ON A PROBLEM OF W. J. LEVEQUE CONCERNING METRIC DIOPHANTINE APPROXIMATION II

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Abstract. In the first part of this series of papers, we solved LeVeque's problem that was to establish a central limit theorem for the number of solutions of the diophantine inequality

$$\left|x-\frac{p}{q}\right|\leq\frac{f(\log q)}{q^2}$$

in unknowns p, q with $q > 0$, where f is a function satisfying special assumptions and x is chosen randomly in the unit interval. In this continuation, we are interested in the almost sure behavior of the solution set. In particular, we obtain a generalized law of the iterated logarithm and we prove a result that gives strong evidence that the law of the iterated logarithm with the standard norming sequence (suggested by the central limit theorem) holds as well. Both results have to be compared with a theorem of W. M. Schmidt; e.g. they imply an inverse to Schmidt's theorem and a strong law of large numbers with an error term that is essentially better than the one provided by Schmidt's result.

1. Introduction and Results

Suppose f is a positive real-valued function defined on the non-negative real numbers satisfying the following conditions

$$f\downarrow 0,\quad \textstyle\sum_{k=1}^{\infty} f(k)=\infty, \tag{1}$$

$$\textstyle\sum_{k=1}^{n} f(k)k^{-\delta_1}\ll\left(\sum_{k=1}^{n} f(k)\right)^{1/2}, \tag{2}$$

$$\textstyle\sum_{k=1}^{n} f(k)^2\ll\left(\sum_{k=1}^{n} f(k)\right)^{1/2}, \tag{3}$$

where $0<\delta_1<1/2$. We are interested in the statistical behavior of the solution set of the diophantine inequality

$$\left|x-\frac{p}{q}\right|\leq\frac{f(\log q)}{q^2}, \tag{4}$$

in unknowns p, q with $q > 0$ when x is randomly chosen in the unit interval (with respect to Lebesgue measure which we are going to denote by λ).

Therefore, we define the following sequence of random variables

$$X_n(x):=\#\{\langle p,q\rangle | 1\leq q\leq n, q\equiv s \bmod r, p/q \text{ is a solution of (4)}\}.$$

(here s, r denote fixed integers with $r \geq 1$) that was investigated by several authors. Especially in [13], LeVeque conjectured a central limit theorem for the above sequence of random variables (if properly normalized) and this conjecture was settled by the author in [3] by proving the following result (thereby notice, that a central limit theorem for the above sequence of random variables was already stated in [12] and [15] but both results turned out to be wrong)

Theorem 1 (see [3])**.** *Set*

$$F(n) := \sum_{k=1}^{n} \frac{f(\log k)}{k}.$$

Then, we have

$$\lim_{n \longrightarrow \infty} \lambda \left[X_n \leq \sigma_1 F(n) + x \left(\sigma_2 F(n) \log F(n)\right)^{1/2} \right] = \Phi(x)$$

where

$$\sigma_1 = \frac{2}{r}, \quad \sigma_2 = \frac{12(s,r)\varphi(r)}{\pi^2 r C(s,r)},$$

and

$$C(s,r) = r^2 \prod_{p|r} \left(1 - \frac{1}{p}\right) \prod_{p|\frac{r}{(s,r)}} \left(1 + \frac{1}{p}\right).$$

(Here and in the following, we use the standard notation Φ to denote the distribution function of the normal law.)

This result describes the statistical behavior of the solution set of (4) in distribution. In this paper, we are interested in the almost sure behavior of the solution set.

In this situation, Erdős obtained a strong law of large numbers (even for a much bigger class of functions than the one considered here but without the restriction that denominators have to be in a arithmetic progression; for details see [2]). This result of Erdős established on the one hand a conjecture of LeVeque [13] and improved on the other hand a famous result due to Khintchine [9]. A few years later Erdős's result was extended by Szüsz [20] (to the case of counting solutions with denominators contained in arithmetic progressions) and W. M. Schmidt [18] who provided a strong law of large numbers with a very sharp error term (even for the multidimensional case but like Erdős without the restriction of denominators in arithmetic progressions). Finally, a few years later, it was once more W. M. Schmidt [19] who improved upon the earlier results by establishing a very general theorem that especially combined the above two lines of developments started by Szüsz and by himself. We content ourself by stating the latter consequence of Schmidt's result (which in the following will be refered to as "Schmidt's theorem").

Corollary 1 (**see** [19])**.** *Let the notation be as in Theorem 1 but here we only assume that f is non-increasing. Then, we have a.s.*

$$X_n = \frac{2}{r}F(n) + \mathcal{O}\left(F(n)^{1/2+\epsilon}\right)$$

for all $\epsilon > 0$.

Remark 1. By further refining Schmidt's method, the above result was in turn extended by Harman [5] to inhomogeneous diophantine approximation with both variables restricted - one to a set with positive lower asymptotic density and the other to an arithmetic progression. While the result in [5] kept the same error term as Schmidt's theorem, in his book Harman gave a different proof leading to the following improvement (again, we just state the result for our particular situation; for the general result see [6])

$$X_n = \frac{2}{r}F(n) + \mathcal{O}\left(F(n)^{1/2}\left(\log F(n)\right)^{2+\epsilon}\right)$$

for all $\epsilon > 0$.

In the following, we will improve upon Schmidt's result (and Harman's result as well) for our more restricted class of functions (in fact, for a class that is a little bit smaller). In particular, we will derive a generalized law of the iterated logarithm for the sequence $(X_n)_{n\geq 1}$ and obtain a result that gives strong evidence that the law of the iterated logarithm with the standard norming sequences (suggested by Theorem 1) holds as well.

In the sequel consider positive real-valued functions f defined on the non-negative real numbers satisfying condition (1), (2), and the following one

$$\sum_{k=1}^{n} f(k)^2 \ll \left(\sum_{k=1}^{n} f(k)\right)^{1/4-\delta_2} \tag{5}$$

where $0 < \delta_2 < 1/4$.

Theorem 2. *Let the notation be as in Theorem 1. Then there exist an increasing sequence $(n_k)_{k\geq 0}$ of non-negative integers with $n_0 = 0$ and positive real constants C_1, C_2 such that, if we define*

$$c_n = \left(F(n_k)\log F(n_k)\log\log F(n_k)\right)^{1/2}$$

for $n_{k-1} < n \leq n_k$, we have a.s.

$$C_1 \leq \limsup_{n\longrightarrow\infty} \frac{|X_n - \sigma_1 F(n)|}{c_n} \leq C_2.$$

Furthermore, the estimate

$$c_n = \mathcal{O}(F(n)^{1/2+\epsilon}) \tag{6}$$

holds for all $\epsilon > 0$.

Remark 2. The above theorem implies the following strong law of large numbers with error term for the sequence X_n

$$X_n = \sigma_1 F(n) + \mathcal{O}(c_n) \quad \text{a.s.} \tag{7}$$

and the error term is essentially best possible.

Remark 3. By using (6), we deduce from (7) (with arbitrary $\epsilon > 0$)

$$X_n = \sigma_1 F(n) + \mathcal{O}(F(n)^{1/2+\epsilon}) \quad \text{a.s..} \tag{8}$$

In particular, we point out that an improvement of the error term in the above relation cannot be obtained from Theorem 2 since estimate (6) is essentially best possible.

Remark 4. The relations (7) and (8) have to be compared with the assertion of Corollary 1. Since the error term in relation (7) is essentially best possible, (7) gives an improvement of Schmidt's theorem. On the other hand relation (8) shows that the error term in relation (7) is not essentially smaller than the one of Corollary 1. Therefore, we will develop another method in order to sharpen relation (8). This approach will finally lead us to an error term that is better than the one in Schmidt's theorem (see Theorem 3 below).

Theorem 1 suggests that a similar result to Theorem 2 should hold with c_n replaced by

$$(F(n) \log F(n) \log\log F(n))^{1/2}$$

(this would be the corresponding law of the iterated logarithmus). Unfortunately, we are not able to prove such a result but the next theorem gives strong evidence for its correctness.

Theorem 3. *Let the notation be as in Theorem 1. Then, we have a.s.*

$$\limsup_{n\longrightarrow\infty} \frac{|X_n - \sigma_1 F(n)|}{(F(n)\log F(n))^{1/2}(\log\log F(n))^{\alpha}} \begin{cases} = 0 & \alpha > 1/2, \\ > 0 & \alpha = 1/2. \end{cases}$$

Remark 5. Notice that the latter result determines the lim sup for all norming sequences

$$(F(n)\log F(n))^{1/2} (\log\log F(n))^{\alpha}$$

with $\alpha \neq 1/2$. In the missing case $\alpha = 1/2$, by the remarks preceding the theorem, one would expect the lim sup to be finite.

The above result yields in the case $\alpha > 1/2$ to the following strong law of large numbers with error term for the sequence X_n

$$X_n = \sigma_1 F(n) + \mathcal{O}((F(n)\log F(n))^{1/2}(\log\log F(n))^{1/2+\epsilon}) \quad \text{a.s.} \tag{9}$$

for all $\epsilon > 0$. Comparing with the assertion in Schmidt's theorem shows, that this improves upon Schmidt's result for our more restricted class of functions.

