

# Decidability of String Graphs

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## ABSTRACT

We show that string graphs can be recognized in nondeterministic exponential time by giving an exponential upper bound on the number of intersections for a drawing realizing the string graph in the plane. This upper bound confirms a conjecture by Kratochvíl and Matoušek [14] and settles the long-standing open problem of the decidability of string graph recognition (Sinden [18], Graham [7]). Finally we show how to apply the result to solve another old open problem: deciding the existence of Euler diagrams, a central problem of topological inference (Grigni, Papadias, Papadimitriou [8]).

## 1. INTRODUCTION

Is it possible that some  $A$  is  $B$ , some  $B$  is  $C$ , but no  $A$  is  $C$ ? Easily, you say, and your mind conjures up a diagram that Euler (and Leibniz, and Sturm before him) would have used to illustrate this situation.

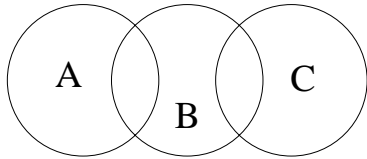


Figure 1: Some  $A$  is  $B$ , some  $B$  is  $C$ , but no  $A$  is  $C$ .

However, it is not always possible to illustrate a situation

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that is logically consistent by an Euler diagram in the plane<sup>1</sup>: we can turn the complete graph on five vertices into such an example with fifteen regions (one for each vertex and edge) [8]. How can we, in general, determine whether there is an Euler diagram for a given set of specifications?

Diagrammatic reasoning is concerned with representability of logical relations in the plane (and other spaces). This area has drawn attention from different research groups including Artificial Intelligence and Geometrical Information Systems [1, 8], Spatial Databases [16], Integrated Circuits [18], and Logic [8, 15]. One of the major open problems in this area is the decidability of the existence of a representation for a given, logically consistent, formula. The special case from the introductory paragraph in which we specified for a collection of regions whether they should intersect or not has been open since the sixties.

This case is captured by the combinatorial notion of string graphs. String graphs are intersection graphs of curves in the plane with a vertex for each curve, and an edge representing an intersection between two curves. The notion was introduced in 1966 by Sinden [18] who stated the main problem thus:

It is specified which pairs of a collection of curves (or connected regions) in the plane cross and which pairs do not cross. When are such specifications consistent?

Sinden was working on the layout problem of integrated circuits (thin film RC circuits to be precise), and the string graph problem arose naturally in this context, since the technology for creating the circuits made it possible for some pairs of conductors to cross. On the theoretical side he observed that all planar graphs are string graphs, and also gave a small example of a graph which is not a string graph. Ron Graham, in 1976, introduced the problem to the combinatorial community [7], and in the same year Ehrlich, Even, and Tarjan [5] showed that computing the chromatic number of string graphs is **NP**-hard. Since then string graphs have become a popular subject in graph theory, and several characterizations of special kinds of string graphs are known, but the general recognition problem remained open. In 1991 Kratochvíl [12] proved that the problem of recognizing string graphs is **NP**-hard, showing that a characterization is not going to be polynomial time computable (unless  $\mathbf{P} = \mathbf{NP}$ ). At the same time Kratochvíl and Matoušek [14] proved the surprising result that some string graphs require

<sup>1</sup>We restrict ourselves to simply connected regions. See Kratochvíl [11, Section 2] for a remark on connected regions.

an exponential number of intersections to be realized in the plane. They conjectured an exponential upper bound on the number of intersections. We show that this conjecture is indeed true, putting the recognition problem of string graphs in **NEXP**. We recently learnt that János Pach and Géza Tóth independently obtained a proof of the decidability of the string graph recognition problem (apparently using different techniques) which they presented at an AMS meeting this spring.

