A Note on the Quicksort Asymptotics

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Abstract

In a recent paper, Bindjeme and Fill obtained a surprisingly easy exact formula for the L_2-distance of the (normalized) number of comparisons of Quicksort under the uniform model to its limit. Shortly afterwards, Neininger proved a central limit theorem for the error. As a consequence, he obtained the asymptotics of the L_3-distance. In this short note, we use the moment transfer approach to re-prove Neininger's result. As a consequence, we obtain the asymptotics of the L_p-distance for all $1 \leq p < \infty$.

1 Introduction

Quicksort, an algorithm proposed by Hoare [6], is one of the most important sorting algorithms. It has been analyzed in many papers under the so-called uniform random model which assumes that the input is a random permutation of size n. One of the most popular characteristics is the number of comparison which we are going to denote by C_n.

First, it is straightforward to show that, as $n \to \infty$,

$$\mathbb{E}(C_n) = 2(n+1)H_n - 4n \sim 2n \log n,$$

were $H_n = \sum_{1 \leq j \leq n}(1/j)$ denotes the n-th harmonic number. Moreover, as $n \to \infty$,

$$\mathrm{Var}(C_n) \sim \left(7 - \frac{2}{3}\pi^2\right) n.$$

As for more refined stochastic properties, Régnier [10] used martingale theory to prove that

$$Y_n := \frac{C_n - \mathbb{E}(C_n)}{n+1}$$

converges to a non-degenerate limit Y both almost surely and in L_p for all $1 \leq p < \infty$, i.e.,

$$\lim_{n \to \infty} \|Y_n - Y\|_p = 0,$$

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where $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}, 1 \le p < \infty$, for a random variable X (here, Y_n and Y are all constructed on the same probability space which, for instance, can be done via random binary search trees). Régnier's result implies weak convergence of Y_n to Y. This was also proved by Rösler [11] who in addition constructed a random variable satisfying a distributional equation and proved that this random variable has the same distribution as Y. Recently, Bindjeme and Fill proved that the random variable constructed by Rösler is even almost surely equal to Y and they constructed random variables $Y^{(0)}$ and $Y^{(1)}$ with

$$Y = UY^{(0)} + (1-U)Y^{(1)} + C(U),$$

where $U, Y^{(0)}$ and $Y^{(1)}$ are independent, $Y^{(0)}$ and $Y^{(1)}$ have the same distribution as Y, U is a uniform distributed random variable on $[0,1]$ and

$$C(x) := 1 + 2x\log x + 2(1-x)\log(1-x).$$

Fill and Janson [3] further refined the above results by studying the rate of convergence of Y_n to Y. They proved that for the minimal L_p metric l_p, we have

$$l_p(Y_n, Y) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \qquad l_p(Y_n, Y) = \Omega\left(\frac{\log n}{n}\right).$$

Moreover, they proved for the Kolmogorov–Smirnov distance ρ that for all $\epsilon > 0$, we have

$$\rho(Y_n, Y) = \mathcal{O}\left(\frac{1}{n^{(1/2)-\epsilon}}\right), \qquad \rho(Y_n, Y) = \Omega\left(\frac{1}{n}\right).$$

Finally, Neininger and Rüschendorf [9] proved that for the Zolotarev metric ζ_3, we have

$$\zeta_3(Y_n, Y) = \Theta\left(\frac{\log n}{n}\right).$$

Very recently, Bindjeme and Fill in [1] obtained the following surprisingly easy exact formula for the L_2-distance of Y_n to Y:

$$\|Y_n - Y\|_2 = \left(\frac{1}{n+1}\left(2H_n + 1 + \frac{6}{n+1}\right) - 4\sum_{j=n+1}^{\infty}\frac{1}{j^2}\right)^{1/2} \sim \sqrt{\frac{2\log n}{n}}. \qquad (1)$$

Also very recently, Neininger [8] proved the following central limit theorem (CLT):

$$\sqrt{\frac{n}{2\log n}}(Y_n - Y) \xrightarrow{d} N(0,1). \qquad (2)$$

As a consequence of his proof, he obtained that, as $n \to \infty$,

$$\|Y_n - Y\|_3 \sim \frac{2}{\pi^{1/6}}\sqrt{\frac{\log n}{n}}.$$

The purpose of this note is two-fold. First, we are going to re-prove Neininger's result (2) with the moment-transfer approach. Second, as a consequence of our proof, we will obtain the following result.