On the other hand, the case $\alpha = 0$ of Theorem 3 gives the following immediate consequence that can be considered as an inverse to Schmidt's theorem.

Corollary 2. *Let the notation be as in Corollary 1. Furthermore, let g be a positive real-valued function defined on the non-negative integers such that*

$$g(n) = o((F(n)\log F(n)\log\log F(n))^{1/2})$$

Then the relation

$$X_n = \frac{2}{r}F(n) + \mathcal{O}(g(n)) \quad a.s.$$

cannot hold.

Remark 6. More general, our result entails an inverse theorem for all theorems that contain the above setting as a special case (like for the results in [5],[18], and [19]; for more examples see [6]).

Before giving a short plan of the paper some words should be said concerning comparison of the method introduced in this paper and the method used by W. M. Schmidt (since our results are closely connected to Schmidt's result). Corollary 1 was just a consequence of Schmidt's main result proved in [19]. In particular, the more general result includes the multidimensional case as well. It is well known, already since Khintchine, that the one dimensional case and the multidimensional case are completely different. In fact, Khintchine proved his famous 0-1 law first in the one dimensional case [9] and gave in a later paper [10] a second method of proof that could be extended to the multidimensional case. The reason for why the extension from the one to the multidimensional case is nontrivial lies in the fact that in the one dimensional case the theory of diophantine approximation is closely connected to the theory of continued fraction expansion. However, in the multidimensional case no proper continued fraction algorithm exists.

The method introduced in this paper can be seen as a further refinement of Khinthine's first method of proof of his famous 0-1 law whereas Schmidt's method is a further refinement of Khintchine's second method of proof. In particular, since our method will essentially rely on tools from the metric theory of continued fraction expansion, an extension of our results to higher dimension is not possible with the method introduced in this paper. In order to establish similar result for the multidimensional setting, the development of a completely new method would be necessary.

We conclude the introduction by giving a short plan of the paper: in the next section, we prove a preparatory result from probability theory, namely that in case of φ-mixing sequences of random variables asymptotic normality always implies a generalized law of the iterated logarithm. This result will be the first step in the proof of Theorem 2. As a second step, we have to take a closer look on the central limit theorem for the sequence $(X_n)_{n\geq 1}$, in particular, we need the central limit theorem with a suitable error term. Such an error term will be provided in Section 3 (and thus, Theorem 1 improved when (3) is replaced by (5))and then used in Section 4 in order to derive Theorem 2. The proof of Theorem 3, too, is based on the central limit theorem for $(X_n)_{n\geq 1}$ with a suitable error term and will finally be performed in Section 5.

2. On the relation between central limit behaviour and the generalized law of the iterated logarithm

This section will be used in order to prove a preparatory result from probability theory. We decided to state this result in a quite general form since it might be useful somewhere else, too.

First recall that a sequence of random variables $(X_n)_{n\geq 1}$ defined on some probability space $(\Omega, \mathcal{A}, P)$ is called φ-mixing if there exists a positive real-valued function φ defined on the positive integers with $\varphi \downarrow 0$ such that

$$|P(A \cap B) - P(A)P(B)| \leq \varphi(n)P(A) \tag{10}$$

for all $A \in \mathcal{F}_1^t, B \in \mathcal{F}_{t+n}^{\infty}$ and $t, n \geq 1$ (thereby $\mathcal{F}_a^b$ denotes the σ-algebra generated by $(X_n)_{a\leq n\leq b}$). We use this opportunity in order to define further mixing properties that will be needed subsequently. The sequence of random variables $(X_n)_{n\geq 1}$ is called $\star$-mixing if in the above definition (10) is replaced by

$$|P(A \cap B) - P(A)P(B)| \leq \varphi(n)P(A)P(B).$$

Furthermore, if we replace (10) by

$$|P(A \cap B) - P(A)P(B)| \leq \varphi(n),$$

then the sequence is called strong mixing. In case of a double sequence of random variables $(X_{k,n})_{k\leq n,1\leq n}$, we use the terminology φ-mixing resp. $\star$-mixing resp. strong mixing if the mixing condition holds for every row. Finally, if the mixing condition holds uniformly over all rows, then the notation of uniformly φ-mixing resp. uniformly $\star$-mixing resp. uniformly strong mixing is used.

In this section, we just use the notation of φ-mixing. Furthermore, we set

$$S_n = \sum_{k=1}^{n} X_k.$$

Theorem 4. *Let $(X_n)_{n\geq 1}$ be a φ-mixing sequence of random variables with $\varphi(1) < 1$ and $(b_n)_{n\geq 1}$ a sequence of positive real numbers satisfying*

$$b_n \uparrow \infty \quad \text{as} \quad n \longrightarrow \infty.$$

Furthermore assume that S_n/b_n satisfies the central limit theorem, i.e. we have, as $n \longrightarrow \infty$,

$$P\left[\frac{S_n}{b_n} \leq x\right] \longrightarrow \Phi(x). \tag{11}$$

Then there exists an increasing sequence $(n_k)_{k\geq 0}$ of non-negative integers with $n_0 = 0$ such that, if we define

$$c_n = b_{n_k}(2\log k)^{1/2}$$

for $n_{k-1} < n \leq n_k$, we have that the set of limit points of the sequence

$$\left(\frac{S_n}{c_n}\right)_{n\geq 1} \tag{12}$$

is a.s. equal to the interval $[-1,1]$.

Remark 7. Notice, that the sequence of random variables considered in Theorem 4 need not to be stationary. If on the other hand one would assume stationarity then the requirements of the theorem can be relaxed. This stems from the case of independent and identically distributed sequences of random variables where it is well known that the theorem already holds if (11) is just assumed to hold along a subsequence (although in this case the exact cluster set of the sequence (12) is not known; for details see [17]).

However, if (11) is just assumed to hold along a subsequence then the method of proof of Theorem 4 (combined with some new ideas) can be used in order to derive a generalized law of the iterated logarithm for S_n centered at $\operatorname{med}(S_n)$ (or even with slightly more general sequences of centerings). Such a result generalizes one half of a well known theorem of Kesten (see [8]). Details will appear elsewhere.

Remark 8. It is easy to see (just from the monotonicity of $\Phi(x)$) that (13) is equivalent to

$$\sup_{x\in\mathbb{R}}\left|P\left[\frac{S_n}{b_n}\le x\right]-\Phi(x)\right|\longrightarrow 0, \tag{13}$$

as $n\longrightarrow\infty$.

Remark 9. Assume in addition to the assumptions of Theorem 4 that we have, as $n\longrightarrow\infty$,

$$b_{n+1}/b_n\longrightarrow 1$$

and

$$\sup_{x\in\mathbb{R}}\left|P\left[\frac{S_n}{b_n}\le x\right]-\Phi(x)\right|=\mathcal{O}\left(\frac{1}{(\log b_n)^{1+\delta}}\right) \tag{14}$$

for some $\delta>0$. Then it easily follows from the proof of Theorem 4 that the normalizing sequence can be chosen as

$$c_n=b_n(2\log\log b_n)^{1/2}$$

i.e. the sequence of random variables $(X_n)_{n\ge1}$ satisfies the corresponding law of the iterated logarithm. This extends a result of Petrov [14] from the independent to the φ-mixing case.

For the proof of Theorem 4, we need the following extension of a lemma due to Petrov (see [14]) that is straightforward to prove (just by applying (10) everywhere where the independence assumption was used)

Lemma 1. *Let* $X_1,\dots,X_n$ *be* φ*-mixing random variables with* $\varphi(1)<c$ *(*$c>0$ *denotes a real constant). Assume that*

$$P\left[\sum_{i=k}^{n}X_i\ge -b\right]\ge c,\quad k=1,\dots,n$$

for some non-negative b. *Then, we have*

$$P[\max_{1\le k\le n}S_k\ge x]\le(c-\varphi(1))^{-1}P[S_n\ge x-b]$$

for all $x \in \mathbb{R}$.

Now, the proof of Theorem 4 follows quite easily by combining some ideas of Petrov [14] with ideas from Kuelbs and Zinn [11] (compare also with Berkes and Philipp [1]). We will provide details for the reader's convenience.

Proof of Theorem 4. First, denote by $\delta > 0$ a real constant satisfying $\varphi(1) < 1 - \delta$. Then, it follows from (13) that there exists a positive integer n_0 and a positive real constant c such that

$$P\left[S_n \geq -\frac{c}{2}b_n\right] \geq 1 - \frac{\delta}{2} \tag{15}$$

and

$$P\left[S_n \geq \frac{c}{2}b_n\right] \leq \frac{\delta}{2} \tag{16}$$

for all $n \geq n_0$. Furthermore, it is clear that it can be achieved that both relations (15) and (16) hold for all $n \geq 1$ (just by increasing c). Now, consider

$$\begin{aligned} P[S_n - S_k \geq -cb_n] &\geq P[S_n \geq -\frac{c}{2}b_n] + P[S_k \leq \frac{c}{2}b_n] - 1 \\ &\geq P[S_n \geq -\frac{c}{2}b_n] - P[S_k \geq \frac{c}{2}b_k] \end{aligned}$$

and by taking (15) and (16) into account, we get

$$P[S_n - S_k \geq -cb_n] \geq 1 - \delta \tag{17}$$

for all $n \geq 1$ and $k \leq n$.