## 2. PRELIMINARIES

Given a graph  $G = (V, E)$  and a set  $R \subseteq \binom{E}{2} = \{\{e, f\} : e, f \in E\}$  on  $E$ , we call a drawing  $D$  of  $G$  in the plane a *weak realization* of  $(G, R)$  if only pairs of edges which are in  $R$  are allowed to intersect in  $D$  (they do not have to intersect, however). In this case we call  $(G, R)$  *weakly realizable*. We say that  $D$  is a *realization* of  $G$  if exactly the pairs of edges in  $R$  intersect in  $D$ .<sup>2</sup> Let us define  $c_w(G, R)$  as the smallest number of intersections in a weak realization of  $(G, R)$ ,  $c_w(G) = \max\{c_w(G, R) : (G, R) \text{ has a weak realization}\}$ , and  $c_w(m) = \max\{c_w(G) : G \text{ has } m \text{ edges}\}$ . Similarly define  $c_r(G, R)$ ,  $c_r(G)$ , and  $c_r(m)$  for realizations.<sup>3</sup> The quantity  $cr(G) = c_w(G, \binom{E}{2})$  is known as the *crossing number* of the graph  $G$ , and was shown to have an **NP**-complete decision problem by Garey and Johnson in the early eighties [6]. The other extreme case,  $c_w(G, \emptyset)$ , is equivalent to planarity testing, and therefore in **P**.

A *curve* (or *string*) is a set homeomorphic to  $[0, 1]$ . Given a collection of curves  $(C_i)_{i \in I}$  in the plane, the corresponding intersection graph is  $(I, \{\{i, j\} : C_i \text{ and } C_j \text{ intersect}\})$ . The *size* of a collection of curves is the number of intersection points (we assume that no three curves intersect in the same point). A graph isomorphic to the intersection graph of a collection of curves in the plane is called a *string graph*. Let  $c_s(G)$  be the size of a smallest (in the sense of size defined above) set of curves whose intersection graph is isomorphic to  $G$ , and define  $c_s(m) = \max\{c_s(G) : G \text{ has } m \text{ edges}\}$ . It is not at all obvious that  $c_s(G)$  is a finite number if  $G$  is a string graph. It is conceivable that an infinite number of intersections might be required to realize a string graph. Lemma 4.2 shows that this is not the case: *a string graph can always be realized with a finite number of intersections*. We postpone the lemma and its proof to the topological part of this paper. The following relationships between the functions we defined are well known:

- (i)  $c_w(m) \leq c_r(m)$ ,
- (ii)  $c_r(m) \leq c_s(3m^2)$ , and
- (iii)  $c_s(m) \leq c_w(m)$ .

The first inequality follows from  $c_w(G, R) \leq \max\{c_r(G, R') : R' \subseteq R, \text{ and } (G, R) \text{ has a realization}\}$ , the second from Kratochvíl's reduction of realizability to string graphs [12, Proposition 1], and the third from his reduction of string graphs to weak realizability [12, Proposition 5].

Kratochvíl and Matoušek [14] showed that  $c_w(m) \geq 2^{cm}$  for some positive constant  $c$ . Our main result shows that

<sup>2</sup>Kratochvíl [13, 11, 12] calls  $(G, R)$  an *abstract topological graph*, and uses the word *feasible* for weakly realizable.

<sup>3</sup>The function definitions here and in the following paragraph are based on similar ones in the papers by Kratochvíl and Matoušek [14, 13].

$c_w(m) \leq m2^m$  (Kratochvíl and Matoušek conjectured an upper bound of  $2^{m^k}$ ). This implies the decidability of string graphs, which was a long-standing open problem in the field.

## 3. BOUNDING THE NUMBER OF INTERSECTIONS

If we assign each curve in a collection of curves a unique number, we can look at the intersections of the curves along a particular curve as a word over an alphabet (we use the fact that the number of intersections is finite). The basic idea of the proof is to show that if such a word is too long, it contains a substructure which allows a drawing of the collection of curves of smaller size (lesser number of intersections). Hence we can bound the number of intersections along each curve, thereby bounding the size of the whole drawing.

LEMMA 3.1. *Every word of length  $\geq 2^n$  over an alphabet of size  $n$  contains a non-trivial subword in which each character occurs an even number of times.*

PROOF. Let  $\Sigma = \{1, \dots, n\}$ , and  $w \in \Sigma^*$ ,  $|w| \geq 2^n$ . To every  $i \in \{0, \dots, 2^n\}$  assign a vector  $v_i$  in  $\mathbb{Z}_2^n$  whose  $j$ -th coordinate is the parity of the number of occurrences of the symbol  $j$  in the prefix of  $w$  of length  $i$ . (In particular  $v_0$  is the all-zero vector.) Since there are  $2^n + 1$  indices, but only  $2^n$  vectors in  $\mathbb{Z}_2^n$ , there are  $0 \leq i < j \leq 2^n$  such that  $v_i = v_j$ . The non-trivial subword of  $w$  starting in position  $i + 1$  and ending in position  $j$  fulfills the conditions of the lemma. ■