Theorem 1. *We have for the L_p-distance of the (normalized) number of comparisons of quicksort Y_n to its limit Y that, as $n \to \infty$,*

$$\|Y_n - Y\|_p \sim \frac{2(\Gamma((p+1)/2))^{1/p}}{\pi^{1/(2p)}}\sqrt{\frac{\log n}{n}}.$$

2 CLT via the Moment-Transfer Approach

The moment-transfer approach was used in many recent papers in the analysis of algorithms. For instance, for Quicksort-type recurrences, it was applied by Hwang and Neininger [7]; see also Fill and Kapur [4] and the very general framework proposed by Chern, Hwang, and Tsai [2].

Before we can start with our proof, we collect some useful results.

First, we need the following result by Bindjeme and Fill [1]: for $n \geq 1$, we have the following (sample-pointwise) recurrence

$$\begin{aligned} Y_n - Y =& \frac{I_n+1}{n+1}\left(Y_{n,0} - Y^{(0)}\right) + \frac{n-I_n}{n+1}\left(Y_{n,1} - Y^{(1)}\right) + \left(\frac{I_n+1}{n+1} - U\right) Y^{(0)} \\ &+ \left(\frac{n-I_n}{n+1} - (1-U)\right) Y^{(1)} + \frac{n}{n+1} C_n(I_n+1) - C(U), \end{aligned} \tag{3}$$

where $C(x), Y^{(0)}, Y^{(1)}$ and U are from the introduction; given $\{U = u\}$ we have that $I_n \stackrel{d}{=} \text{Binom}(n-1, u)$; given $\{I_n = j\}$ we have that $Y_{n,0}$ and $Y_{n,1}$ are independent and distributed as Y_j and Y_{n-1-j}, respectively; and

$$C_n(i) := \frac{1}{n}\left(\mathbb{E}(C_{i-1}) + \mathbb{E}(C_{n-i}) - \mathbb{E}(C_n) + n - 1\right).$$

For the sake of simplicity, we will use the notation

$$T_n := \left(\frac{I_n+1}{n+1} - U\right) Y^{(0)} + \left(\frac{n-I_n}{n+1} - (1-U)\right) Y^{(1)} + \frac{n}{n+1} C_n(I_n+1) - C(U). \tag{4}$$

Next, we recall the following lemma which was obtained by Neininger in [8].

Lemma 1. *We have,*

$$\|T_n\|_p = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Proof. In Lemma 2.2 of [8], this was proved for $p = 3$. However, a careful inspection of the proof shows that it holds in fact for all $1 \leq p < \infty$. ∎

Using the above notation, (3) becomes

$$(n+1)(Y_n - Y) = (I_n+1)\left(Y_{n,0} - Y^{(0)}\right) + (n - I_n)\left(Y_{n,1} - Y^{(1)}\right) + (n+1)T_n. \tag{5}$$

Now, set

$$A_n^{[k]} = \mathbb{E}\left((n+1)^k (Y_n - Y)^k\right).$$

Raising the above equation to the k-th power and taking expectation yields, for $n \geq 1$,

$$A_n^{[k]} = \frac{2}{n}\sum_{j=0}^{n-1} A_j^{[k]} + B_n^{[k]}, \tag{6}$$

where

$$B_n^{[k]} = \sum_{\substack{i_1+i_2+i_3=k \\ 0\leq i_1,i_2<k}} \binom{k}{i_1, i_2, i_3} \mathbb{E}\left((I_n+1)^{i_1}\left(Y_{n,0} - Y^{(0)}\right)^{i_1}(n-I_n)^{i_2}\left(Y_{n,1} - Y^{(1)}\right)^{i_2}(n+1)^{i_3} T_n^{i_3}\right). \tag{7}$$

The above recurrence for $A_n^{[k]}$ was extensively studied. For instance, in [7], [4], and [2], the authors derived very general (asymptotic) transfer theorems. We recall one result which we will need in the sequel.