Next, it follows from (13) that there exists an increasing sequence $(n_k)_{k \geq 1}$ of positive integers such that

$$\sup_{x \in \mathbb{R}} \left| P\left[\frac{S_{n_k}}{b_{n_k}} \leq x\right] - \Phi(x) \right| \leq \frac{1}{k^2} \tag{18}$$

for all $k \geq 1$. Furthermore it is plain that the sequence $(n_k)_{k \geq 1}$ can be chosen in such a fashion that

$$P(|S_{n_{k-1}}| \geq k^{-1}b_{n_k}) \leq k^{-2} \tag{19}$$

for all $k \geq 2$. Consequently, we get, as $k \longrightarrow \infty$,

$$\frac{|S_{n_{k-1}}|}{b_{n_k}} \longrightarrow 0 \tag{20}$$

by applying the Borel-Cantelli lemma.

Now, define for $n_{k-1} < n \leq n_k$

$$c_n = b_{n_k}(2 \log k)^{1/2}$$

(compare with the assertion of Theorem 4). An application of Lemma 1 gives

$$\begin{aligned} P\left[\max_{n_{k-1}<n\le n_k}\frac{S_n}{c_n}\ge 1+\epsilon\right] &\le P\left[\max_{n\le n_k} S_n \ge (1+\epsilon)b_{n_k}(2\log k)^{1/2}\right]\\ &\ll P\left[S_{n_k}\ge (1+\epsilon)b_{n_k}(2\log k)^{1/2}-cb_{n_k}\right]\\ &\ll P\left[S_{n_k}\ge (1+\bar\epsilon)b_{n_k}(2\log k)^{1/2}\right]\end{aligned}$$

for suitable $\bar\epsilon > 0$ and k large enough. Consequently, by using (18) and the well known asymptotic $1-\Phi(x)\sim(2\pi)^{-1/2}x^{-1}\exp(-x^2/2)$, we obtain

$$P\left[\max_{n_{k-1}<n\le n_k}\frac{S_n}{c_n}\ge 1+\epsilon\right]\ll\left(k^{-(1+\bar\epsilon)^2}\right)$$

and applying the Borel-Cantelli lemma yields

$$\limsup_{n\longrightarrow\infty}\frac{S_n}{c_n}\le 1 \quad \text{a.s.} \tag{21}$$

Furthermore, by using the same method, it is straightforward to obtain

$$\liminf_{n\longrightarrow\infty}\frac{S_n}{c_n}\ge -1 \quad \text{a.s.} \tag{22}$$

and hence by combining (21) and (22)

$$\limsup_{n\longrightarrow\infty}\left|\frac{S_n}{c_n}\right|\le 1 \quad \text{a.s.} \tag{23}$$

i.e. the set of limit points of the sequence S_n/c_n is a.s. contained in the interval $[-1,1]$.

Next consider for $0<a<b<1$ and k large enough

$$\begin{aligned} &P\Big[a-\frac{1}{k}\le\frac{S_{n_k}-S_{n_{k-1}}}{b_{n_k}(2\log k)^{1/2}}\le b+\frac{1}{k}\Big]\\ &\qquad\ge P\Big[a\le\frac{S_{n_k}}{b_{n_k}(2\log k)^{1/2}}\le b\Big]\cap\left[\frac{|S_{n_{k-1}}|}{b_{n_k}(2\log k)^{1/2}}\le\frac{1}{k}\right]\\ &\qquad\ge P\Big[a\le\frac{S_{n_k}}{b_{n_k}(2\log k)^{1/2}}\le b\Big]-P\left[\frac{|S_{n_{k-1}}|}{b_{n_k}(2\log k)^{1/2}}\ge\frac{1}{k}\right]\\ &\qquad\ge\Phi(b(2\log k)^{1/2})-\Phi(a(2\log k)^{1/2})-\frac{1}{k^2}\\ &\qquad\ge k^{-\bar a}\end{aligned}$$

with $a<\bar a<1$. Here, (19) and once more the well known asymptotic $1-\Phi(x)\sim(2\pi)^{-1}x^{-1}\exp(-x^2/2)$ was used. Applying the Borel-Cantelli lemma on the φ-mixing sequence $S_{n_k}-S_{n_{k-1}}$ gives that a.s.

$$\left(\frac{S_{n_k}-S_{n_{k-1}}}{b_{n_k}(2\log k)^{1/2}}\right)_{k\ge 2}$$

has at least one limit point in the interval $[a,b]$. Because of (20) the same is true for the sequence S_{n_k}/c_{n_k}. Since a similar argument gives the analogues result for intervals having the form $[a, b]$ with $-1 < a < b < 0$, we obtain that a.s. the set of limit points of the sequence S_{n_k}/c_{n_k} contains the interval $[-1, 1]$. Obviously, the same is true for S_n/c_n and together with (23), we are done. □

Remark 10. In order to proof the result posed in third remark succeeding Theorem 4 one has to choose an increasing sequence $(n_k)_{k\geq 1}$ in such a fashion that $b_{n_k} \sim \tau^k$ with suitable $\tau > 1$. Then, it easily follows that the left hand side of (18) is bounded by a function $f(k)$ that satisfies $\sum_{k=1}^{\infty} f(k) < \infty$ (this is thanks to (14)). The same holds for the left hand side of (19). By defining the normalizing sequence as

$$c_n = b_n(2 \log \log b_n)^{1/2}$$

and applying more or less the same arguments than in the proof of Theorem 4, it is straightforward to obtain the result.

3. The central limit theorem with error term for the sequence $(X_n)_{n\geq 1}$

In this section, we use the method introduced in [3] combined with some standard tools in order to sharpen Theorem 1. In details, we are going to obtain the following central limit theorem with error term

Theorem 5. *Let the notation be as in Theorem 1. Then, we have*

$$\sup_{x\in\mathbb{R}} \left| \lambda \left[\frac{X_n - \sigma_1 F(n)}{(\sigma_2 F(n) \log F(n))^{1/2}} \leq x \right] - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{(\log F(n))^{\tau}} \right)$$

with $0 < \tau < 1$.

Remark 11. The error term provided by the above theorem seems to be rather weak. But on the other hand taking a closer look on the proof method in [3] shows that X_n behaves somehow like a sum of random variables having infinite second moments. And in this situation it is due to Heyde [7] that in general the convergence to the normal law cannot be too fast.

In order to prove Theorem 5, we briefly recall the approximation process that was applied in [3]. By the law of the iterated logarithm for the denominators of the convergents in the continued fraction expansion (see [4]), we have that for each $\epsilon > 0$ there exist κ large enough and a subset F of $[0, 1]$ with $\lambda(F) \geq 1 - \epsilon$ such that

$$k \log \gamma - \kappa k^{1-\delta_1} \leq \log q_k \leq k \log \gamma + \kappa k^{1-\delta_1}, \quad k \geq 1$$

for all $x \in F$ (here γ denotes the Khintchine-Levy constant). Using this, we set

$$f_1(k) := f((k+1)\log \gamma + \kappa(k+1)^{1-\delta_1}), \quad f_2(k) := f(k \log \gamma - \kappa k^{1-\delta_1}),$$

and define sequences of random variables as

$$Z_k^{(j)}(x) := \#\{1 \leq c | cq_k \equiv s \mod r, c^2 \leq (a_{k+1} + 2\delta_{2,j}) f_j(k)\} \quad j = 1, 2,$$

where $\delta_{i,j}$ is the Kronecker function. From [3], we know that the above sequences can be used in order to approximate X_n. In details, we have

$$\sum_{q_{k+1}\le n} Z_k^{(1)}(x) \le X_n(x) \le \sum_{q_k\le n} Z_k^{(2)}(x)$$

for all $x \in F$.

Since the $Z_k^{(j)}$ have infinite second moments truncation was used in [3] in order to prove asymptotic normality of those sequences. Thereby, the sequences $(Z_k^{(j)})_{k\ge 0}$ where approximated by double sequences of random variables by truncating the first $n+1$ random variables at the level $[F(n)^{1/2}\log F(n)^{1/2-\rho}]$. Here, a slightly different truncation technique is used that can still be handled by the method introduced in [3]. Thus, we put

$$F_j(n) := \sum_{k=1}^{n} f_j(k), \quad j = 1, 2,$$

and define for $j = 1, 2$

$$Z_{k,n}^{(j)}(x) = \#\{1 \le c \le \phi_n^{(j)} | cq_k \equiv s \mod r, c^2 \le (a_{k+1} + 2\delta_{2,j}) f_j(k)\}$$

where $\phi_n^{(j)} = [(F_j(n)\log F_j(n))^{1/2}(\log\log F_j(n))^{-1-\rho}]$.

It is immediate that for this truncation Lemma 6 of [3] continues to hold.

