The Variance for Partial Match Retrievals in k-dimensional Bucket Digital Trees

($\approx$ 1/2 joint with Hsien-Kuei Hwang and Vytas Zacharovas)

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Introduced by Coffman and Eve (1970).

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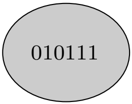

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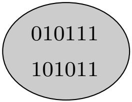

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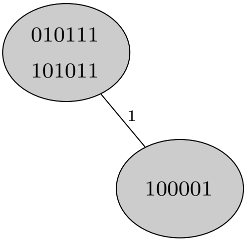

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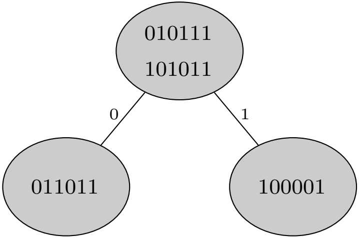

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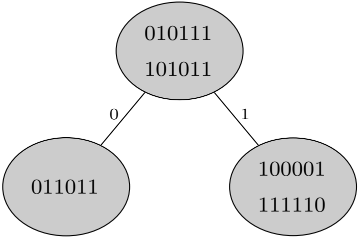

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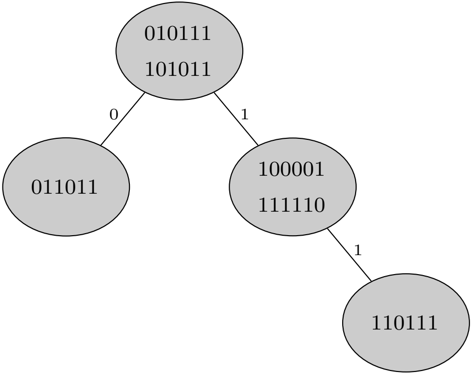

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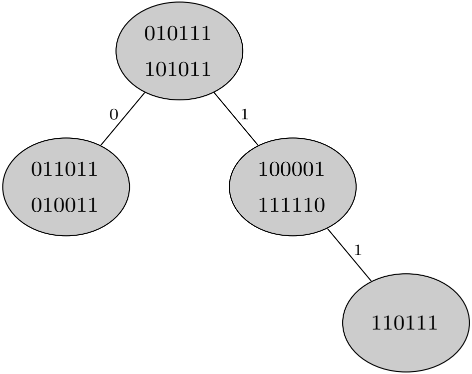

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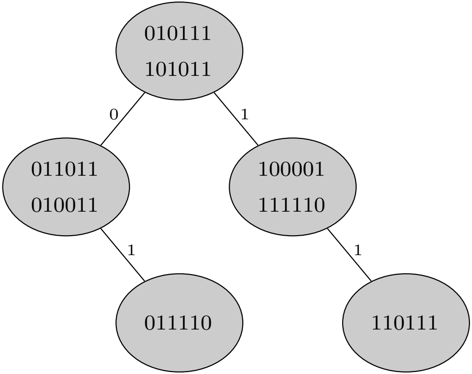

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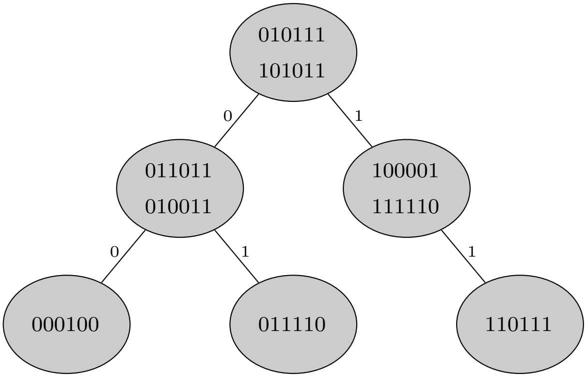

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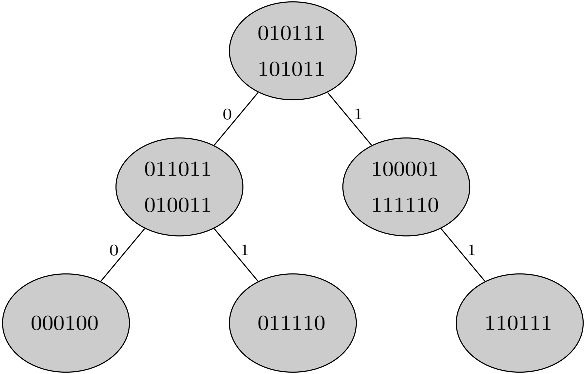

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Random Model: 0-1 are generated independently and equally likely.