Lemma 2. *Let $(b_n)_{n\geq 1}$ be a given sequence and define a sequence $(a_n)_{n\geq 0}$ by*

$$a_n = \frac{2}{n}\sum_{j=0}^{n-1} a_j + b_n$$

for $n \geq 1$ with arbitrary initial value a_0. Let $\alpha > 1$ and β be positive real numbers.

(i) If $b_n = n^\alpha \log^\beta n$, then

$$a_n = \frac{\alpha+1}{\alpha-1} n^\alpha \log^\beta n + \mathcal{O}\left(n^\alpha \log^{\beta-1} n\right).$$

(ii) If $b_n = \mathcal{O}\left(n^\alpha \log^\beta n\right)$, then

$$a_n = \mathcal{O}\left(n^\alpha \log^\beta n\right).$$

Using the above two lemmas and (6), we will prove the following result.

Proposition 1. *For integers $m \geq 1$, we have*

$$A_n^{[2m-1]} = \mathcal{O}\left(n^{m-(1/2)} \log^{m-1} n\right)$$

and

$$A_n^{[2m]} = g_m n^m \log^m n + \mathcal{O}\left(n^m \log^{m-1} n\right),$$

where $g_m = (2m)!/m!$.

Proof. We prove this result by induction on m, where in addition we prove that

$$\mathbb{E}\left((n+1)^{2m-1}|Y_n - Y|^{2m-1}\right) = \mathcal{O}\left(n^{m-(1/2)} \log^{m-(1/2)} n\right). \tag{8}$$

For $m = 1$, observe that the claim trivially holds for $A_n^{[1]}$ and by (1) holds for $A_n^{[2]}$. Also, (8) follows from (1) since

$$\|Y_n - Y\|_1 \leq \|Y_n - Y\|_2 = \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right).$$

Now, assume that the claim is true for all $m' < m$. We are going to prove it for m. We will only present the proof for $A_n^{[2m-1]}$ and $A_n^{[2m]}$. The proof of (8) is slightly different and will be done in an appendix (actually, (8) has to be proved first since it will be used below).

We start with $A_n^{[2m-1]}$. First consider (7) which we break into two parts

$$B_n^{[2m-1]} = \Sigma_0 + \Sigma_1$$

according to whether in the summation $i_3 = 0$ or $i_3 \geq 1$, respectively.

For Σ_0, we obtain

$$\begin{aligned}\Sigma_0 &= \sum_{i=1}^{2m-2} \binom{2m-1}{i} \mathbb{E}\left((I_n+1)^i \left(Y_{n,0} - Y^{(0)}\right)^i (n - I_n)^{2m-1-i} \left(Y_{n,1} - Y^{(1)}\right)^{2m-1-i}\right) \\ &= \sum_{i=1}^{2m-2} \binom{2m-1}{i} \frac{1}{n} \sum_{j=0}^{n-1} A_j^{[i]} A_{n-1-j}^{[2m-1-i]}.\end{aligned}$$

Note that either i is odd or $2m-1-i$ is odd. Consequently, by using the induction hypothesis,

$$\Sigma_0 = \mathcal{O}\left(n^{m-(1/2)}\log^{m-1}\right).$$

Next, we consider Σ_1. Here, by an application of Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left((I_n+1)^{i_1}\left|Y_{n,0}-Y^{(0)}\right|^{i_1}(n-I_n)^{i_2}\left|Y_{n,1}-Y^{(1)}\right|^{i_2}(n+1)^{i_3}|T_n|^{i_3}\right)\\
&\quad\le \|(I_n+1)\left(Y_{n,0}-Y^{(0)}\right)\|_{2m-1}^{i_1}\|(n-I_n)\left(Y_{n,1}-Y^{(1)}\right)\|_{2m-1}^{i_2}\|(n+1)T_n\|_{i_3(2m-1)/(2m-1-i_1-i_2)}^{i_3}
\end{aligned} \tag{9}$$

Consequently, by using (8) and Lemma 1, we obtain

$$\Sigma_1 = \mathcal{O}\left(n^{m-(1/2)}\log^{m-1}\right).$$

Putting the above two estimates for Σ_0 and Σ_1 together gives

$$B_n^{[2m-1]} = \mathcal{O}\left(n^{m-(1/2)}\log^{m-1}\right).$$

From this, the claim for $A_n^{[2m-1]}$ follows from Lemma 2.