THEOREM 3.2. *Let  $G$  be a graph with  $m$  edges,  $R \subseteq \binom{E}{2}$  such that  $(G, R)$  is weakly realizable, and let  $D$  be a weak realization of  $(G, R)$  with the minimal number of intersections. Then for any edge  $e \in G$  there are less than  $2^m$  intersections on the curve realizing  $e$  in  $D$ .*

PROOF. Suppose not. Let  $D$  be a weak realization of  $(G, R)$  with the minimal number of intersections and let  $e$  be an edge of  $G$  which has more than  $2^m - 1$  intersections in  $D$ . Lemma 4.2 tells us that the number of intersections in the realization is finite. By Lemma 3.1 we can choose a non-trivial segment of this edge which is intersected an even number of times by any other edge. Draw a window around this part which contains no other intersections. For each edge  $f$  assign numbers  $1, 2, \dots, 4n_f$  ( $n_f \in \mathbb{N}$ ) to intersections with the window in the order they appear on  $f$  (choose an arbitrary orientation of  $f$ ). For an example see Figure 2.

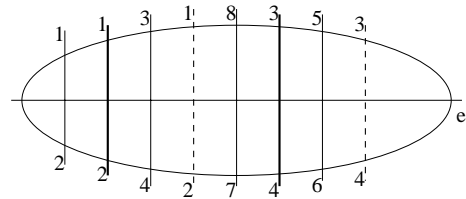


Figure 2: Part of  $e$  with surrounding window.

We can assume that the window is a circle, that  $e$  within the window is a straight line passing through the center, and that for every  $f$  intersections  $2i - 1$  and  $2i$  are mirror images of each other (with  $e$  as the mirror),  $i = \{1, \dots, 2n_f\}$ .

Clear the inside of the window with the exception of  $e$ . For each edge  $f$  there is connection between intersection  $4i - 2$  and  $4i - 1$  lying completely outside the window,  $i \in \{1, \dots, n_f\}$ . Look at all of these connections and use circular inversion to bring them inside the circle. Now mirror everything inside the window along  $e$ .

This yields for every edge  $f$  a connection between  $4i - 3$  and  $4i$ ,  $i \in \{1, \dots, n_f\}$ , inside the window. Since  $4n_f$  is the last intersection of  $f$  with the window we have that the edge  $f$  still connects its endpoints (here we needed that  $f$  intersects  $e$  an even number of times). Note that we have reduced the number of intersections of  $f$  with the window from  $4n_f$  to  $2n_f$ . Every intersection between curves inside the circle corresponds to an intersection outside and hence this new realization respects  $R$ . We now move the part of  $e$  in the circle to coincide with one of the two arcs into which  $e$  separates the full circle. We choose the one which results in the smaller number of intersections with  $e$ . Since each edge  $f$  causes at most  $2n_f$  intersections with the window, this means that the number of intersections on  $e$  within the area of the window has been halved, and hence the total number of intersections of the drawing has been decreased, a contradiction. ■

COROLLARY 3.3. *String graph recognition is in NEXP.*

PROOF. Theorem 3.2, and the fact that  $c_s(m) \leq c_w(m)$  (see the preliminaries) shows that if  $G$  is a string graph, there is a collection of curves of size at most  $m2^m$  whose intersection graph is isomorphic to  $G$ . We can consider the drawing of the collection of curves as a planar graph (each intersection point becoming a vertex) with at most  $m2^m$  vertices. By a result of Schnyder [17], and de Fraysseix, Pach, and Pollack [3] there is a drawing of this graph on a planar straight-line grid of at most  $(m2^m)^2$  vertices. Hence in NEXP we can guess a graph on such a grid and verify whether its intersection graph is isomorphic to  $G$ . ■

The same argument shows that we can decide the (weak) realizability of a topological graph  $(G, R)$  in NEXP.

## 4. PLANE TOPOLOGY

We introduce some basic plane topology. A *homeomorphism* is a bijective continuous mapping whose inverse is also continuous. A *region* is a subset of the plane homeomorphic to the closed unit disc. Note that both a region and its boundary are compact (the homeomorphic image of a compact set is compact). The boundary of a region is a simple closed curve, i.e. it can be parameterized by a continuous function  $\gamma : [0, 1] \mapsto \mathbb{R}^2$  which is injective (apart from  $\gamma(0) = \gamma(1)$ ).