Shape Parameters

- **Depth**

  Konheim, Newman, Knuth, Devroye, Louchard, Szpankowski

- **Partial Match Queries**

  Flajolet, Puech, Kirschenhofer, Prodinger, Szpankowski, Schachinger

- **# of Occurrences of Patterns**

  Knuth, Flajolet, Sedgewick, Prodinger, Kirschenhofer

- **Key-Wise Path Length**

  Flajolet, Sedgewick, Prodinger, Kirschenhofer, Szpankowski, Hubalek

- **Node-Wise Path Length**

  Fuchs, Hwang, Zacharovas

Previous Approaches

- **Rice Method**

  Introduced by Flajolet and Sedgewick for digital search trees with bucket size one.

- **Approach of Flajolet and Richmond**

  Introduced for the analysis of bucket digital search trees. Based on Euler transform, Mellin transform, and singularity analysis.

- **Approach via Analytic Depoissonization**

  Introduced by Jacquet & Regnier and Jacquet & Szpankowski. Based on saddle point method and Mellin transform.

- **Schachinger's Approach**

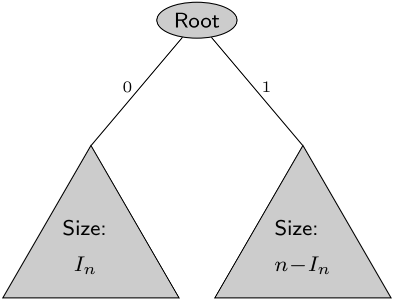
  Largely elementary.

Distributional Recurrence

Shape parameters X_n satisfy the recurrence:

$$X_{n+b} \stackrel{d}{=} X_{I_n} + X^*_{n-I_n} + T_n$$

- $I_n \stackrel{d}{=} \text{Binomial}(n, 1/2)$;
- $X_n \stackrel{d}{=} X^*_n$;
- X_n, X^*_n, I_n independent.
- T_n toll-function.



Poissonization

Moments satisfy the recurrence:

$$f_{n+b} = 2^{1-n} \sum_{j=0}^{n} \binom{n}{j} f_j + g_n.$$

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Consider Poisson-generating function of f_n and g_n, i.e.,

$$\tilde{f}(z) := e^{-z} \sum_{n \geq 0} f_n \frac{z^n}{n!}, \qquad \tilde{g}(z) := e^{-z} \sum_{n \geq 0} g_n \frac{z^n}{n!}.$$

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Then,

$$\sum_{j=0}^{b} \binom{b}{j} \tilde{f}^{(j)}(z) = 2\tilde{f}(z/2) + \tilde{g}(z).$$

Poissonized Variance

Poisson Heuristic:

$$f_n \text{ sufficiently smooth} \quad \Longrightarrow \quad \tilde{f}(n) \approx f_n.$$

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$$\tilde{V}(z) = \tilde{f}_2(z) - \tilde{f}_1(z)^2 - z\tilde{f}_1'(z)^2.$$

Jacquet-Szpankowski-admissibility (JS-admissibility)

$\tilde{f}(z)$ is called JS-admissible if

$(\mathbf{I})$ Uniformly for $|\arg(z)| \le \epsilon$,

$$\tilde{f}(z) = \mathcal{O}\left(|z|^{\alpha} \log^{\beta} |z|\right),$$

$(\mathbf{O})$ Uniformly for $\epsilon < |\arg(z)| \le \pi$,

$$f(z) := e^{z} \tilde{f}(z) = \mathcal{O}\left(e^{(1-\epsilon)|z|}\right).$$

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Theorem (Jacquet and Szpankowski)

If $\tilde{f}(z)$ is JS-admissible, then

$$f_n \sim \tilde{f}(n) - \frac{n}{2}\tilde{f}''(n) + \cdots.$$

Depoissonization

JS-admissibility satisfies closure properties:

(i) $\tilde{f}, \tilde{g}$ JS-admissible, then $\tilde{f} + \tilde{g}$ JS-admissible.

(ii) $\tilde{f}$ JS-admissible, then $\tilde{f}'$ JS-admissible. Etc.