Next, we consider $A_n^{[2m]}$. We again start from (7) which we now break into three parts

$$B_n^{[2m]} = \Sigma_0 + \Sigma_1 + \Sigma_2$$

according to whether in the summation $i_3 = 0$, $i_3 = 1$ or $i_3 \ge 2$, respectively.

For Σ_0, we have

$$\begin{aligned}
\Sigma_0 = &\sum_{i=1}^{m-1}\binom{2m}{2i}\mathbb{E}\left((I_n+1)^{2i}\left(Y_{n,0}-Y^{(0)}\right)^{2i}(n-I_n)^{2m-2i}\left(Y_{n,1}-Y^{(1)}\right)^{2m-2i}\right)\\
&+\sum_{i=0}^{m-1}\binom{2m}{2i+1}\mathbb{E}\left((I_n+1)^{2i+1}\left(Y_{n,0}-Y^{(0)}\right)^{2i+1}(n-I_n)^{2m-2i-1}\left(Y_{n,1}-Y^{(1)}\right)^{2m-2i-1}\right).
\end{aligned}$$

Plugging the induction hypothesis into the first term on the right-hand side yields

$$\begin{aligned}
\Sigma_{00} :=& \sum_{i=1}^{m-1}\binom{2m}{2i}\mathbb{E}\left((I_n+1)^{2i}\left(Y_{n,0}-Y^{(0)}\right)^{2i}(n-I_n)^{2m-2i}\left(Y_{n,1}-Y^{(1)}\right)^{2m-2i}\right)\\
=& \sum_{i=1}^{m-1}\binom{2m}{2i}\frac{1}{n}\sum_{j=0}^{n-1}A_j^{[2i]}A_{n-1-j}^{[2m-2i]}\\
=& \sum_{i=1}^{m-1}\binom{2m}{2i}g_i g_{m-i}\frac{1}{n}\sum_{j=0}^{n-1}j^i(\log j)^i(n-1-j)^{m-i}(\log(n-1-j))^{m-i} + \mathcal{O}\left(n^m\log^{m-1}n\right).
\end{aligned}$$

Now, observe that by an application of the Euler-Maclaurin summation formula (see Section 4.5 in Flajolet and Sedgewick [5]), we have

$$\begin{aligned}
&\frac{1}{n}\sum_{j=0}^{n-1}j^i(\log j)^i(n-1-j)^{m-i}(\log(n-1-j))^{m-i}\\
&\qquad = n^m\log^m n\int_0^1 x^i(1-x)^{m-i}\mathrm{d}x + \mathcal{O}\left(n^m\log^{m-1}n\right)\\
&\qquad = \frac{i!(m-i)!}{(m+1)!}n^m\log^m n + \mathcal{O}\left(n^m\log^{m-1}n\right).
\end{aligned}$$

Consequently, by a simple computation,

$$\Sigma_{00} = \frac{m-1}{m+1} g_m n^m \log^m n + \mathcal{O}\left(n^m \log^{m-1} n\right).$$

As for the second term on the right-hand side of Σ_0, again by the induction hypothesis,

$$\begin{aligned}\Sigma_{01} &:= \sum_{i=0}^{m-1} \binom{2m}{2i+1} \mathbb{E}\left((I_n+1)^{2i+1}\left(Y_{n,0}-Y^{(0)}\right)^{2i+1}(n-I_n)^{2m-2i-1}\left(Y_{n,1}-Y^{(1)}\right)^{2m-2i-1}\right) \\ &= \sum_{i=0}^{m-1} \binom{2m}{2i+1} \frac{1}{n} \sum_{j=0}^{n-1} A_j^{[2i+1]} A_{n-1-j}^{[2m-2i-1]} = \mathcal{O}\left(n^m \log^{m-1} n\right).\end{aligned}$$

