r_4 > 0$)

$$B^2(r_1 + r_4)^2 > \sqrt{r_1 r_4}. \tag{9.14}$$

We first identify the regions where (9.14) is hard to prove. Note that

$$B^2(r_1^2 + 2r_1 r_4 + r_4^2) \geq 4B^2 r_1 r_4,$$

so (9.14) is true if $4B^2 r_1 r_4 > \sqrt{r_1 r_4}$. Thus we may assume that

$$\frac{1}{16B^4} \geq r_1 r_4. \tag{9.15}$$

Since $r_1 r_4 > 1$, by item (2) of Lemma 9.6

$$B^2 r_1^2 > \frac{4^2}{9^2 B^2}, \quad (9.16)$$

so that (9.14) is true if

$$\frac{4^2}{9^2 B^2} \geq \sqrt{r_1 r_4}.$$

Thus we may assume that

$$r_1 r_4 > \frac{4^4}{9^4 B^4}. \quad (9.17)$$

We are now ready to prove (9.14). Using (9.15), (9.16) and (9.17) we obtain

$$B^2(r_1^2 + 2r_1 r_4 + r_4^2) \geq B^2(r_1^2 + 2r_1 r_4) > \frac{1}{B^2} \left(\frac{4^2}{9^2} + 2 \cdot \frac{4^4}{9^4} \right) > \frac{1}{B^2} \cdot \frac{1}{4} \geq \sqrt{r_1 r_4}. \quad \square$$

Proof of Lemma 2.5. The arguments in this section established the required analogue of Lemma 5.6, that is, the only critical points of $\phi_2(\gamma, \delta)$ satisfy $\gamma = \alpha^2$, $\delta = \beta^2$. Now, we can conclude using Lemmas 5.7 and 5.8, just as in the proof of Lemma 2.4 in Section 5.3. \square

9.5.1. Proof of Lemma 9.5. By the remarks in the end of Section 3, for the Ising model with no external field it holds that

$$\frac{\alpha}{1-\alpha} = \frac{1-\beta}{\beta}$$

and $Q^+ Q^- = 1$. For ease of presentation, let $x = Q^+$, $y = Q^-$. By (3.3) and (3.6), we have

$$\frac{\alpha}{1-\alpha} = \frac{x(By+1)}{y+B}, \quad \text{where } x = \left(\frac{By+1}{y+B} \right)^2 \quad \text{and } x > 1 > y > 0.$$

Since $xy = 1$, we have that

$$\frac{\alpha}{1-\alpha} = \frac{By+1}{y(y+B)}, \quad \text{where } \frac{1}{y^3} = \left(\frac{By+1}{y(y+B)} \right)^2 \quad \text{and } 0 < y < 1. \quad (9.18)$$

Proof of Lemma 9.5. For

$$0 < B < \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2},$$

we want to prove that

$$\frac{\alpha}{1-\alpha} > \frac{4}{9} \cdot \frac{1}{B^3},$$

which, given (9.18), is equivalent to $y < (3/2) \sqrt[3]{3/2B^2}$. For convenience, let $c := (3/2) \sqrt[3]{3/2}$. The equation for y in (9.18) may be rewritten as

$$(y+B)^2 = y(By+1)^2,$$

which in turn yields that

$$B^2y^2 + (B^2 + 2B - 1)y + B^2 = 0,$$

since, by (9.18), we are interested in $0 < y < 1$.

Let $f(z) := B^2z^2 + (B^2 + 2B - 1)z + B^2$. By Lemma 3.3, we know that y is the unique root of $f(z)$ for $0 < z < 1$. Observe that $f(0) = B^2 > 0$, so the desired inequality will follow if we prove that $f(B^2c) < 0$. Now, f may be rewritten as

$$f(z) = B^2z \left(z + \frac{1}{z} - \frac{1 - 2B - B^2}{B^2} \right).$$

For $z = B^2c$ and $0 < B < 3 - 2\sqrt{2}$, we have

$$\begin{aligned} z + \frac{1}{z} &= \frac{B^4c^2 + 1}{B^2c} \leq \frac{(3 - 2\sqrt{2})^4c^2 + 1}{c} \cdot \frac{1}{B^2}, \\ \frac{1 - 2B - B^2}{B^2} &\geq \frac{1 - 2(3 - 2\sqrt{2}) - (3 - 2\sqrt{2})^2}{B^2} = \frac{16\sqrt{2} - 22}{B^2}. \end{aligned}$$