The *Hausdorff distance*  $\text{dist}(A, B)$  of two sets is defined as

$$\text{dist}(A, B) = \max\left\{\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y)\right\},$$

where  $d(x, y)$  is the Euclidean distance of two points in the plane. The Hausdorff distance is a metric for compact sets, i.e. it is symmetric, satisfies the triangle inequality and  $\text{dist}(A, B) = 0$  iff  $A = B$ . We let

$$d(A, B) = \inf_{x \in A} \inf_{y \in B} d(x, y).$$

Note that for closed, nonempty sets  $d(A, B) > 0$  iff  $A \cap B = \emptyset$ . For sets  $d$  is not a metric.

A *simple curve* is any homeomorphic image of the interval  $[0, 1]$ . It is called a *polygonal arc* if it is made up of a finite number of line segments. Consider a simple curve  $C$  with parameterization  $\gamma : [0, 1] \mapsto C$ . We call a *polygonal  $\delta$ -skeleton* for a simple curve  $C$  any polygonal arc described by a sequence  $(\gamma(r_i))_{1 \leq i \leq n}$ , where  $0 = r_1 < \dots < r_n = 1$  such that  $r_{i+1} - r_i < \delta$ . The points  $\gamma(r_i)$  are the *vertices* of the polygonal skeleton.

The homeomorphism  $\gamma : [0, 1] \mapsto C$  is uniformly continuous (being defined on a compact set) which immediately implies the following result (for a proof see [19]).

PROPOSITION 4.1. *Given a simple curve  $C$  and  $\varepsilon > 0$  there is  $\delta > 0$  such that every polygonal  $\delta$ -skeleton  $P$  of  $C$  fulfills  $\text{dist}(P, C) < \varepsilon$ .*

The idea for showing that string graphs can be realized with finitely many intersections is to substitute each curve with a skeleton that approximates it closely. To maintain intersections we introduce witness points that belong to more than one curve. Finally we have to guard against line segments of different curves overlapping which we do by moving the points into general position.

LEMMA 4.2. *A string graph can be realized by a family of polygonal arcs with a finite number of intersections.*

PROOF. Assume we have a string graph realized by a family of curves  $(C_i)_{i \in I}$ . For each  $(i, j)$  such that  $C_i \cap C_j \neq \emptyset$  we select a witness point  $p_{i,j} \in C_i \cap C_j$ , and let  $W$  be the (finite) set of these witness points. Define

$$\begin{aligned} \varepsilon_1 &= \min\{d(C_i, C_j) : C_i \cap C_j = \emptyset\}, \\ \varepsilon_2 &= \min\{d(p, q) : p \neq q, \text{ and } p, q \in W\}, \end{aligned}$$

and  $\varepsilon = 1/2 \min\{\varepsilon_1, \varepsilon_2\}$ . Then  $\varepsilon > 0$ , since all the curves are compact sets, and  $W$  is finite. Choose  $\delta$  according to Proposition 4.1 such that all polygonal  $\delta$ -skeletons of the curve  $C_i$  are within Hausdorff distance  $\varepsilon/2$  of  $C_i$  (for all  $i \in I$ ). Fix such a polygonal  $\delta$ -skeleton  $P_i$  for each curve  $C_i$  and include on it all the witness points in  $C_i \cap W$ .

Consider the multiset  $M$  of vertices of the polygonal skeletons that are not witness points ( $M$  is a multiset because a point may belong to several skeletons). Substitute each point in  $M$  by a point within distance  $\varepsilon/2$  of the original point such that each point in  $M$  ends up in general position with regard to all points (including the ones in  $W$ ), i.e. not on the same line with any other two points from  $M \cup W$ . This defines new polygonal arcs  $P'_i$  for which  $\text{dist}(P_i, P'_i) < \varepsilon/2$ , and hence  $\text{dist}(P'_i, C_i) < \varepsilon$  (for  $i \in I$ ). If  $C_i \cap C_j = \emptyset$ , then  $d(C_i, C_j) \geq 2\varepsilon$ , hence  $d(P'_i, P'_j) \geq d(C_i, C_j) - \text{dist}(C_i, P'_i) - \text{dist}(C_j, P'_j) > 0$ , and therefore  $P'_i$  and  $P'_j$  do not intersect. If on the other hand  $C_i \cap C_j \neq \emptyset$ , then we chose a witness points  $p_{i,j}$  which belongs to both  $P'_i$  and  $P'_j$ , hence  $P'_i$  and  $P'_j$  do intersect in this case.