Lemma 2. *For the random variables $Z_{k,n}^{(j)}$ introduced above, we have*

$$\mu_{j,n} := \mathbf{E}\sum_{k\le n} Z_{k,n}^{(j)} = \frac{\pi^2}{6r\log 2} F_j(n) + \mathcal{O}\left(F_j(n)^{1/2}\right), \tag{24}$$

and

$$\tau_{j,n}^2 := \mathbf{V}\sum_{k\le n} Z_{k,n}^{(j)} = \sigma F_j(n)\log F_j(n) + \mathcal{O}(F_j(n)\log\log F_j(n)) \tag{25}$$

with

$$\sigma = \frac{(s,r)\varphi(r)}{rC(s,r)\log 2}$$

where $C(s,r)$ is as in the introduction.

Furthermore if we use the normalization

$$\eta_{k,n}^{(j)} := (Z_{k,n}^{(j)} - \mathbf{E}Z_{k,n}^{(j)})/\tau_{j,n}$$

then the properties of Lemma 7 in [3] remain true. We only state the properties that we will use in the sequel.

Lemma 3. *The double sequence $\eta_{k,n}^{(j)}$ satisfies the following properties*

(1) $\eta_{k,n}^{(j)}$ *is uniformly strong mixing,*

(2)

$$\mathbf{V}\sum_{k\le n}\eta_{k,n}^{(j)} = 1,$$

(3)
$$\mathbf{V}\sum_{k\in I}\eta_{k,n}^{(j)}=\sum_{k\in I}\mathbf{V}\eta_{k,n}^{(j)}+\mathcal{O}\left(\frac{1}{(F_j(n))^{3/4+\delta_2}}\right),$$
where $I\subseteq\{0,1,\dots,n\}$ *and the implied constant doesn't depend on* I.

Remark 12. In [3], the above lemma was stated without exact error term on the right hand side of (3). However, the error terms posed in the above lemma is implicitly contained in the proof of the corresponding result in [3].

The next goal is the following central limit theorem with error term for the sequence $\eta_{k,n}^{(j)}$.

Lemma 4. *We have*
$$\sup_{x\in\mathbb{R}}\left|\lambda\left[\sum_{k\leq n}\eta_{k,n}^{(j)}\leq x\right]-\Phi(x)\right|=\mathcal{O}\left(\frac{1}{\log F_j(n)(\log\log F_j(n))^{1+\rho}}\right).$$

The proof will be performed in several steps and carried out by following the proof of the central limit theorem in [3] combined with some classical ideas in proving central limit theorems with error terms.

We start by introducing a suitable blocking. Therefore, fix n and define a sequence of integers by
$$m_{0,n}:=0,$$
and for $l=0,1,2,\dots$ by
$$m_{2l+1,n}:=\min\Big\{m>m_{2l}\Big|\sum_{k=m_{2l}+1}^{m}\mathbf{V}\eta_{k,n}^{(j)}\geq(F_j(n))^{-\alpha}\Big\},\tag{26}$$
$$m_{2l+2,n}:=m_{2l+1,n}+\Big[-\frac{c}{\log q}\log F_j(n)\Big].\tag{27}$$
where $3/4<\alpha<3/4+\delta_2$ and c are real constants to be spezified later. Thereby, two possibilities occur how this construction can stop: on the one hand if we are in step (27) and there are no random variables for (26) left or on the other hand if the sum of the variances of the remaining random variables is too small to be at least $(F_j(n))^{-\alpha}$. In the first case, we put $m_{2l+2}:=n$ and in the second case, we increase m_{2l} by the number of remaining random variables. Using the above sequence, we define
$$\begin{aligned}I_{l,n}&=\{k|m_{2l}<k\leq m_{2l+1}\},\\ J_{l,n}&=\{k|m_{2l+1}<k\leq m_{2l+2}\},\end{aligned}$$
and finally
$$\begin{aligned}\xi_{l,n}^{(j)}&=\sum_{k\in I_{l,n}}\eta_{k,n}^{(j)},\\ \zeta_{l,n}^{(j)}&=\sum_{k\in J_{l,n}}\eta_{k,n}^{(j)},\end{aligned}$$

where $0 \le l < w_n$ and w_n is the number of $I_{l,n}$ (resp. $J_{l,n}$) obtained from the construction above.

We gather some properties of this blocking

Lemma 5. *We have*

(1)
$$w_n \ll F_j(n)^\alpha$$

(2)
$$\sum_{l<w_n} \mathbf{V}\zeta_{l,n}^{(j)} \ll F_j(n)^{\alpha-3/4-\delta_2}$$

(3)
$$\sum_{l<w_n} \mathbf{V}\xi_{l,n}^{(j)} = 1 + \mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})$$

(4)
$$\sum_{l<w_n} \mathbf{E}|\xi_{l,n}^{(j)}|^3 \ll \frac{1}{\log F_j(n)(\log\log F_j(n))^{1+\rho}}$$

Proof. In order to prove (1), observe
$$w_n F_j(n)^{-\alpha} \le \sum_{l<w_n}\sum_{k\in I_{l,n}} \mathbf{V}\eta_{k,n}^{(j)} \le \sum_{k\le n} \mathbf{V}\eta_{k,n}^{(j)} \ll 1$$
where (2) and (3) of Lemma 3 were used.

For (2), we apply (3) of Lemma 3 and obtain
$$\begin{aligned}\sum_{l<w_n} \mathbf{V}\zeta_{l,n}^{(j)} &= \sum_{l<w_n}\sum_{k\in J_{l,n}} \mathbf{V}\eta_{k,n}^{(j)} + \mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})\\ &\ll w_n \log F_j(n) \max_{k\le n} \mathbf{V}\eta_{k,n}^{(j)} + F_j(n)^{-\alpha} + F_j(n)^{\alpha-3/4-\delta_2}.\end{aligned}$$

By the following estimate that can be proved by the same method as used for the variance in Lemma 6 of [3]
$$\mathbf{V}\eta_{k,n}^{(j)} = c\frac{f_j(k)}{F_j(n)} + \mathcal{O}\left(\frac{f_j(k)\log\log F_j(n)}{F_j(n)\log F_j(n)}\right) \ll \frac{1}{F_j(n)} \tag{28}$$
the desired result follows.

For (3), we apply once more (3) of Lemma 3 together with (2) of Lemma 3 and the estimate used above
$$\begin{aligned}\sum_{l<w_n} \mathbf{V}\xi_{l,n}^{(j)} &= \sum_{l<w_n}\sum_{k\in I_{l,n}} \mathbf{V}\eta_{k,n}^{(j)} + \mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})\\ &= \sum_{k\le n} \mathbf{V}\eta_{k,n}^{(j)} + \mathcal{O}\left(\sum_{l<w_n}\sum_{k\in J_{l,n}} \mathbf{V}\eta_{k,n}^{(j)}\right) + \mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})\\ &= 1 + \mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2}).\end{aligned}$$

Finally, in order to prove (4), we start by observing

$$F_j(n)^{-\alpha} \le \sum_{k\in I_{l,n}} \mathbf{V}\eta_{k,n}^{(j)} \le F_j(n)^{-\alpha} + \max_{k\le n} \mathbf{V}\eta_{k,n}^{(j)}$$

and therefore

$$\sum_{k\in I_{l,n}} \mathbf{V}\eta_{k,n}^{(j)} = F_j(n)^{-\alpha} + \mathcal{O}(F_j(n)^{-1}).$$

Furthermore, we get by using (28)

$$\sum_{k\in I_{l,n}} \mathbf{V}\eta_{k,n}^{(j)} = \sum_{k\in I_{l,n}} f_j(k)\left(\frac{c}{F_j(n)} + \mathcal{O}\left(\frac{\log\log F_j(n)}{F_j(n)\log F_j(n)}\right)\right)$$

and consequently

$$\begin{aligned}\sum_{k\in I_{l,n}} f_j(k) &= \frac{1}{c}F_j(n)\left(1+\mathcal{O}\left(\frac{\log\log F_j(n)}{\log F_j(n)}\right)\right)(F_j(n)^{-\alpha}+\mathcal{O}(F_j(n)^{-1}))\\ &\ll F_j(n)^{1-\alpha}.\end{aligned}$$

We finish the prove by applying the multinomial theorem on

$$\mathbf{E}\left(\sum_{k\in I_{l,n}} |\eta_{k,n}^{(j)}|\right)^3$$

and treating each sum separately. We start with the following one

$$\begin{aligned}\sum_{k_1,k_2\in I_{l,n},k_1<k_2} \mathbf{E}(\eta_{k_1,n}^{(j)})^2|\eta_{k_2,n}^{(j)}| &\ll \sum_{k_1,k_2\in I_{l,n},k_1<k_2} \mathbf{V}\eta_{k_1,n}^{(j)}\mathbf{E}|\eta_{k_2,n}^{(j)}|\\ &\ll \sum_{k\in I_{l,n}} \mathbf{V}\eta_{k,n}^{(j)} \sum_{k\in I_{l,n}} \mathbf{E}|\eta_{k,n}^{(j)}| \ll \frac{1}{\tau_{j,n}}F_j(n)^{-\alpha}\sum_{k\in I_{l,n}} \mathbf{E}Z_{k,n}^{(j)}\\ &\ll \frac{1}{\tau_{j,n}}F_j(n)^{-\alpha}\sum_{k\in I_{l,n}} f_j(k) \ll F_j(n)^{1/2-2\alpha}.\end{aligned}$$

where (1) of Lemma 3, (14) in [3], and the estimates above were used. The sum

$$\sum_{k_1,k_2\in I_{l,n},k_1<k_2} \mathbf{E}|\eta_{k_1,n}^{(j)}|(\eta_{k_2,n}^{(j)})^2$$

is treated in the same manner. Next, by using similar estimates then before, we obtain

$$\begin{aligned}\sum_{k_1,k_2,k_3\in I_{l,n},k_1<k_2<k_3}\mathbf{E}|\eta_{k_1}^{(j)}\eta_{k_2}^{(j)}\eta_{k_3}^{(j)}| &\ll \left(\sum_{k\in I_{l,n}}\mathbf{E}|\eta_{k,n}^{(j)}|\right)^3\\ &\ll \frac{1}{\tau_{j,n}^3}\left(\sum_{k\in I_{l,n}}\mathbf{E}Z_{k,n}^{(j)}\right)^3 \ll \frac{1}{\tau_{j,n}^3}\left(\sum_{k\in I_{l,n}}f_j(k)\right)^3\\ &\ll F_j(n)^{3/2-3\alpha}.\end{aligned}$$