A straightforward calculation gives

$$\frac{(3 - 2\sqrt{2})^4c^2 + 1}{c} < \frac{3}{5} < 16\sqrt{2} - 22,$$

thus proving $f(B^2c) < 0$. The lemma follows. □

9.5.2. Proof of Lemma 9.6. We need the following lemma.

Lemma 9.9. *When $1 > B_1B_2 > 0$ and r_1, r_4, c_1, c_4 satisfy (5.15) and (5.16), it holds that*

$$\frac{\alpha}{1 - \alpha} < \frac{1 + r_1/B_2}{1 + B_2r_4}, \quad \frac{1 + B_1c_1}{1 + c_4/B_1} < \frac{\beta}{1 - \beta}.$$

Assuming Lemma 9.9 for the moment, we give the proof of Lemma 9.6.

Proof of Lemma 9.6. When $B_1 = B_2 = B$, Lemma 9.9 gives

$$\frac{\alpha}{1 - \alpha} < \frac{1 + r_1/B}{1 + Br_4}, \quad \frac{1 + Bc_1}{1 + c_4/B} < \frac{\beta}{1 - \beta}.$$

Since $r_4, c_1 > 0$, it follows that

$$r_1 > B \left(\frac{\alpha}{1 - \alpha} - 1 \right) > B \left(\frac{4}{9} \cdot \frac{1}{B^3} - 1 \right) \quad \text{and} \quad c_4 > B \left(\frac{1 - \beta}{\beta} - 1 \right) > B \left(\frac{4}{9} \cdot \frac{1}{B^3} - 1 \right),$$

where we have used Lemma 9.5 to bound $\alpha/(1 - \alpha)$ and $(1 - \beta)/\beta$. This proves the first item of the lemma, after observing that

$$\frac{4}{9} \cdot \frac{1}{B^3} - 1 > \frac{1}{3} \cdot \frac{1}{B^3}$$

for any $B < 1/3$.

To prove the second item of the lemma, note that when $r_1 r_4, c_1 c_4 > 1$ we have

$$\frac{1 + r_1/B}{1 + Br_4} < \frac{1 + r_1/B}{1 + B/r_1} = \frac{r_1}{B} \quad \text{and} \quad \frac{1 + Bc_1}{1 + c_4/B} > \frac{1 + B/c_4}{1 + c_4/B} = \frac{B}{c_4}.$$

It follows that

$$r_1 > B \cdot \frac{\alpha}{1 - \alpha} \quad \text{and} \quad c_4 > B \cdot \frac{1 - \beta}{\beta}.$$

The desired bounds now follow after using Lemma 9.5 to bound $\alpha/(1 - \alpha)$ and $(1 - \beta)/\beta$. □

We next give the proof of Lemma 9.9.

Proof of Lemma 9.9. Define

$$\begin{aligned} C'_1 &= \frac{B_1^2 c_1 + 2B_1 + c_4}{B_1 c_1 + (B_1 B_2 + 1) + B_2 c_4}, & C'_4 &= \frac{c_1 + 2B_2 + B_2^2 c_4}{B_1 c_1 + (B_1 B_2 + 1) + B_2 c_4}, \\ R'_1 &= \frac{B_1^2 r_1 + 2B_1 + r_4}{B_1 r_1 + (B_1 B_2 + 1) + B_2 r_4}, & R'_4 &= \frac{r_1 + 2B_2 + B_2^2 r_4}{B_1 r_1 + (B_1 B_2 + 1) + B_2 r_4}. \end{aligned}$$

Using C'_1, C'_4, R'_1, R'_4 we can rewrite the equations in (5.15), (5.16) as

$$r_1 C'_1 = \frac{\gamma}{\alpha - \gamma}, \quad r_4 C'_4 = \frac{1 - 2\alpha + \gamma}{\alpha - \gamma}, \tag{9.19}$$

$$c_1 R'_1 = \frac{\delta}{\beta - \delta}, \quad c_4 R'_4 = \frac{1 - 2\beta + \delta}{\beta - \delta}. \tag{9.20}$$

Eliminating γ, δ from (9.19) and (9.20) respectively, we obtain

$$\frac{\alpha}{1 - \alpha} = \frac{1 + r_1 C'_1}{1 + r_4 C'_4}, \quad \frac{\beta}{1 - \beta} = \frac{1 + c_1 R'_1}{1 + c_4 R'_4}. \tag{9.21}$$

We have the following loose bounds on R'_1, R'_4, C'_1, C'_4 :

$$B_1 < R'_1, \quad R'_4 < \frac{1}{B_1}, \quad C'_1 < \frac{1}{B_2}, \quad B_2 < C'_4. \tag{9.22}$$

These bounds may be proved by picking any of the inequalities, multiplying out and using $B_1 B_2 < 1$. It is immediate now to check that the bounds in Lemma 9.9 follow after combining (9.21) and (9.22). □

9.6. Proof of Lemma 2.6

To prove Lemma 2.6, we will need to establish the following lemma.