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Proposition

Consider

$$\sum_{j=0}^{b} \binom{b}{j} \tilde{f}^{(j)}(z) = 2\tilde{f}(z/2) + \tilde{g}(z).$$

We have,

$$\tilde{g}(z) \text{ JS-admissible} \quad \Longrightarrow \quad \tilde{f}(z) \text{ JS-admissible.}$$

Laplace-Mellin Approach (i)

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Applying Laplace transform,

$$(s+1)^b \mathscr{L}[\tilde{f}(z); s] = 4\mathscr{L}[\tilde{f}(z); 2s] + \mathscr{L}[\tilde{g}(z); s] + p(s).$$

with $p(s)$ a polynomial.

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with $p(s)$ a polynomial.

Define,

$$Q(s) := \sum_{l \geq 1} \left(1 - \frac{s}{2^l}\right)$$

and $Q_\infty := Q(1)$.

Laplace-Mellin Approach (ii)

Set

$$\bar{f}(s) := \frac{\mathscr{L}[\tilde{f}(z); s]}{Q(-s)^b}, \qquad \bar{g}(s) := \frac{\mathscr{L}[\tilde{g}(z); s] + p(s)}{Q(-2s)^b}.$$

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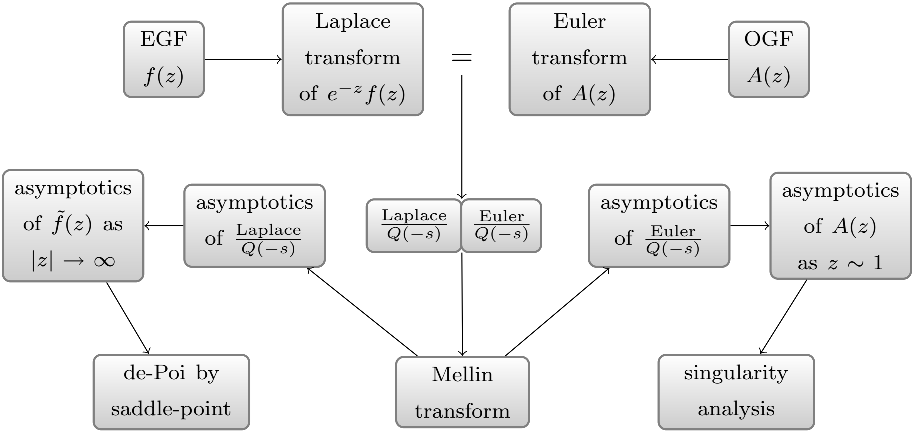
From this, an asymptotic expansion of $\tilde{f}(z)$ as $z \to \infty$ is obtained via inverse Mellin transform and inverse Laplace transform.

Our Approach vs. Flajolet-Richmond



Variance for Key-Wise Path Length

Theorem

We have,

$$\mathrm{Var}(X_n) \sim nP_2(\log_2 n),$$

where $P_2(z)$ is a one-periodic function with Fourier coefficients

$$\frac{1}{L\Gamma(2+2\pi ir/L)} \int_0^\infty \frac{s^{1+2\pi ir/L}}{Q(-2s)^b} \int_0^\infty e^{-zs}\tilde{h}(z)\mathrm{d}z\mathrm{d}s$$

with $L := \log 2$.

Here, $\tilde{h}(z)$ is a function of the Poisson generating function of the mean.

Fourier Coefficients

Theorem

For $b = 1$,

$$\frac{Q_\infty}{L\Gamma(2+2\pi ir/L)} \sum_{j_1,j_2,j_3\geq 0} \frac{(-1)^{j_1}2^{-\binom{j_1+1}{2}+2\pi irj_1/L}}{Q_{j_1}Q_{j_2}Q_{j_3}2^{j_2+j_3}}\varphi(2+2\pi ir/L; 2^{-j_1-j_2}+2^{-j_1-j_3})$$

with $Q_j = \prod_{l=1}^{j}(1-2^l)$ *and*

$$\varphi(\omega; x) = \frac{\pi(1+x^{\omega-2}((\omega-2)x+1-\omega))}{(x-1)^2\sin(\pi\omega)}.$$