We finally have to prove that the number of intersections of the polygonal arcs is finite. Assume  $P'_i$  and  $P'_j$  intersect in some point  $p$  which is not a vertex of either arc (there are only finitely many vertices, so we can ignore those). Then  $p$  lies on two line segments (one belonging to  $P'_i$ , the other to  $P'_j$ ) with four (not necessarily distinct) endpoints, at most one of which can be in  $W$ , since  $\varepsilon$  is a lower bound on the

distance between points of  $W$ . Hence three of the endpoints belong to  $M$ , and are therefore in general position with regard to  $M \cup W$ . Hence  $p$  is an isolated point in  $P'_i \cap P'_j$ . However,  $P'_i \cap P'_j$  as a compact set can only be made up of a finite number of isolated points, implying it is finite. Since this is true for every  $i, j \in I$  we have shown that the overall number of intersection points is finite. ■

## 5. TOPOLOGICAL INFERENCE

We mentioned earlier that settling the problem of recognizing string graphs solves an old open problem from topological inference [2, 18]: if we specify for a collection of simply connected regions  $(A_i)_{i \in I}$  which pairs may intersect and which may not, can these regions be drawn in the plane so as to fulfill the requirements? Since the existence of such a drawing does not change if we change the universe of discourse regions to curves, the problem is equivalent to the string graph problem, and therefore solvable in **NEXP**.

Topological inference works over a larger set of predicates than overlap and disjoint. Egenhofer determined all eight possible relationships of two simply connected regions based on whether the intersection of their interior, boundary and exterior is empty or not [4]. The relations are disjoint, equal, inside, contains, cover, covered, meet, and overlap. See Figure 3 for definitions. We note that for any two simply connected regions  $A$  and  $B$  exactly one of these predicates will be true.

disjoint( $A, B$ )	the boundaries and interiors of $A$ and $B$ do not intersect.
equal( $A, B$ )	$A$ and $B$ have the same interior.
inside( $A, B$ ) ( $\equiv$ contains( $B, A$ ))	the interior and boundary of $A$ are contained in the interior of $B$ .
covered( $A, B$ ) ( $\equiv$ cover( $B, A$ ))	the interior of $A$ is properly contained in the interior of $B$ , and the boundaries intersect.
meet( $A, B$ )	the interior of $A$ is contained in the exterior of $B$ , and the boundaries intersect.
overlap( $A, B$ )	the interior of $A$ intersects the exterior of $B$ , and vice versa.

**Figure 3: The eight relationships between regions (Egenhofer).**

We call a Boolean combination of the topological predicates a *topological expression*. A topological expression is *explicit*, if it specifies the relationship between any pair of variables, meaning it is of the form  $\bigwedge_{A, B \in I} \mathcal{P}_{A, B}(A, B)$ , where  $I$  is the set of variables, and  $\mathcal{P}_{A, B}$  is one of the eight basic predicates (for each  $A, B \in I$ ). We can always assume that the expression does not contain the predicates contains or cover, because we can substitute them by inside and covered. Quantifying topological expressions we obtain *topological formulas*. Determining the truth of these (were the universe is the set of all regions in the plane) is the goal of *topological inference* [8]. Of main interest are the purely existential formulas, since they express the existence of diagrammatic representations of logical relationships (Euler

diagrams). In this case we also speak of the *realizability* of a topological expression.

In this section we will show how the decidability of the existential theory of topological expressions follows from the decidability of string graphs. More precisely we show that the realizability of topological expressions can be decided in **NEXP**.

Talking about a realization of meet( $A, B$ ), or covered( $A, B$ ) we call any point belonging to  $\partial A \cap \partial B$  a *contact point* of  $A$  and  $B$ . In the other cases points belonging to the intersection of  $\partial A$  and  $\partial B$  we simply call *intersection points*.

We will now show how to redraw a realization of an explicit topological expression to bound the number of contact points in the drawing. Note that for any explicit expression there is always an equivalent explicit expression not containing equal.