Lemma 9.10. *For the hard-core model (that is, $B_1 = 0, B_2 = 1$) for $\Delta = 3, 4, 5$ and $(\alpha, \beta) = (p^+, p^-)$ the solution of (5.9), (5.10) and (5.11) satisfies $\gamma = \alpha^2$ and $\delta = \beta^2$.*

Assuming for now Lemma 9.10, let us conclude Lemma 2.6.

Proof of Lemma 2.6. By Lemma 9.10, the only critical points of $\phi_2(\gamma, \delta)$ satisfy $\gamma = \alpha^2$, $\delta = \beta^2$. The result now follows by using Lemmas 5.7 and 5.8, just as in the proof of Lemma 2.4 in Section 5.3. \square

Finally, we give the proof of Lemma 9.10.

Proof of Lemma 9.10. Let $d := \Delta - 1$ and let

$$c_1 := C_1/C_2, \quad c_4 := C_4/C_2, \quad r_1 := R_1/R_2, \quad r_4 := R_4/R_2$$

(the same notation as we used in the proof of Lemma 5.6). Equations (5.19), for $B_1 = 0$ and $B_2 = 1$, become

$$(c_1c_4)^{1/d} - 1 = \frac{r_1r_4 - 1}{(1 + r_4)^2} \quad \text{and} \quad (r_1r_4)^{1/d} - 1 = \frac{c_1c_4 - 1}{(1 + c_4)^2}. \tag{9.23}$$

Let

$$x = (r_1r_4)^{1/d}, \quad y = (c_1c_4)^{1/d}, \quad a = 1/r_4, \quad \text{and} \quad b = 1/c_4.$$

We can rewrite equation (9.23) as follows:

$$y = \frac{1 + 2a + a^2x^d}{1 + 2a + a^2} \quad \text{and} \quad x = \frac{1 + 2b + b^2y^d}{1 + 2b + b^2}. \tag{9.24}$$

Solving for a and b we obtain

$$a = \frac{y - 1 + q_a}{x^d - y} \quad \text{and} \quad b = \frac{x - 1 + q_b}{y^d - x}, \tag{9.25}$$

where

$$q_a^2 = (y - 1)(x^d - 1) \quad \text{and} \quad q_b^2 = (x - 1)(y^d - 1). \tag{9.26}$$

Note that $a > 0$ and $b > 0$. Assume $y > 1$. Then $x > 1$ (otherwise the sides of (9.23) would have different signs) and also $y \leq x^d$ (the right-hand side of (9.24) is a convex combination of 1 and x^d) and hence $|q_a| > y - 1$. Thus, using $a > 0$ in equation (9.25), we obtain $q_a > 0$ and similarly $q_b > 0$. Now assume $y < 1$. Then $x < 1$ and also $y \geq x^d$ (again, the right-hand side of (9.24) is a convex combination of 1 and x^d) and hence $|q_a| > 1 - y$. Thus, using $a > 0$ in equation (9.25), we obtain $q_a < 0$ and similarly $q_b < 0$.

Because of symmetry between x and y we only need to consider two cases (in the case $x = y = 1$ we have $\gamma = \alpha^2$ and $\delta = \beta^2$):

- I $1 < y \leq x \leq y^d$, $q_a := \sqrt{(y - 1)(x^d - 1)}$, $q_b := \sqrt{(x - 1)(y^d - 1)}$, and
- II $0 < y^d \leq x \leq y < 1$, $q_a := -\sqrt{(y - 1)(x^d - 1)}$, $q_b := -\sqrt{(x - 1)(y^d - 1)}$.