Therefore, we are left with the sum of the third moments. Here, an easy but lengthy calculation gives (compare with the calculation of the variance in Lemma 6 of [3])

$$\mathbf{E}|\eta_{k,n}^{(j)}|^3 \ll \frac{f_j(k)}{F_j(n)\log F_j(n)(\log\log F_j(n))^{1+\rho}}.$$

Now, we can put all parts together and obtain

$$\begin{aligned}\sum_{l<w_n}\mathbf{E}|\xi_{l,n}^{(j)}|^3 &\ll \sum_{l<w_n}\sum_{k\in I_{l,n}}\mathbf{E}|\eta_{k,n}^{(j)}|^3 + w_nF_j(n)^{1/2-2\alpha} + w_nF_j(n)^{3/2-3\alpha}\\ &\ll \frac{1}{\log F_j(n)(\log\log F_j(n))^{1+\rho}}\end{aligned}$$

which finishes the proof of (4). □

Next, we are going to consider the sequence $\zeta_{l,n}^{(j)}$.

Lemma 6. *We have*

$$\lambda\Big[\Big|\sum_{l<w_n}\zeta_{l,n}^{(j)}\Big| \geq \epsilon\Big] \ll \frac{1}{F_j(n)^{\alpha-1}\log F_j(n)\epsilon^2}$$

for all $\epsilon > 0$.

Proof. An application of (3) of Lemma 3 together with estimates used in the proof of the last lemma yields

$$\begin{aligned}\mathbf{V}\sum_{l<w_n}\zeta_{l,n}^{(j)} &= \sum_{l<w_n}\sum_{k\in J_{l,n}}\mathbf{V}\eta_{k,n}^{(j)} + \mathcal{O}(F_j(n)^{-3/4-\delta_2})\\ &\ll F_j(n)^{\alpha-1}\log F_j(n) + F_j(n)^{-\alpha} + F_j(n)^{-3/4-\delta_2}\\ &\ll F_j(n)^{\alpha-1}\log F_j(n).\end{aligned}$$

From that the assertion follows by using Chebyshev's inequality. □

Now, define

$$g_n^3 := \sum_{l<w_n}\mathbf{E}|\xi_{l,n}^{(j)}|^3.$$

Using this notation, we have the following result for the sequence $\xi_{l,n}^{(i)}$ that is proved by classical arguments (for instance see Petrov [14]).

Lemma 7. *For large enough n and $0 \leq 2t \leq g_n^{-3}$, we have*

$$\prod_{l<w_n} \mathbf{E}\exp\{it\xi_{l,n}^{(j)}\} - e^{-t^2/2} \ll t^2(tg_n^3 + \mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2}))e^{-t^2/8}.$$

Proof. First consider the case $tg_n \geq 1$. Observe

$$t^2 \geq g_n^{-2} \gg (\log F_j(n))^{2/3}.$$

and consequently

$$t^2(tg_n^3 + \mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})) \geq c > 0$$

for some constant c and n large enough. Therefore, it suffices to prove that

$$\left|\prod_{l<w_n} \mathbf{E}\exp\{it\xi_{l,n}^{(j)}\}\right| \leq e^{-t^2/8}. \tag{29}$$

In order to to this, consider

$$|\mathbf{E}\exp\{it\xi_{l,n}^{(j)}\}| \leq 1 - \frac{t^2}{2}\mathbf{V}\xi_{l,n}^{(j)} + \frac{t^3}{6}\mathbf{E}|\xi_{l,n}^{(j)}|^3$$

and thus, by using the elementary inequality $1 + x \leq e^x$ and (3) of Lemma 5

$$\left|\prod_{l<w_n} \mathbf{E}\exp\{it\xi_{l,n}^{(j)}\}\right| \leq \exp\Big\{ -\frac{t^2}{2}(1 + \mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})) + \frac{t^3}{6}g_n^3\Big\}$$

which entails (29) for large enough n.

Now, we turn to the case $tg_n < 1$. Here, we consider

$$\mathbf{E}\exp\{it\xi_{l,n}^{(j)}\} = 1 - \frac{t^2}{2}\mathbf{V}\xi_{l,n}^{(j)} + \frac{t^3}{6}\theta_1\mathbf{E}|\xi_{l,n}^{(j)}|^3 =: 1 - r_{l,n}^{(j)} \tag{30}$$

where θ_1 is a suitable constant with $|\theta_1| \leq 1$. Since we have

$$|r_{l,n}^{(j)}| \leq \frac{1}{2}t^2\mathbf{V}\xi_{l,n}^{(j)} + \frac{1}{6}t^3\mathbf{E}|\xi_{l,n}^{(j)}|^3 \leq \frac{2}{3},$$

(by using the estimate $\mathbf{V}\xi_{l,n}^{(j)} \leq g_n^2$) we can apply a suitable branch of the log on (30) and thus

$$\log \mathbf{E}\exp\{it\xi_{l,n}^{j}\} = -r_{l,n}^{(j)} + \theta_2(r_{l,n}^{(j)})^2$$

with a suitable constant θ_2 satisfying $|\theta_2| \leq 1$. By a straightforward calculation, we get

$$\log \mathbf{E}\exp\{it\xi_{l,n}^{j}\} = -\frac{t^2}{2}\mathbf{V}\xi_{l,n}^{(j)} + \theta_3\frac{11}{18}t^3\mathbf{E}|\xi_{l,n}^{(j)}|^3$$

with a suitable constant θ_3 satisfying $|\theta_3| \le 1$. Summing up and using (3) of Lemma 5 yields

$$\log \prod_{l<w_n} \mathbf{E}\exp\{it\xi_{l,n}^{(j)}\} = -\frac{t^2}{2}(1+\mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})) + \theta_3 \frac{11}{18} t^3 g_n^3.$$

Now, the result follows immediately from the fact $e^a = 1 + ae^{\theta_4 a}$ (with θ_4 a suitable constant satisfying $|\theta_4| \le 1$) together with some computation. □

The next lemma is taken from [16].

Lemma 8. *We have for* $0 \le t \le 1$

$$\prod_{l<w_n} \mathbf{E}\exp\{it\xi_{l,n}^{(j)}\} - \mathbf{E}\exp\left\{it\sum_{l<w_n}\xi_{l,n}^{(j)}\right\} \ll t^2.$$

Proof. Since the proof is nearly trivial, we are going to repeat it. Just observe

$$\begin{aligned}\prod_{l<w_n}\left(1-\frac{t^2}{2}\theta_l \mathbf{V}\xi_{l,n}^{(j)}\right) &= 1-\frac{t^2}{2}\sum_{l<w_n}\theta_l\mathbf{V}\xi_{l,n}^{(j)}\prod_{l<k<w_n}(1-\frac{t^2}{2}\theta_k\mathbf{V}\xi_{k,n}^{(j)})\\ &= 1-\frac{t^2}{2}\theta(1+\mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2}))\end{aligned}$$

for suitable constants $\theta_1, \dots, \theta_{w_n-1}, \theta$. Hence

$$\begin{aligned}&\prod_{l<w_n} \mathbf{E}\exp\{it\xi_{l,n}^{(j)}\} - \mathbf{E}\exp\left\{it\sum_{l<w_n}\xi_{l,n}^{(j)}\right\}\\ &\qquad \ll \frac{t^2}{2}(1+\mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})) + \frac{t^2}{2}(1+\mathcal{O}(F_j(n)^{\alpha-3/4-\delta-2}))\\ &\qquad \ll t^2\end{aligned}$$

as it was claimed. □

Now, we can put all parts together in order to prove Lemma 4. Thereby, we use some ideas from [16] together with standard techniques (see for instance [14]).