Next, we consider Σ_1, where we plug (4) into Σ_1 and break the expectation into three parts according to the three terms in the definition of T_n. For the first part, we obtain

$$\begin{aligned}&\mathbb{E}\left((I_n+1)^{i_1}\left(Y_{n,0}-Y^{(0)}\right)^{i_1}(n-I_n)^{i_2}\left(Y_{n,1}-Y^{(1)}\right)^{i_2}\left(I_n+1-(n+1)U\right)Y^{(0)}\right) \\ &= \sum_{j=0}^{n-1} (j+1)^{i_1} \mathbb{E}\left((Y_j-Y)^{i_1} Y\right) A_{n-1-j}^{[i_2]} \int_0^1 (j+1-(n+1)u) \binom{n-1}{j} u^j (1-u)^{n-1-j} \mathrm{d}u.\end{aligned}$$

Note that

$$\int_0^1 (j+1-(n+1)u) \binom{n-1}{j} u^j (1-u)^{n-1-j} \mathrm{d}u = \frac{j+1}{n} - \frac{j+1}{n} = 0.$$

Hence, the first part vanishes. Similarly, the second part vanishes. As for the third part, we have

$$\begin{aligned}&\mathbb{E}\left((I_n+1)^{i_1}\left(Y_{n,0}-Y^{(0)}\right)^{i_1}(n-I_n)^{i_2}\left(Y_{n,1}-Y^{(1)}\right)^{i_2}\left(nC_n(I_n+1)-(n+1)C(U)\right)\right) \\ &= \sum_{j=0}^{n-1} A_j^{[i_1]} A_{n-1-j}^{[i_2]} \int_0^1 \left(nC_n(j+1)-(n+1)C(u)\right) \binom{n-1}{j} u^j (1-u)^{n-1-j} \mathrm{d}u \\ &= \sum_{j=0}^{n-1} A_j^{[i_1]} A_{n-1-j}^{[i_2]} \left(C_n(j+1) - (n+1)\binom{n-1}{j} \int_0^1 C(u) u^j (1-u)^{n-1-j} \mathrm{d}u\right).\end{aligned}$$

We will show that

$$c_{j,n} := C_n(j+1) - (n+1)\binom{n-1}{j} \int_0^1 C(u) u^j (1-u)^{n-1-j} \mathrm{d}u = \mathcal{O}\left(\frac{1}{n}\right) \tag{10}$$

uniformly in j. Then, by using the induction hypothesis and the fact that $i_1 + i_2 = 2m-1$, we have

$$\Sigma_1 = \mathcal{O}\left(n^{m-(1/2)} \log^{m-1} n\right).$$

In order to show (10), we use

$$\mathbb{E}(C_n) = 2(n+1)\log n + cn + \mathcal{O}(1)$$

for some constant c. Consequently,

$$\begin{aligned}C_n(j+1) &= \frac{1}{n}\left(\mathbb{E}(C_j) + \mathbb{E}(C_{n-j-1}) - \mathbb{E}(C_n) + n - 1\right) \\ &= \frac{2}{n}(j+1)\log j + \frac{2}{n}(n-j)\log(n-j-1) - \frac{2}{n}(n+1)\log n + 1 + \mathcal{O}\left(\frac{1}{n}\right).\end{aligned} \tag{11}$$