$$
\begin{aligned}
&-\frac{28}{3L}-\frac{39}{4}-2\sum_{l\ge1}\frac{l2^l}{(2^l-1)^2}+\frac{2}{L}\sum_{l\ge1}\frac{1}{2^l-1}+\frac{\pi^2}{2L^2}+\frac{2}{L^2}\\
&-\frac{2}{L}\sum_{l\ge3}\frac{(-1)^{l+1}(l-5)}{(l+1)l(l-1)(2^l-1)}\\
&+\frac{2}{L}\sum_{l\ge1}(-1)^l2^{-\binom{l+1}{2}}\left(\frac{L(1-2^{-l+1})/2-1}{1-2^{-l}}-\sum_{r\ge2}\frac{(-1)^{r+1}}{r(r-1)(2^{r+l}-1)}\right)\\
&-\frac{2Q(1)}{L}+\sum_{l\ge2}\frac{1}{2^lQ_l}\sum_{r\ge0}\frac{(-1)^r2^{-\binom{r+1}{2}}}{Q_r}Q_{r+l-2}\cdot\\
&\cdot\left(-\sum_{j\ge1}\frac{1}{2^{j+r+l+2}-1}\left(2^{l+1}-2l-4+2\sum_{i=2}^{l-1}\binom{l+1}{i}\frac{1}{2^{r+i-1}-1}\right)\right.\\
&\quad+\frac{2}{(1-2^{-l-r})^2}+\frac{2l+2}{(1-2^{1-l-r})^2}-\frac{2}{L}\frac{1}{1-2^{1-l-r}}+\frac{2}{L}\sum_{j=1}^{l+1}\binom{l+1}{j}\frac{1}{2^{r+j}-1}\\
&\quad\left.-2\sum_{j=2}^{l+1}\binom{l+1}{j}\frac{1}{2^{r+j-1}-1}+\frac{2}{L}\sum_{j=0}^{l+1}\binom{l+1}{j}\sum_{i\ge1}\frac{(-1)^i}{(i+1)(2^{r+j+i}-1)}\right)\\
&+\sum_{l\ge3}\sum_{r=2}^{l-1}\binom{l+1}{r}\frac{Q_{r-2}Q_{l-r-1}}{2^lQ_l}\sum_{j\ge l+1}\frac{1}{2^j-1}-2[FH]_0-[F^2]_0.
\end{aligned}
$$

k-d Bucket Digital Search Trees

Let $R_1, \ldots, R_n$ be k-dimensional data, i.e., R_i consist of k 0-1 strings

$$\begin{aligned} R_{i,1} &= \left(R_{i,1}^{[1]}, R_{i,1}^{[2]}, R_{i,1}^{[3]}, \ldots\right), \\ &\vdots \\ R_{i,k} &= \left(R_{i,k}^{[1]}, R_{i,k}^{[2]}, R_{i,k}^{[3]}, \ldots\right). \end{aligned}$$

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Shuffling yields

$$\tilde{R}_i = \left(R_{i,1}^{[1]}, \ldots, R_{i,k}^{[1]}, R_{i,1}^{[2]}, \ldots, R_{i,k}^{[2]}, \ldots\right).$$

Use $\tilde{R}_1, \ldots, \tilde{R}_n$ to construct the usual bucket digital search tree.

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$\longrightarrow$ k-d bucket digital search tree.

Partial Match Queries

- Let $Q = (Q_1, \ldots, Q_k)$ be a *partial match query*, where Q_i is either a 0-1 string or a an undefined string.

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- The cost only depends on the *partial match pattern* $q \in \{S, \star\}^k$, where

$$q_i = \begin{cases} S, & \text{if } Q_i \text{ is a } 0\text{-}1 \text{ string;} \\ \star, & \text{if } Q_i \text{ is unspecified.} \end{cases}$$

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- Cost will be denoted by $X_{q,n}$, where n is the size of the tree.

Distributional Recurrence

Similar as before,

$$X_{q,n+b} \stackrel{d}{=} \begin{cases} X_{q',I_n} + X^*_{q',n-I_n} + 1, & \text{if } q = (\star, \dots); \\ X_{q',I_n} + 1, & \text{if } q = (S, \dots), \end{cases}$$

where q' denotes the cyclic shift to the left.

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Applying Poisson generating function gives for the moments

$$\sum_{j=0}^{b} \binom{b}{j} \tilde{f}^{(j)}_{q^{(l)}}(z) = \delta_{l+1} \tilde{f}_{q^{(l+1)}}(z/2) + \tilde{g}_{q^{(l)}}(z),$$

where $q^{(l)}$ is the cyclic shift applied l times and $\delta_{l+1} \in \{1, 2\}$.

Mean Value

Theorem (Kirschenhofer and Prodinger 1994)

For $b = 1$ and u the number of unspecified coordinates,

$$\mathbf{E}(X_{q,n}) \sim u^{u/k} P_1(\log_2 n^{1/k}),$$

where $P_1(z)$ is one-periodic with Fourier coefficients

$$\frac{\mathscr{M}[1/(sQ(-2s)^b); \omega_r]}{kL\Gamma(1+\omega_r)} \sum_{l=0}^{k-1} \delta_1 \dots \delta_l 2^{-\omega_r l}$$

with $w_r = u/k + 2\pi i r/(kL)$.