Proof of Lemma 4. In order to apply the so called basic inequality, we have to estimate

$$\int_0^T \left|\mathbf{E}\exp\left\{it\sum_{l<w_n}\xi_{l,n}^{(j)}\right\} - e^{t^2/2}\right| t^{-1}dt$$

where we set $T=\min\{F_j(n),2^{-1}g_n^{-3}\}$. Therefore, consider

$$\begin{aligned}\int_0^T &\left|\mathbf{E}\exp\left\{it\sum_{l<w_n}\xi_{l,n}^{(j)}\right\}-e^{t^2/2}\right|t^{-1}dt\\ &\le\int_0^T\left|\mathbf{E}\exp\left\{it\sum_{l<w_n}\xi_{l,n}^{(j)}\right\}-\prod_{l<w_n}\mathbf{E}\exp\{it\xi_{l,n}^{(j)}\}\right|t^{-1}dt\\ &\quad+\int_0^T\left|\prod_{l<w_n}\mathbf{E}\exp\left\{it\xi_{l,n}^{(j)}\right\}-e^{-t^2/2}\right|t^{-1}dt\\ &=: J_1+I_1\end{aligned}$$

and we break J_1 into three parts

$$J_1=\int_0^{F_j(n)^{-1}}+\int_{F_j(n)^{-1}}^1+\int_1^T=:I_2+I_3+I_4.$$

We are going to estimate each part separately.

For I_1, we apply Lemma 7 and obtain

$$\begin{aligned}I_1&\ll g_n^3\int_0^\infty t^2e^{-t^2/8}dt+\mathcal{O}(F_j(n)^{\alpha-3/4-\delta_2})\int_0^\infty te^{-t^2/8}dt\\ &\ll\frac{1}{\log F_j(n)(\log\log F_j(n))^{1+2\rho}}\end{aligned}$$

where in the last step (4) of Lemma 5 was used.

For I_2, an application of Lemma 8 gives

$$I_2\ll\int_0^{F_j(n)^{-1}}tdt\ll F_j(n)^{-2}.$$

For I_3 and I_4, we start by observing that by using the mixing proberty of $\eta_{k,n}^{(j)}$ (see (1) of Lemma 3), we obtain

$$\mathbf{E}\exp\left\{it\sum_{l<w_n}\xi_{l,n}^{(j)}\right\}-\prod_{l<w_n}\mathbf{E}\exp\{it\xi_{l,n}^{(j)}\}\ll w_nF_j(n)^{-c}\ll F_j(n)^{\alpha-c}$$

where (1) of Lemma 5 and the definition of the blocking were used. Hence

$$\begin{aligned}I_3&\ll F_j(n)^{\alpha-c}\log F_j(n)\\ I_4&\ll F_j(n)^{\alpha-c}\log T\ll F_j(n)^{\alpha-c}\log F_j(n).\end{aligned}$$

By setting $c=2\alpha$ and combining all estimates, we finally obtain

$$\int_0^T\left|\mathbf{E}\exp\left\{it\sum_{l<w_n}\xi_{l,n}^{(j)}\right\}-e^{t^2/2}\right|\ll\frac{1}{\log F_j(n)(\log\log F_j(n))^{1+2\rho}}$$

and consequently, by the basic inequality

$$\sup_{x\in R}\left|\lambda\left[\sum_{l<w_n}\xi^{(j)}_{l,n}\le x\right]-\Phi(x)\right|=\mathcal{O}\left(\frac{1}{\log F_j(n)(\log\log F_j(n))^{1+2\rho}}\right).$$

Next, consider

$$\lambda\left[\sum_{k\le n}\eta^{(j)}_{k,n}\le x\right]\le\lambda\left[\sum_{l<w_n}\xi^{(j)}_{l,n}\le x+\epsilon\right]+\lambda\left[\left|\sum_{l<w_n}\zeta^{(j)}_{l,n}\right|\ge\epsilon\right]$$

and by using Lemma 6, the easy fact

$$\Phi(x+\epsilon)-\Phi(x)\ll\epsilon, \tag{31}$$

and choosing $\epsilon=F_j(n)^{-(\alpha-1)/4}$, we obtain

$$\lambda\left[\sum_{k\le n}\eta^{(j)}_{k,n}\le x\right]-\Phi(x)\le C_1\frac{1}{\log F_j(n)(\log\log F_j(n))^{1+2\rho}}$$

for some constant $C_1>0$ which is not depending on x. Similarly, it is proved that

$$\lambda\left[\sum_{k\le n}\eta^{(j)}_{k,n}\le x\right]-\Phi(x)\ge -C_2\frac{1}{\log F_j(n)(\log\log F_j(n))^{1+2\rho}}$$

for some constant $C_2>0$ which is not depending on x either. Combining the last two inequalities finishes the proof of Lemma 4. □

We finish the section by showing that Lemma 4 entails Theorem 5. Therefore, observe that by the method used in section 4 of [3] the following result can be easily obtained

Lemma 9. *We have*

$$\lambda\left[\left|\Big(\sum_{k\le n}Z^{(j)}_k-\mathbf{E}Z^{(j)}_{k,n}\Big)/\tau_{j,n}-\sum_{k\le n}\eta^{(j)}_{k,n}\right|\ge\epsilon\right]\ll\frac{(\log\log F_j(n))^{1+\rho}}{\epsilon\log F_j(n)}. \tag{32}$$

Furthermore, we will use the following result that is contained in [3]

Lemma 10. *Let g_1 (resp. g_2) be the inverse function of $\gamma^{k+1}\exp(\kappa(k+1)^{1-\delta})$ (resp. $\gamma^k\exp(-\kappa k^{1-\delta})$). Then, we have*

$$F_j(g_j(n))=\frac{1}{\log\gamma}F(n)+\mathcal{O}(F(n)^{1/2}).$$

Now, we are ready to prove Theorem 5.

Proof of Theorem 5. By using the same arguments that were applied at the end of the proof of Lemma 4 together with Lemma 4 and (32), we obtain

$$\sup_{x\in R}\left|\lambda\left[\Big(\sum_{k\le n}Z^{(j)}_k-\mathbf{E}Z^{(j)}_{k,n}\Big)/\tau_{j,n}\le x\right]-\Phi(x)\right|=\mathcal{O}\left(\frac{1}{(\log F_j(n))^{\eta}}\right) \tag{33}$$

for some constant $0 < \eta < 1$ (thereby, we have chosen $\epsilon = (\log F_j(n))^{-1/2}$). Next observe

$$\frac{\tau_{j,n}^2}{\sigma F_j(n) \log F_j(n)} = 1 + \mathcal{O}\left(\frac{\log\log F_j(n)}{\log F_j(n)}\right)$$

and together with the easy fact

$$\Phi(\epsilon x) - \Phi(x) \ll \begin{cases} \epsilon - 1 & \text{if } \epsilon \geq 1 \\ \frac{1}{\epsilon} - 1 & \text{if } 0 < \epsilon < 1, \end{cases} \tag{34}$$

we can replace (33) by

$$\sup_{x \in R} \left| \lambda \left[\frac{\sum_{k \leq n} Z_k^{(j)} - \mathbf{E} Z_{k,n}^{(j)}}{(\sigma F_j(n) \log F_j(n))^{1/2}} \leq x \right] - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{(\log F_j(n))^\eta} \right).$$

By using (28) and (31), this can in turn be replaced by

$$\sup_{x \in R} \left| \lambda \left[\frac{\sum_{k \leq n} Z_k^{(j)} - (\pi^2/(6r \log 2)) F_j(n)}{(\sigma F_j(n) \log F_j(n))^{1/2}} \leq x \right] - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{(\log F_j(n))^\eta} \right).$$

Now, by using Lemma 10 and once more (31) and (34), the above relation entails

$$\sup_{x \in R} \left| \lambda \left[\frac{\sum_{k \leq g_j(n)} Z_k^{(j)} - \sigma_1 F(n)}{(\sigma_2 F(n) \log F(n))^{1/2}} \leq x \right] - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{(\log F(n))^\eta} \right)$$

and the rest of the proof is performed as in section 4 in [3] by applying standard arguments. □

4. Proof of Theorem 2

For the proof of Theorem 2, we use the method and notation introduced in the last section. First notice that the following estimate is contained in the proof of Theorem 5

$$\sup_{x \in \mathbb{R}} \left| \lambda \left[\frac{\sum_{k \leq g_j(n)} Z_k^{(j)} - (\pi^2/(6r \log 2)) F_j(g_j(n))}{(\sigma_2 F(n) \log F(n))^{1/2}} \leq x \right] - \Phi(x) \right|$$
$$= \mathcal{O}\left(\frac{1}{(\log F(n))^\eta} \right).$$

Furthermore, a theorem of Szüsz (Satz 3.1 in [20]) immediatly implies that the sequence $Z_k^{(j)}$ is φ-mixing with an exponential mixing rate (actually, the cited result implies that the sequence is even $\star$-mixing). Unfortunately, we are not able to ensure that $\varphi(1) < 1$ and therefore, we cannot apply Theorem 4 directly. But since the assumption $\varphi(1) < 1$ is only used in one part of the proof of Theorem 4, the other part is still valid in our situation. Therefore, define $n_0 := 0$ and n_k as the largest integer that satisfies

$$\log F(n) \leq k^{2/\eta}$$

which immediately gives

$$\log F(n_k) = k^{2/\eta} + \mathcal{O}(1). \tag{35}$$

Then, by defining a sequence of positive integers $(d_n)_{n\geq 1}$ as

$$d_n = (\sigma_2 \eta F(n_k) \log F(n_k) \log\log F(n_k))^{1/2}$$

for $n_{k-1} < n \leq n_k$, the above mentioned second part of the proof of Theorem 4 implies a.s.

$$\limsup_{n\longrightarrow\infty} \frac{|\sum_{k\leq g_j(n)} Z_k^{(j)} - (\pi^2/(6r\log 2))F_j(g_j(n))|}{d_n} \geq 1.$$

By taking Lemma 10 into account this can be replaced by

$$\limsup_{n\longrightarrow\infty} \frac{|\sum_{k\leq g_j(n)} Z_k^{(j)} - \sigma_1 F(n)|}{d_n} \geq 1$$

and the method used in section 4 in [3] entails the corresponding result for the sequence $(X_n)_{n\geq 1}$. Therefore, we have proved one half of Theorem 2.