**LEMMA 5.1.** *Let  $\varphi$  be an explicit topological expression not containing equal. If there is a drawing realizing  $\varphi$ , then there is a drawing realizing  $\varphi$  in which the number of contact points on each boundary is bounded by the square of the number of variables in  $\varphi$ .*

**PROOF.** Let  $A_1, \dots, A_{|I|}$  be the family of variables occurring in  $\varphi$ . We can assume that the variables are sorted such that for  $i < j$  there is no covered( $A_i, A_j$ ). If such an ordering does not exist, then  $\varphi$  has no realization. For each meet( $A, B$ ) and covered( $A, B$ ) in  $\varphi$  we choose a witness point  $p_{A, B} \in \partial A \cap \partial B$ . Let

$$\begin{aligned} \varepsilon_1 &= \min\{d(\partial A, \partial B) : \varphi \text{ contains disjoint}(A, B), \text{ or} \\ &\quad \text{inside}(A, B)\} \\ \varepsilon_2 &= \min\{\text{dist}(A \cap \partial B, \partial A) : \varphi \text{ contains overlap}(A, B)\} \end{aligned}$$

Note that  $\varepsilon_1 > 0$ , since boundaries are closed and disjoint. Also  $\varepsilon_2 > 0$ , since there is a point in  $A \cap \partial B$  which is inside  $A$ . Let  $\varepsilon = 1/2 \min\{\varepsilon_1, \varepsilon_2\}$ . If  $B$  is a region with  $\text{dist}(B, A_i) \leq \varepsilon$  then

$$\begin{aligned} \text{inside}(A_i, A_j) &\Rightarrow \text{inside}(B, A_j) \\ \text{inside}(A_j, A_i) &\Rightarrow \text{inside}(A_j, B) \\ \text{disjoint}(A_i, A_j) &\Rightarrow \text{disjoint}(B, A_j) \\ \text{overlap}(A_i, A_j) &\Rightarrow \text{overlap}(B, A_j) \end{aligned} \tag{1}$$

Unfortunately the same is not true for meet and covered. We will redraw the regions one by one, removing unnecessary contact points while preserving the meet and covered relationships.

Suppose then that for  $A_1, \dots, A_{i-1}$  the only contact points on their boundaries are witness points. We will show how to redraw  $A_i$  to make this true for  $A_1, \dots, A_i$  while preserving that  $A_1, \dots, A_{|I|}$  realize  $\varphi$ .

Let  $\psi : D \rightarrow A_i$  be the homeomorphism of the unit closed disc to  $A_i$ . Using the Jordan-Schoenflies theorem we extend  $\psi$  to a homeomorphism of the whole plane to itself which we call  $\psi$  again. Since  $\psi$  is uniformly continuous, there exists  $\eta$  such that if  $(1 - \eta)D \subseteq E \subseteq D$  then  $\text{dist}(\psi(E), A_i) < \varepsilon$ . Let  $F$  be the union of  $\psi^{-1}(A_j)$  for which there is covered( $A_i, A_j$ ). By our assumption  $F \cap \partial D$  contains only witness points. Choose  $E$  such that  $F \cup (1 - \eta)D \subseteq E \subseteq D$  and  $E$  intersects  $\partial D$  only in witness points. Replace  $A_i$  by  $\psi(E)$ . By the implications in (1) all inside, disjoint and overlaps are preserved. Because  $E$  contains all witness points for region  $A_i$ , all covered and meet relations are satisfied, and only the witness points are contact points of  $A_i$ .

Since contact points of  $A_j$ ,  $j < i$  did not change, this will be true after redrawing all regions. ■

Before we prove the main result we need to introduce a variant of realizability. Let  $(G, R, S)$  be such that  $R, S \subseteq \binom{E}{2}$ , and  $R \cap S = \emptyset$ . We call  $(G, R, S)$  realizable if  $G$  can be drawn in the plane, such that only the pairs of edges in  $R \cup S$  intersect, and all the pairs of edges in  $S$  do intersect. It is easy to see that this variant can also be decided in **NEXP**, since the same exponential upper bounds on the intersection number applies.

**THEOREM 5.2.** *The realizability of a topological expression can be decided in **NEXP**.*

**PROOF.** Given a topological expression  $\varphi$  over variables  $(A_i)_{i \in I}$  we have to decide whether it can be realized by regions in the plane. We begin by simplifying  $\varphi$ , and then we show how to reduce the problem to a realizability problem  $(G, R, S)$  which we know to be decidable in **NEXP** by the remarks preceding the theorem.