Next, to evaluate the integral in (10), we need

$$\begin{aligned}\int_0^1 u^{\alpha}(\log u)(1-u)^{\beta}\mathrm{d}u &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 u^{x}(1-u)^{\beta}\mathrm{d}u\bigg|_{x=\alpha}\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\Gamma(x+1)\Gamma(\beta+1)}{\Gamma(x+\beta+2)}\bigg|_{x=\alpha}\\ &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\left(\Psi(\alpha+1)-\Psi(\alpha+\beta+2)\right)\end{aligned}$$

for $\alpha, \beta > -1$, where Ψ is the digamma function. Recall that, as $x \to \infty$,

$$\Psi(x) = \log x + \mathcal{O}\left(\frac{1}{x}\right).$$

Using these results, we obtain for the integral in (10), uniformly in j,

$$\begin{aligned}&(n+1)\binom{n-1}{j}\int_0^1 C(u)u^{j}(1-u)^{n-1-j}\mathrm{d}u\\ &\qquad = 1+\frac{2}{n}(j+1)(\log j-\log n)+\frac{2}{n}(n-j)(\log(n-j-1)-\log n)+\mathcal{O}\left(\frac{1}{n}\right).\end{aligned}$$

Combining this with (11) proves (10).

Finally, we consider Σ_2. Here, again from (9) together with (8) and Lemma 1, we obtain

$$\Sigma_2 = \mathcal{O}\left(n^m \log^{m-1} n\right).$$

Overall, by combining the above estimates for Σ_0, Σ_1 and Σ_2,

$$B_n^{[2m]} = \frac{m-1}{m+1}g_m n^m \log^m n + \mathcal{O}\left(n^m \log^{m-1} n\right).$$

From this, by applying Lemma 2,

$$A_n^{[2m]} = g_m n^m \log^m n + \mathcal{O}\left(n^m \log^{m-1} n\right).$$

This is the claimed result. Hence, the proof is finished. ∎

The latter proposition together with the Fréchet-Shohat theorem implies (2).

3 Proof of Theorem 1

Here, we prove Theorem 1. First, observe that by Proposition 1, we have

$$\left\|\sqrt{\frac{n}{2\log n}}(Y_n-Y)\right\|_p \le \left\|\sqrt{\frac{n}{2\log n}}(Y_n-Y)\right\|_{2\lceil p/2\rceil} = \mathcal{O}(1). \tag{12}$$

Consequently,

$$\left(\frac{n}{2\log n}\right)^{p/2}|Y_n-Y|^p$$

is uniformly integrable, for all $p \geq 1$. This together with (2) implies that

$$\lim_{n\to\infty} \left(\frac{n}{2\log n}\right)^{p/2} \mathbb{E}|Y_n - Y|^p = \mathbb{E}|N(0,1)|^p.$$

A standard computation yields

$$\mathbb{E}|N(0,1)|^p = \frac{2^{p/2}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right).$$

Overall, as $n \to \infty$, we have

$$\|Y_n - Y\|_p \sim \frac{2}{\pi^{1/(2p)}}\Gamma\left(\frac{p+1}{2}\right)^{1/p}\sqrt{\frac{\log n}{n}}.$$

This proves the claimed result.

Remark 1. The property (12) was mentioned without proof on page 11 in [8].

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Appendix

Throughout this appendix, we will use the notation $\log_+(x) = \max\{\log x, 1\}$.

The goal of this appendix is to prove (8). We will use the notation

$$\bar{A}_n^{[k]} = \mathbb{E}\left((n+1)^k |Y_n - Y|^k\right).$$

Note that $\bar{A}_n^{[2m]} = A_n^{[2m]}$ and (8) becomes

$$\bar{A}_n^{[2m-1]} \le c n^{m-(1/2)} \log_+^{m-(1/2)} n \tag{13}$$

for a suitable constant $c > 1$.

The proof will proceed by induction. Thus, we assume that (13) and the claims of Proposition 1 hold for all $m' < m$. Moreover, we assume that (13) holds for all $n' < n$ where we can assume that n is sufficiently large. We will show how to choose c such that (13) holds for all n.