In order to prove the other half of Theorem 2, we start by choosing an integer l in such a fashion that if φ is the function involved in the mixing condition of $Z_k^{(j)}$, we have $\varphi(l) < 1$. Next, we point out that the method of proof introduced in the last section can be used as well to derive the following estimate

$$\sup_{x\in\mathbb{R}} \left| \lambda\left[\frac{\sum_{i+lk\leq n} Z_k^{(j)} - (\pi^2/(6r\log 2))F_j^*(n)}{(\sigma F_j^*(n)\log F_j^*(n))^{1/2}} \leq x \right] - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{(\log F_j^*(n))^{\eta}} \right)$$

where $F_j^*(n) = \sum_{i+lk\leq n} f_j(k)$ and $0 \leq i < l$. We make an easy observation

Lemma 11. *We have*

$$F_j^*(n) = \frac{1}{l} F_j(n) + \mathcal{O}(1).$$

Proof. Observe

$$F_j^*(n) = \frac{1}{l} \sum_{i+lk\leq n} l f_j(k) = \frac{1}{l}\int_0^n f_j(x)dx + \mathcal{O}(1) = \frac{1}{l}F_j(n) + \mathcal{O}(1).$$

□

By using this lemma together with Lemma 10, we can replace the above estimate by

$$\sup_{x\in\mathbb{R}}\left|\lambda\left[\frac{\sum_{i+lk\le g_j(n)} Z_k^{(j)}-(\pi^2/(6r\log 2))F_j^*(g_j(n))}{((\sigma_2/l)F(n)\log F(n))^{1/2}}\le x\right]-\Phi(x)\right| = \mathcal{O}\left(\frac{1}{(\log F(n))^\eta}\right).$$

Due to the choice of l, we can apply Theorem 4. Consequently with $(d_n)_{n\ge 1}$ defined as before, we have a.s.

$$\limsup_{n\longrightarrow\infty}\frac{|\sum_{i+lk\le g_j(n)} Z_k^{(j)}-(\pi^2/(6r\log 2))F_j^*(g_j(n))|}{d_n}=l^{1/2}.$$

By using

$$\begin{aligned}\Big|\sum_{k\le g_j(n)} Z_k^{(j)}-(\pi^2/(6r\log 2))F_j(g_j(n))\Big| &\le \Big|\sum_{lk\le g_j(n)} Z_k^{(j)}-(\pi^2/(6r\log 2))\sum_{lk\le g_j(n)} f_j(k)\Big| \\ &\quad+\Big|\sum_{1+lk\le g_j(n)} Z_k^{(j)}-(\pi^2/(6r\log 2))\sum_{1+lk\le g_j(n)} f_j(k)\Big|\dots \\ &\quad+\Big|\sum_{l-1+lk\le g_j(n)} Z_k^{(j)}-(\pi^2/(6r\log 2))\sum_{l-1+lk\le g_j(n)} f_j(k)\Big|,\end{aligned}$$

we get

$$\limsup_{n\longrightarrow\infty}\frac{|\sum_{k\le g_j(n)} Z_k^{(j)}-(\pi^2/(6r\log 2))F(g_j(n))|}{d_n}\le l^{3/2}\quad\text{a.s.}$$

and hence by using Lemma 10 and the definition of d_n

$$\limsup_{n\longrightarrow\infty}\frac{|\sum_{k\le g_j(n)} Z_k^{(j)}-\sigma_1 F(n)|}{d_n}\le l^{3/2}\quad\text{a.s.}$$

This gives (by using once more the method introduced in section 4 in [3]) the second part of Theorem 2.

Finally, in order to prove (6), just observe that (35) implies

$$(F(n_k)\log F(n_k)\log\log F(n_k))^{1/2}\ll F(n_{k-1})^{1/2+\epsilon}$$

for all $\epsilon>0$. Consequently, for $n_{k-1}<n\le n_k$

$$c_n\ll F(n_{k-1})^{1/2+\epsilon}\ll F(n)^{1/2+\epsilon}$$

which is the desired result.

5. Proof of Theorem 3

Throughout this section, we use the notation introduced in section 3 and section 4. We can concentrate in the sequel on the case $\alpha > 1/2$ since in case $\alpha = 1/2$ the assertion of Theorem 3 immediately follows from Theorem 2.

We will again use the approximation method introduced in section 3. Furthermore, we define

$$\begin{aligned}\phi_{n,1}^{(j)} &= [(F_j(n)\log F_j(n))^{1/2}(\log\log F_j(n))^{-1-\rho_1}],\\ \phi_{n,2}^{(j)} &= [(F_j(n)\log F_j(n))^{1/2}(\log\log F_j(n))^{1+\rho_2}]\end{aligned}$$

for real constants $\rho_1, \rho_2 > 0$ and

$$\begin{aligned}U_{k,n}^{(j)}(x) &:= \#\{1 \le c \le \phi_{n,1}^{(j)} | cq_k \equiv s\ (r), c^2 \le (a_{k+1} + 2\delta_{2,j})f_j(n)\},\\ V_{k,n}^{(j)}(x) &:= \#\{\phi_{n,1}^{(j)} < c \le \phi_{n,2}^{(j)} | cq_k \equiv s\ (r), c^2 \le (a_{k+1} + 2\delta_{2,j})f_j(n)\}.\end{aligned}$$

We start by considering the double sequence $U_{k,n}^{(j)}$. Therefore, we choose, as in the second part of the proof of Theorem 2, an integer l such that if φ is the function involved in the mixing condition, we have $\varphi(l) < 1/2$. Then, by a similar reasoning than in section 4, we obtain the following result that is proved by the method of section 3.

Lemma 12. *We have*

$$\left|\lambda\left[\frac{\sum_{i+lk\le n} U_{k,n}^{(j)} - \mathbf{E}U_{k,n}^{(j)}}{\tau_{j,n}^*} \le x\right] - \Phi(x)\right| = \mathcal{O}\left(\frac{1}{\log F_j(n)(\log\log F_j(n))^{1+\rho_1}}\right)$$

where $(\tau_{j,n}^*)^2 := \sum_{i+lk\le n} \mathbf{V}U_{k,n}^{(j)}$ *and* $0 \le i < l$.

Furthermore, we need the following result that is contained in [15].

Lemma 13 (Lemma 1.3.3 in [15]**).** *Let $X_1, \cdots, X_n$ be φ-mixing random variables centered at expectation with a mixing rate satisfying $\varphi(1) < 1/2$. Then, we have*

$$P[\max_{1\le k\le n} |S_k| \ge x] \le (1/2 - \varphi(1))^{-1} P[|S_n| \ge x - \sqrt{2}s_n]$$

where $s_n^2 = \max_{1\le k\le n} \mathbf{V}S_k$.

In the following, we define the integer n_p as the largest integer such that

$$\log F_j(n) \le p$$

which immediately entails

$$\log F_j(n_p) = p + \mathcal{O}(1).$$

Using this and the last two lemmas, we get for the sequence $U_{k,n}^{(j)}$.

Lemma 14. *We have a.s.*

$$\sum_{k\le n} U_{k,n_p}^{(j)} = \frac{\pi^2}{6r\log 2}F_j(n) + \mathcal{O}((F_j(n)\log F_j(n)\log\log F_j(n))^{1/2})$$

for $n_{p-1} < n \le n_p$.