We can assume that  $\varphi$  is a logically consistent formula in conjunctive normal form, and does not contain any negations (substitute any negation with a disjunction of all the other predicates). In **NEXP** we can verify consistency, and do not need to worry about the possibly exponential blow-up in formula length in converting a formula to conjunctive normal form. Now realizability of  $\varphi$  means realizability of any of its clauses, hence we need only consider the case that  $\varphi$  is a single clause, and an explicit clause at that (we can guess the relationships that are not given). Furthermore we remove the relation of equality from  $\varphi$  by renaming of variables, and substitute any occurrence of  $\text{cover}(B, A)$  with  $\text{covered}(A, B)$ , and  $\text{contains}(B, A)$  by  $\text{inside}(A, B)$ .

Summarizing the steps, we can assume that  $\varphi$  is an explicit, conjunctive formula containing only positive occurrences of the relations disjoint, meet, covered, overlap, inside.

Suppose that a topological graph  $(G, R, S)$  satisfies:

- (◇) There are vertices  $z, z_1, z_2, z_3$  in  $G$  connected to each other by edges which may not intersect any other edges.
- For each region  $R_i$  there is a vertex  $c_i$  (center) and a circle graph  $B_i$  (boundary) with at least 3 vertices, and no two edges of  $B_i$  may intersect.
- (♣) Each vertex in  $B_i$  is connected to  $c_i, z_1, z_2, z_3$ ; these edges are not allowed to intersect the boundary  $B_i$ , and no edge with endpoint  $c_i$  may intersect an edge with endpoint  $z_1, z_2$ , or  $z_3$ .
- (▽) The boundaries  $B_i, B_j$  may share vertices unless  $\text{disjoint}(R_i, R_j)$ , or  $\text{inside}(R_i, R_j)$  is contained in  $\varphi$ .
- (△) Edges of  $B_i, B_j$  may intersect only if  $\varphi$  contains  $\text{overlap}(R_i, R_j)$ .
- We say that a vertex  $v$  is an *in- $R_i$ -witness* (*out- $R_i$ -witness*) if it does not belong to  $B_i$  and is adjacent to  $c_i$  ( $z_1, z_2$ , and  $z_3$ , resp.) using an edge (edges, resp.) which are not allowed to intersect  $B_i$ .
- (♠) If  $\varphi$  contains  $\text{meet}(R_i, R_j)$  or  $\text{cover}(R_i, R_j)$  then  $B_i$  and  $B_j$  share at least one common vertex.

- If  $\text{disjoint}(R_i, R_j)$  is in  $\varphi$ , then there is an out- $R_i$ -witness on  $B_j$ , and an out- $R_j$ -witness on  $B_i$ . If  $\text{inside}(R_i, R_j)$  then there is in- $R_j$ -witness on  $B_i$ . If  $\text{meet}(R_i, R_j)$ , then there is an out- $R_i$ -witness on  $B_j$  between any two vertices shared with  $B_i$ , and an out- $R_j$ -witness on  $B_i$  between any two vertices shared with  $B_j$ . If  $\text{covered}(R_i, R_j)$  then there is an in- $R_i$ -witness on the boundary of  $B_j$  between any two vertices shared with  $B_i$ . If  $\text{overlap}(R_i, R_j)$  then there is an in- $R_i$  witness and an out- $R_i$  witness on the boundary of  $R_j$ , and vice versa.

We claim that if  $(G, R, S)$  has a weak realization then  $\varphi$  can be realized as an Euler diagram. Take the weak realization of  $(G, R, S)$ . We can assume that  $z$  lies outside the triangle  $z_1, z_2, z_3$ . Hence by (◇) all other vertices and edges lie inside the triangle. Because of (♣) vertex  $c_i$  must lie inside of  $B_i$  ( $z_1, z_2$ , and  $z_3$  being outside). Let region  $R_i$  be the interior of  $B_i$  together with its boundary. Clearly any in- $R_i$ -witness lies inside  $R_i$ , and any out- $R_i$ -witness lies outside  $R_i$ . For  $\text{inside}(R_i, R_j)$ , and  $\text{disjoint}(R_i, R_j)$  boundaries cannot intersect and therefore the in/out-witnesses guarantee the correct relationship. For  $\text{overlap}(R_i, R_j)$  we have in/out-witnesses of overlap. For  $\text{meet}(R_i, R_j)$  the interior of  $R_i$  cannot intersect  $R_j$ , and vice versa because of the out-witnesses; similarly for  $\text{covered}(R_i, R_j)$ .