From equation (5.11) we obtain the following expressions for α and β (see equation (9.21)):

$$\frac{\alpha}{1 - \alpha} = \frac{\frac{c_4}{1+c_4}r_1 + 1}{\frac{c_1+2+c_4}{1+c_4}r_4 + 1} \quad \text{and} \quad \frac{\beta}{1 - \beta} = \frac{\frac{r_4}{1+r_4}c_1 + 1}{\frac{r_1+2+r_4}{1+r_4}c_4 + 1}.$$

Solving for α, β and using the parametrization with x, y, a, b , we have

$$\begin{aligned}\alpha &= \frac{y(1+a)^2 - 1 - a + ab}{x(1+b)^2 + y(1+a)^2 + 2ab - 1}, \\ \beta &= \frac{x(1+b)^2 - 1 - b + ab}{x(1+b)^2 + y(1+a)^2 + 2ab - 1}.\end{aligned}\tag{9.27}$$

Equations (3.1), (3.6) for $B_1 = 0, B_2 = 1, (\alpha, \beta) = (p^+, p^-)$ easily imply

$$\alpha(1-\alpha)^d = \beta(1-\beta)^d.\tag{9.28}$$

Equation (9.28) is equivalent to (assuming $\alpha \neq \beta$)

$$\frac{\alpha(1-\alpha)^d - \beta(1-\beta)^d}{\alpha - \beta} = 0,\tag{9.29}$$

which using the expressions for α and β from (9.27) becomes

$$\frac{(y(1+a)^2 - 1 - a + ab)(x(1+b)^2 + ab + a)^d - (x(1+b)^2 - 1 - b + ab)(y(1+a)^2 + ab + b)^d}{x(b+1)^2 - y(a+1)^2 + a - b} = 0.\tag{9.30}$$

The final part of the proof will require some computational assistance (just to manipulate polynomials). First we are going to plug the expression for a and b from (9.25) into (9.30). Then we are going to use (9.26) to reduce the powers of q_a and q_b , obtaining an expression of the form

$$c_{00} + c_{10}q_a + c_{01}q_b + c_{11}q_aq_b,$$

where $c_{00}, c_{01}, c_{10}, c_{11}$ are polynomials in x and y . Finally we will show that in case I we have $c_{00} > 0, c_{01} > 0, c_{10} > 0, c_{11} > 0$ and in case II we have $c_{00} > 0, c_{01} < 0, c_{10} < 0, c_{11} > 0$ (and hence (9.30) cannot be zero). This will be accomplished by reparametrizing $x = (ty + y^d)/(t + 1)$, factoring the expressions, and observing that:

- factor $y - 1$ occurs with even power in c_{00} and c_{11} ,
- factor $y - 1$ occurs with odd power in c_{01} and c_{10} , and
- all the other factors have all coefficients positive.

The details of the argument appear in Appendix A of the online version [9]. □

References

- [1] Achlioptas, D. and Naor, A. (2005) The two possible values of the chromatic number of a random graph. *Ann. of Math.* **162** 1333–1349.
- [2] Achlioptas, D. and Peres, Y. (2004) The threshold for random k -SAT is $2^k \log 2 - O(k)$. *J. Amer. Math. Soc.* **17** 947–973.
- [3] Beffara, V. and Duminil-Copin, H. (2012) The self-dual point of the two-dimensional random cluster model is critical for $q \geq 1$. *Probab. Theory Rel. Fields* **153** 511–542.
- [4] de Bruijn, N. G. (1981) *Asymptotic Methods in Analysis*, Dover.
- [5] Cai, J.-Y., Galanis, A., Goldberg, L. A., Guo, H., Jerrum, M., Štefankovič, D. and Vigoda, E. (2014) #BIS-hardness for 2-spin systems on bipartite bounded degree graphs in the tree non-uniqueness region. In *Proc. 18th International Workshop on Randomization and Computation (RANDOM)*, pp. 582–595.