Depoissonization

Since mean value is sublinear, Poissonized variance is given by

$$\tilde{V}_q(z) := \tilde{f}_{q,2}(z) - \tilde{f}_{q,1}(z)^2.$$

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Depoissonization is done via the following result + closure properties.

Proposition

Consider

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We have,

$$\tilde{g}_{q^{(l)}}(z) \text{ JS-admissible} \quad \Longrightarrow \quad \tilde{f}_{q^{(l)}}(z) \text{ JS-admissible.}$$

Mean, Second Moment and Variance

We have,

$$\sum_{j=0}^{b} \binom{b}{j} \tilde{f}_{q,1}^{(j)}(z) = \delta_1 \tilde{f}_{q',1}(z/2) + 1$$

Mean, Second Moment and Variance

We have,

$$\sum_{j=0}^{b}\binom{b}{j}\tilde{f}_{q,1}^{(j)}(z)=\delta_1\tilde{f}_{q',1}(z/2)+1$$

and

$$\sum_{j=0}^{b}\binom{b}{j}\tilde{f}_{q,2}^{(j)}(z)=\delta_1\tilde{f}_{q',2}(z/2)+\tilde{g}_{q,2}(z).$$

where

$$\tilde{g}_{q,2}(z)=\begin{cases}4\tilde{f}_{q',1}(z/2)+2\tilde{f}_{q',1}(z/2)^2+1, & \text{if } q^{(l)}=(\star,\ldots);\\ 2\tilde{f}_{q',1}(z/2)+1, & \text{if } q^{(l)}=(S,\ldots).\end{cases}$$

Mean, Second Moment and Variance

We have,

$$\sum_{j=0}^{b}\binom{b}{j}\tilde{f}_{q,1}^{(j)}(z)=\delta_1\tilde{f}_{q',1}(z/2)+1$$

and

$$\sum_{j=0}^{b}\binom{b}{j}\tilde{f}_{q,2}^{(j)}(z)=\delta_1\tilde{f}_{q',2}(z/2)+\tilde{g}_{q,2}(z).$$

where

$$\tilde{g}_{q,2}(z)=\begin{cases}4\tilde{f}_{q',1}(z/2)+2\tilde{f}_{q',1}(z/2)^2+1, & \text{if } q^{(l)}=(\star,\dots);\\ 2\tilde{f}_{q',1}(z/2)+1, & \text{if } q^{(l)}=(S,\dots).\end{cases}$$

$\tilde{f}_{q,1}(z), \tilde{f}_{q,2}(z)$ are JS-admissible and hence $\tilde{V}_q(n)\sim \mathrm{Var}(X_n)$.

Laplace-Mellin Approach (i)

We start from,

$$\sum_{j=0}^{b} \binom{b}{j} \tilde{f}_{q^{(l)}}^{(j)}(z) = \delta_{l+1} \tilde{f}_{q^{(l+1)}}(z/2) + \tilde{g}_{q^{(l)}}(z).$$

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Applying Laplace transform,

$$(s+1)^b [\tilde{f}_{q^{(l)}}(z); s] = 2\delta_{l+1} \mathscr{L}[\tilde{f}_{q^{(l+1)}}(z); 2s] + \mathscr{L}[\tilde{g}_{q^{(l)}}(z); s] + p(s),$$

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Next normalize,

$$\bar{f}_{q^{(l)}}(s) = \frac{\mathscr{M}[\tilde{f}_{q^{(l)}}(z); s]}{Q(-s)^b}, \qquad \bar{g}_{q^{(l)}}(s) = \frac{\mathscr{M}[\tilde{g}_{q^{(l)}}(z); s] + p(s)}{Q(-2s)^b}.$$

Laplace-Mellin Approach (ii)

This yields,

$$\bar{f}_{q^{(l)}}(s) = 2\delta_{l+1}\bar{f}_{q^{(l+1)}}(2s) + \bar{g}_{q^{(l)}}(s).$$

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By iteration

$$\bar{f}_q(s) = 2^{k+u}\bar{f}_q(2^k s) + \sum_{l=0}^{k-1} \delta_1 \cdots \delta_l 2^l \bar{g}_{q^{(l)}}(2^l s).$$

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Applying Mellin transform

$$\mathscr{M}[\bar{f}_q(s);\omega] = \frac{1}{1-2^{k-\omega k+u}} \sum_{l=0}^{k-1} \delta_1 \cdots \delta_l 2^{l-\omega l} \mathscr{M}[\bar{g}_{q^{(l)}}(s);\omega].$$

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The analysis is completed by Inverse Mellin + Laplace.