In order to prove our claim, we start from (5). First, observe that

$$(n+1)|Y_n - Y| \le (I_n + 1)\left|Y_{n,0} - Y^{(0)}\right| + (n - I_n)\left|Y_{n,1} - Y^{(1)}\right| + (n+1)|T_n|.$$

Raising this to the $2m-1$-st power gives

$$\bar{A}_n^{[2m-1]} \le \frac{2}{n}\sum_{j=0}^{n-1} A_j^{[2m-1]} + \bar{B}_n^{[2m-1]}, \tag{14}$$

where $\bar{B}_n^{[2m-1]}$ is given by

$$\sum_{\substack{i_1+i_2+i_3=2m-1\\ 0\le i_1,i_2<2m-1}} \binom{2m-1}{i_1, i_2, i_3} \mathbb{E}\left((I_n+1)^{i_1}\left|Y_{n,0} - Y^{(0)}\right|^{i_1}(n-I_n)^{i_2}\left|Y_{n,1} - Y^{(1)}\right|^{i_2}(n+1)^{i_3}|T_n|^{i_3}\right).$$

As in the proof of Proposition (1) we will break $\bar{B}_n^{[2m-1]}$ into three parts

$$\bar{B}_n^{[2m-1]} = \Sigma_0 + \Sigma_1 + \Sigma_2$$

according to whether $i_3 = 0, i_3 = 1$, or $i_3 \ge 2$, respectively.

For Σ_0, by using the induction hypothesis, we have

$$\Sigma_0 = \sum_{i=1}^{2m-2}\binom{2m-1}{i}\frac{2}{n}\sum_{j=0}^{n-1}\bar{A}_j^{[i]}\bar{A}_{n-1-j}^{[2m-1-i]} \le d_1 n^{m-(1/2)}\log_+^{m-(1/2)} n$$

for a suitable constant d_1.

Next, for Σ_1, we use (9). From this, another application of the induction hypothesis and using Lemma (1), we obtain

$$\begin{aligned}\Sigma_1 &\le \bar{d}_2\sqrt{n}(2m-1)4^{m-1}\left(\frac{1}{n}\sum_{j=0}^{n-1}\bar{A}_j^{[2m-1]}\right)^{(2m-2)/(2m-1)}\\ &\le \bar{d}_2 c^{(2m-2)/(2m-1)}(2m-1)4^{m-1}n^{m-(1/2)}\log_+^{m-1} n \le d_2 c n^{m-(1/2)}\log_+^{m-1} n\end{aligned}$$

for suitable constants $\bar{d}_2$ and d_2.

Finally, for Σ_2, we again use (9) where $2m-1$ is replaced by $2m-2$. Consequently, again by the induction hypothesis

$$\Sigma_3 \le d_3 n^{m-(1/2)} \log_+^{m-(3/2)} n \le d_3 n^{m-(1/2)} \log_+^{m-(1/2)} n$$

for a suitable constant d_3.

Putting the last three estimates together yields

$$\bar{B}_n^{[2m-1]} \le \left(\frac{d_2 c}{\sqrt{\log_+ n}} + d_1 + d_3 \right) n^{m-(1/2)} \log_+^{m-(1/2)} n.$$

Plugging this into (14) and using once more the induction hypothesis yields

$$\begin{aligned}
\bar{A}_n^{[2m-1]} &\le 2c\frac{1}{n}\sum_{j=0}^{n-1} j^{m-(1/2)} \log_+^{m-(1/2)} j + \left(\frac{d_2 c}{\sqrt{\log_+ n}} + d_1 + d_3 \right) n^{m-(1/2)} \log_+^{m-(1/2)} n \\
&\le \left(2c \int_0^1 u^{m-(1/2)} \mathrm{d}u \right) n^{m-(1/2)} \log_+^{m-(1/2)} n + \left(\frac{d_2 c}{\sqrt{\log_+ n}} + d_1 + d_3 \right) n^{m-(1/2)} \log_+^{m-(1/2)} n \\
&= \left(\frac{2c}{m+(1/2)} + \frac{d_2 c}{\sqrt{\log_+ n}} + d_1 + d_3 \right) n^{m-(1/2)} \log_+^{m-(1/2)} n.
\end{aligned}$$

Note that for n large enough, we have

$$\frac{2}{m+(1/2)} + \frac{d_2}{\sqrt{\log_+ n}} < 1.$$

Hence, we can choose c such that

$$\frac{2c}{m+(1/2)} + \frac{d_2 c}{\sqrt{\log_+ n}} + d_1 + d_3 \le c$$

which concludes our proof.