Now we will show that if  $\varphi$  has Euler diagram then there is  $(G, R, S)$  satisfying above conditions which is small. This would imply that the problem is in **NEXP**, because we can guess  $(G, R, S)$ .

First redraw the graph using Lemma 5.1 so that the number of contact points is at most  $|I|^2$ . Enclose the diagram with large region  $Z$ , on  $\partial Z$  choose three points  $z_1, z_2, z_3$ , choose  $z$  outside  $Z$  and connect  $z$  to  $z_1, z_2, z_3$  with edges outside  $Z$ . Choose  $c_i$  inside each  $R_i$ . Now we will choose vertices on each  $\partial R_i$  and connect them to  $z_1, z_2, z_3$  with edges inside  $Z - R_i$  and to  $c_i$  with edges inside  $R_i$  (thus (♣) is satisfied). Clearly (▽) is satisfied. All contact points will be chosen on each  $\partial R_i$ . This satisfies (♠) and also (△), because we know that if two edges intersect then they intersect in an intersection point of their boundaries. If less than 3 points were chosen on  $\partial R_i$ , choose some more. Now it is routine check to see that we can choose in/out witnesses for  $\text{disjoint}(R_i, R_j)$ ,  $\text{inside}(R_i, R_j)$ ,  $\text{meet}(R_i, R_j)$ ,  $\text{covered}(R_i, R_j)$ , and  $\text{overlap}(R_i, R_j)$ . Note that we chose at most  $|I|^2$  witnesses and at most  $|I|^4$  in/out witnesses. Hence  $(G, R, S)$  is small and we can guess it in **NEXP**. ■

## 6. OPEN QUESTIONS

While it is satisfying to know that string graphs can be effectively recognized, the gap between **NP** and **NEXP** is large, and a more precise classification is called for. We know that we will not be able to reduce the upper bound on the number of intersections significantly, so it might seem that **NEXP** is the best upper bound we can expect from a combinatorial argument, but this intuition might be misleading as was demonstrated in the case of recognizing the unknot (Hass, Lagarias, Pippingier [10]).

Kratochvíl [13] suggested a different approach to obtaining an exponential upper bound. He conjectured that in any smallest weak realization of a  $(G, R)$  any edge crossed at least once is crossed exactly once by some other edge. He shows that his conjecture implies  $c_w(m) \leq m(2^{m-1} - 1)/2$ .

A topological generalization of the string graph problem remains open: is the recognition problem decidable on surfaces of higher genus? Our proof essentially relies on the inversion operation which will not be available to us (at least not in the straightforward manner we used it) in surfaces other than the 2-sphere.

From a logical point of view we have shown that the existential theories of strings and diagrams are decidable (in **NEXP**). The natural question here is what happens if additional quantifiers are allowed? As it turns out both theories become undecidable (indeed as hard as second-order arithmetic), as we will show in the journal version of the paper.

We mentioned earlier the problem of computing the crossing number of a graph (the smallest number of intersections necessary to draw the graph in the plane). This problem has long been known to be **NP**-complete. Martin Grohe [9] recently showed it to be solvable in time  $O(f(k)n^2)$  (where  $k$  is the number of intersections, and  $f(k) = O(2^{2^{p(k)}})$ , for some polynomial  $p$ ), implying that it is fixed parameter tractable, since for fixed  $k$  the complexity is quadratic. We can obtain an interesting variant of the crossing number problem by asking for the smallest number of *pairs* of edges that need to intersect to draw the graph in the plane (where each such pair can intersect any number of times). Call this the *crossing pairs number* of a graph  $G$ . Our proof technique implies that if there is a drawing of  $G$  in which at most  $k$  pairs of edges intersect, then there is a drawing of  $G$  with at most  $2k2^{2^k}$  intersections. We can then use Grohe's result to conclude that the crossing pairs number of a graph  $G$  is fixed parameter tractable. We do not know, however, whether this problem is **NP**-hard, or even in **NP**.

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