Variance

Theorem

For u *the number of unspecified coordinates,*

$$\mathrm{Var}(X_{q,n}) \sim n^{u/k} P_2(\log_2 n^{1/k}),$$

where $P_2(z)$ *is one-periodic with Fourier coefficients*

$$\frac{1}{kL\Gamma(1+\omega_r)} \sum_{l=0}^{k-1} \delta_1 \dots \delta_l 2^{-\omega_r l} \int_0^\infty \frac{s^{\omega_r}}{Q(-2s)^b} \left(\int_0^\infty e^{-zs} \tilde{h}_q^l(z) \mathrm{d}z + p(s) \right) \mathrm{d}s$$

with $p(s)$ *a polynomial and* $\tilde{h}_q^l(z)$ *a function of the Poisson generating function of the mean.*

Fourier coefficients

Corollary

For $b = 1$,

$$\frac{1}{kLQ_\infty\Gamma(1+\omega_r)}\sum_{l=0}^{k-1}\delta_1\cdots\delta_l 2^{-\omega_r l}$$

$$\sum_{j_1,j_2,j_3\geq 0}\frac{(-1)^{j_1}\bar{\delta}_{q^{(l)},j_2}\bar{\delta}_{q^{(l)},j_3}2^{-\binom{j_1}{2}+(1-\omega_r)j_1}}{2^{j_2+j_3}Q_{j_1}Q_{j_2}Q_{j_3}}\varphi(\omega_r;2^{j_1-j_2}+2^{j_1-j_3})$$

with

$$\bar{\delta}_{q,j}=\sum_{l\geq 0}\frac{(-1)^l 2^{-\binom{l+1}{2}}}{Q_l}\prod_{h=1}^{l+j}\delta_h.$$

and $\varphi(\omega;x)=\pi(x^\omega-1)/(\sin(\pi\omega)(x-1))$.

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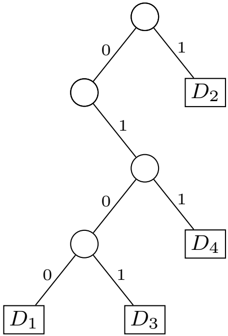

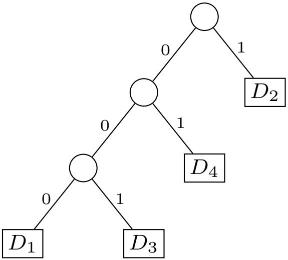

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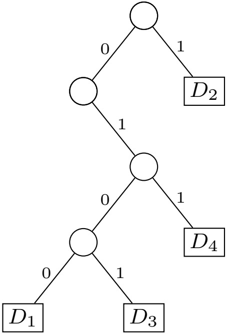

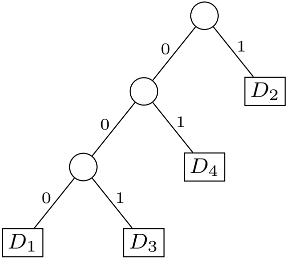

$X_{q,n} = \#$ of internal nodes visited by the query.

Previous Results for Tries

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- Schachinger proved the above conjecture.

Variance for Tries

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where $P_2(z)$ is one-periodic with Fourier coefficients

$$\frac{\Gamma(-\omega_r)}{kL}\left(\delta(2^{-\omega_r})\left(\binom{-\omega_r+b}{b} - 2^{\omega_r}\sum_{j_1,j_2=0}^{b}\binom{j_1+j_2}{j_1}\binom{-\omega_r+j_1+j_2}{j_1+j_2}2^{-j_1-j_2}\right)\right.$$
$$\left. - \sum_{l\geq b+1}\binom{-l+b}{b}\binom{-\omega_r+l+b}{b}\binom{\omega_r}{l}\frac{2^{1-l}\sigma(2^{-\omega_r},2^{-l}))}{1-2^{-lk+u}}\right),$$

where

$$\delta(z) = \sum_{j=0}^{k-1}\delta_1\cdots\delta_j z^j, \qquad \sigma(z_1,z_2) = \sum_{j_1,j_2=0}^{k-1}\delta_1\cdots\delta_{j_1+j_2+1} z_1^{j_1} z_2^{j_2}.$$