Proof. Define

$$\chi_j(n_{p-1}) = (F_j(n_{p-1})\log F_j(n_{p-1})\log\log F_j(n_{p-1}))^{1/2}$$

and observe

$$\begin{aligned}
&\lambda\left[\max_{n_{p-1}<n\le n_p}\Big|\sum_{k\le n} U_{k,n_p}^{(j)} - \mathbf{E}U_{k,n_p}^{(j)}\Big| \ge c\chi_j(n_{p-1})\right]\\
&\le \sum_{0\le i<l}\lambda\left[\max_{n_{p-1}<n\le n_p}\Big|\sum_{i+lk\le n} U_{k,n_p}^{(j)} - \mathbf{E}U_{k,n_p}^{(j)}\Big| \ge (c/l)\chi_j(n_{p-1})\right]\\
&\ll \sum_{0\le i<l}\lambda\Bigg[\Big|\sum_{i+lk\le n_p} U_{k,n_p}^{(j)} - \mathbf{E}U_{k,n_p}^{(j)}\Big| \ge (c/l)\chi_j(n_{p-1})\\
&\qquad\qquad - \mathcal{O}((F_j(n_p)\log F_j(n_p))^{1/2})\Bigg]\\
&\ll \Big(1-\Phi\Big((\bar{c}/\tau_{j,n_p}^{*})\chi_j(n_{p-1})\Big)\Big) + \frac{1}{p(\log p)^{1+\rho_1}}.
\end{aligned}$$

where in the third step Lemma 13 was used (which is possible thanks to the definition of l; thereby s_n of Lemma 13 is computed by applying the method that was used in order to compute (25)) and in the last step Lemma 12 was applied. Since an easy computation gives

$$(\tau_{j,n}^{*})^2 \sim (\sigma/l)F_j(n)\log F_j(n)$$

(this has to be compared with (25)) and by the well known asymptotic

$$1-\Phi(x) \sim (2\pi)^{-1/2}x^{-1}\exp(-x^2/2),$$

we finally arrive at

$$\lambda\left[\max_{n_{p-1}<n\le n_p}\Big|\sum_{k\le n} U_{k,n_p}^{(j)} - \mathbf{E}U_{k,n_p}^{(j)}\Big| \ge c\chi_j(n_{p-1})\right] \ll \frac{1}{p(\log p)^{1+\rho_1}}$$

if c is chosen suitable. Applying the Borel-Cantelli lemma and computing $\sum_{k\le n}\mathbf{E}U_{k,n_p}^{(j)}$ (which is done in the same manner than (24)) gives the desired result. □

Next, we turn to the sequence $V_{k,n}^{(j)}$ for which we have the following result.

Lemma 15. *We have a.s.*

$$\sum_{k\le n} V_{k,n_p}^{(j)} = \mathcal{O}((F_j(n)\log F_j(n))^{1/2}(\log\log F_j(n))^{1/2+\epsilon})$$

for $n_{p-1} < n \le n_p$ *and all* $\epsilon > 0$.

Proof. First notice that an easy but lengthy computation gives

$$\mathbf{E}(V_{k,n}^{(j)})^2 \ll f_j(k) \log\log\log F_j(n)$$

(this has to be compared with the proof method of (25)). Furthermore, by taking the mixing property of $V_{k,n}$ into account, it is straightforward to obtain

$$\mathbf{V} \sum_{k\le n} V_{k,n}^{(j)} \ll F_j(n) \log\log\log F_j(n)$$

(again compare with the proof method of (25)). Now, observe

$$\lambda\left[\Big| \sum_{k\le n_p} V_{k,n_p}^{(j)}\Big| \ge (F_j(n_{p-1}) \log F_j(n_{p-1}))^{1/2} (\log\log F_j(n_{p-1}))^{1/2+\epsilon}\right]$$
$$\ll \frac{F_j(n_p) \log\log\log F_j(n_p)}{F_j(n_{p-1}) \log F_j(n_{p-1}) (\log\log F_j(n_{p-1}))^{1+2\epsilon}} \ll \frac{1}{p(\log p)^{1+\bar{\epsilon}}}$$

and using the Borel-Cantelli lemma implies a.s.

$$\sum_{k\le n_p} V_{k,n_p}^{(j)} \ll (F_j(n_{p-1}) \log F_j(n_{p-1}))^{1/2} (\log\log F_j(n_{p-1}))^{1/2+\epsilon}.$$

Therefore, for $n_{p-1} < n \le n_p$, we have a.s.

$$\begin{aligned}\sum_{k\le n} V_{k,n_p}^{(j)} &\le \sum_{k\le n_p} V_{k,n_p}^{(j)} \\ &\ll (F_j(n_{p-1}) \log F_j(n_{p-1}))^{1/2} (\log\log F_j(n_{p-1}))^{1/2+\epsilon} \\ &\ll (F_j(n) \log F_j(n))^{1/2} (\log\log F_j(n))^{1/2+\epsilon}\end{aligned}$$

which finishes the proof of the result. □

Finally, we need the following result.

Lemma 16. *We have a.s.*

$$\sum_{k\le n} Z_k^{(j)} = \sum_{k\le n} (U_{k,n_p}^{(j)} + V_{k,n_p}^{(j)}) + \mathcal{O}((F_j(n) \log F_j(n))^{1/2})$$

for $n_{p-1} < n \le n_p$.

Proof. It is straightforward to obtain

$$\mathbf{E}\Big| \sum_{k\le n_p} Z_k^{(j)} - \sum_{k\le n_p} (U_{k,n_p}^{(j)} + V_{k,n_p}^{(j)})\Big| \ll \frac{F_j(n_p)^{1/2}}{(\log F_j(n_p))^{1/2} (\log\log F_j(n_p))^{1+\rho_2}}$$

(this has to be compared with the proof of Lemma 9 in section 3). Consequently

$$\lambda\left[\Big|\sum_{k\le n_p} Z_k^{(j)} - \sum_{k\le n_p}(U_{k,n_p}^{(j)}+V_{k,n_p}^{(j)})\Big| \ge (F_j(n_{p-1})\log F_j(n_{p-1}))^{1/2}\right]$$
$$\ll \frac{1}{p(\log p)^{1+\rho_2}}$$

and applying the Borel-Cantelli lemma gives a.s.

$$\sum_{k\le n_p} Z_k^{(j)} - \sum_{k\le n_p}(U_{k,n_p}^{(j)}+V_{k,n_p}^{(j)}) \ll (F_j(n_{p-1})\log F_j(n_{p-1}))^{1/2}.$$

This immediately entails the desired result. □

Now, by combining Lemma 14, Lemma 15, and Lemma 16, we obtain a.s.

$$\limsup_{n\longrightarrow\infty} \frac{|\sum_{k\le n} Z_k^{(j)} - (\pi^2/(6r\log 2))F_j(n)|}{(F_j(n)\log F_j(n))^{1/2}(\log\log F_j(n))^{\alpha}} = 0$$

and applying Lemma 10 entails a.s.

$$\limsup_{n\longrightarrow\infty} \frac{|\sum_{k\le g_j(n)} Z_k^{(j)} - \sigma_1 F(n)}{(F(n)\log F(n))^{1/2}(\log\log F(n))^{\alpha}} = 0.$$

From this the result is easily obtained by using once more the method introduced in section 4 in [3].

Acknowledgement. The author is indebted to the referee for valuable suggestions leading to an improvement of the paper. Furthermore, we acknowledge financial support by the Austrian Science Foundation FWF, grant S8302-MAT.

References

[1] I. Berkes and W. Philipp. Limit theorems for mixing sequences without rate assumptions. Ann. Probab. 26 (1998), 805-831.
[2] P. Erdős. Some results on diophantine approximation. Acta Arith. 5 (1959), 359-369.
[3] M. FUCHS. On a problem of W. J. LeVeque concerning metric diophantine approximation. *Trans. Amer. Math. Soc.* **355** (2003), 1787-1801.
[4] M. I. Gordin and M. H. Reznik. The law of the iterated logarithm for denominators of continued fractions. Vestnik Leningrad Univ. 25 (1970), 28-33.
[5] G. Harman. Metric diophantine approximation with two restricted variables. I. Two square-free integers, or integers in arithmetic progressions. Math. Proc. Cambridge Philos. Soc. 103 (1988), 197-206.
[6] G. Harman. Metric Number Theory. London Mathematical Society Monographs. New Series 18 (Oxford University Press, 1998).
[7] C. C. Heyde. On the implication of a certain rate of convergence to normality. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 16 (1970), 151-156.
[8] H. Kesten. Sums of independent random variables - without moment conditions. Ann. Math. Statist. 43 (1972), 701-732.

[9] A. YA. KHINTCHINE. Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.* **92** (1924), 115-125.

[10] A. Ya. Khintchine. Zur metrischen Theorie der diophantischen Approximationen. Math. Z. 24 (1926), 706-714.

[11] J. Kuelbs and J. Zinn. Some results on LIL behavior. Ann. Probab. 11 (1983), 506-557.

[12] W. J. Leveque. On the frequency of small fractional parts in certain real sequences I. Trans. Amer. Math. Soc. 87 (1958), 237-260.

[13] W. J. Leveque. On the frequency of small fractional parts in certain real sequences II. Trans. Amer. Math. Soc. 94 (1959), 130-149.

[14] V. V. Petrov. A certain theorem on the law of the iterated logarithm. Teor. Verojatnost. i Primenen. 16 (1971), 715-718.

[15] W. Philipp. Mixing sequences of random variables and probablistic number theory. Mem. Amer. Math. Soc. 114 (1971), Providence, Rhode Island.

[16] W. Philipp. The remainder in the central limit theorem for mixing stochastic processes. Ann. Math. Statist. 40 (1969), 601-609.

[17] W. E. Pruitt. General one-sided laws of the iterated logarithm. Ann. Probab. 9 (1981), 1-48.

[18] W. M. Schmidt. A metrical theorem in diophantine approximation. Canad. J. Math. 12 (1960), 619-631.

[19] W. M. SCHMIDT. Metrical theorems on fractional parts of sequences. *Trans. Amer. Math. Soc.* **110** (1964), 493-518.

[20] P. Szüsz. Verallgemeinerung und Anwendung eines Kusminschen Satzes. Acta Arith. 7 (1962), 149-160.

[21] P. Szüsz. Über die metrische Theorie der Diophantischen Approximation II. Acta Arith. 8 (1963), 225-241.