- [6] Dembo, A., Montanari, A. and Sun, N. (2013) Factor models on locally tree-like graphs. *Ann. Probab.* **41** 4162–4213.
- [7] Dyer, M., Frieze, A. M. and Jerrum, M. (2002) On counting independent sets in sparse graphs. *SIAM J. Comput.* **31** 1527–1541.
- [8] Galanis, A., Ge, Q., Štefankovič, D., Vigoda, E. and Yang, L. (2014) Improved inapproximability results for counting independent sets in the hard-core model. *Random Struct. Alg.* **45** 78–110.
- [9] Galanis, A., Štefankovič, D. and Vigoda, E. (2012) Inapproximability of the partition function for the antiferromagnetic Ising and hard-core models. [arXiv.org:1203.2226](https://arxiv.org/abs/1203.2226)
- [10] Galanis, A., Štefankovič, D. and Vigoda, E. (2014) Inapproximability for antiferromagnetic spin systems in the tree non-uniqueness region. In *Proc. 46th Annual ACM Symposium on Theory of Computing (STOC)*. **62** pp. 823–831. <http://dl.acm.org/citation.cfm?doid=2785964>
- [11] Georgii, H.-O. (1988) *Gibbs Measures and Phase Transitions*, Vol. 9 of *De Gruyter Studies in Mathematics*, Walter de Gruyter.
- [12] Goldberg, L. A., Jerrum, M. and Paterson, M. (2003) The computational complexity of two-state spin systems. *Random Struct. Alg.* **23** 133–154.
- [13] Greenhill, C. (2000) The complexity of counting colorings and independent sets in sparse graphs and hypergraphs. *Comput. Complexity* **9** 52–72.
- [14] Janson, S. (1995) Random regular graphs: Asymptotic distributions and contiguity. *Combin. Probab. Comput.* **4** 369–405.
- [15] Janson, S., Luczak, T. and Rucinski, A. (2000) *Random Graphs*, Wiley-Interscience.
- [16] Jerrum, M. and Sinclair, A. (1993) Polynomial-time approximation algorithms for the Ising model. *SIAM J. Comput.* **22** 1087–1116.
- [17] Kelly, F. P. (1985) Stochastic models of computer communication systems. *J. Royal Statist. Soc. Ser. B* **47** 379–395.
- [18] Li, L., Lu, P. and Yin, Y. (2013) Correlation decay up to uniqueness in spin systems. In *Proc. 24th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA)*, pp. 47–66.
- [19] Martinelli, F., Sinclair, A. and Weitz, D. (2004) Glauber dynamics on trees: Boundary conditions and mixing time. *Comm. Math. Phys.* **250** 301–334.
- [20] Martinelli, F., Sinclair, A. and Weitz, D. (2007) Fast mixing for independent sets, colorings, and other models on trees. *Random Struct. Alg.* **31** 134–172.
- [21] Mezard, M. and Montanari, A. (2009) *Information, Physics, and Computation*, Oxford University Press.
- [22] Molloy, M., Robalewska, H., Robinson, R. W. and Wormald, N. C. (1997) 1-factorizations of random regular graphs. *Random Struct. Alg.* **10** 305–321.
- [23] Mossel, E. (2004) Survey: Information flow on trees. In *Graphs, Morphisms, and Statistical Physics*, Vol. 63 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pp. 155–170.
- [24] Mossel, E., Weitz, D. and Wormald, N. (2009) On the hardness of sampling independent sets beyond the tree threshold. *Probab. Theory Rel. Fields* **143** 401–439.
- [25] Restrepo, R., Shin, J., Tetali, P., Vigoda, E. and Yang, L. (2013) Improved mixing condition on the grid for counting and sampling independent sets. *Probab. Theory Rel. Fields* **156** 75–99.
- [26] Sinclair, A., Srivastava, P. and Thurley, M. (2014) Approximation algorithms for two-state anti-ferromagnetic spin systems on bounded degree graphs. *J. Statist. Phys.* **155** 666–686.
- [27] Sly, A. (2010) Computational transition at the uniqueness threshold. In *Proc. 51st Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pp. 287–296.
- [28] Sly, A. and Sun, N. (2014) The computational hardness of counting in two-spin models on d -regular graphs. *Ann. Probab.* **42** 2383–2416.
- [29] Talagrand, M. (2003) *Spin Glasses: A Challenge for Mathematicians*, Springer.
- [30] Valiant, L. G. (1979) The complexity of enumeration and reliability problems. *SIAM J. Comput.* **8** 410–421.

- [31] Weitz, D. (2006) Counting independent sets up to the tree threshold. In *Proc. 38th Annual ACM Symposium on Theory of Computing (STOC)*, pp. 140–149.
- [32] Wormald, N. C. (1999) Models of random regular graphs. In *Surveys in Combinatorics 1999* (J. D. Lamb and D. A. Preece, eds), Cambridge University Press, pp. 239–298.
- [33] Zhang, J., Liang, H. and Bai, F. (2011) Approximating partition functions of two-state spin systems. *Inform. Process. Lett.* **111** 702–710.