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On Kurzweil's 0-1 law in inhomogeneous Diophantine approximation

by

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1. Introduction and results. This paper is concerned with metric inhomogeneous Diophantine approximation. More precisely, we consider the inhomogeneous Diophantine approximation problem

$$\|n\theta - s\| < \psi(n) \tag{1.1}$$

whose number of solutions in $n \in \mathbb{N}$ is sought. Here and throughout this paper, $\theta, s \in \mathbb{R}$, $\|\cdot\|$ denotes the distance to the nearest integer and $\psi(n)$ is a (fixed) positive, non-increasing sequence which is called an *approximation sequence*. In addition, we will sometimes assume that $\psi(n)$ is a *Khintchine sequence*, which means that $n\psi(n)$ is non-increasing.

There are two different ways of looking at (1.1): (i) s is fixed and one is interested in the number of solutions for almost all θ (with respect to the Lebesgue measure), or (ii) θ is fixed and one is interested in the number of solutions for almost all s. Alternatively, one can also consider the number of solutions for almost all (θ, s) (with respect to the two-dimensional Lebesgue measure). However, we will not consider this "double-metric" situation in this paper.

First, we recall what is known for case (i). Here, it was proved by Khintchine [8] for Khintchine sequences and $s = 0$ (homogeneous Diophantine approximation) that (1.1) has either finitely many solutions in $n \in \mathbb{N}$ for almost all θ or infinitely many solutions in $n \in \mathbb{N}$ for almost all θ with the latter happening if and only if

$$\sum_{n\geq 1} \psi(n) = \infty.$$

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This result was extended to general s (inhomogeneous Diophantine approximation) by Szüsz [19]. Another extension was given by Schmidt [17] whose (very general) result in particular implies that the previous results of Khintchine and Szüsz hold for all non-increasing approximation sequences. This line of research was then extended in many different directions; see the monograph [6].

Case (ii) was considerably less studied. Here, in his pioneering work, Kurzweil [14] showed that also a 0-1 law holds.

THEOREM 1.1 (Kurzweil's 0-1 law [14]). *Let $\psi(n)$ be a positive, non-increasing sequence and θ be an irrational number. Then*

$$\|n\theta - s\| < \psi(n) \quad \textit{for infinitely many } n \in \mathbb{N} \tag{1.2}$$

either for almost all s or for almost no s.

It is an immediate consequence of the lemma of Borel–Cantelli that $\sum_{n\geq 1} \psi(n) = \infty$ is a necessary condition for (1.2). Thus, it is natural to ask when this is also sufficient. The answer to this question was also given by Kurzweil in [14] where he showed that the above condition is necessary and sufficient exactly for the set of badly approximable θ (Kurzweil's theorem). It seems that his paper was forgotten for a long time. However, in recent years, there was a revival of interest in his study with many follow-up papers; see for instance [4], [10], [20].

The main goal of this paper is to give a necessary and sufficient condition for (1.2) to hold for all θ (not only badly approximable θ). For Khintchine sequences such a result was already proved in [10] with a condition which cannot be used in the general case. In the present paper, we will find a new condition which does work for all positive, non-increasing sequences $\psi(n)$. More precisely, our main result reads as follows.

THEOREM 1.2. *Let $\psi(n)$ be a positive, non-increasing sequence and θ be an irrational number with principal convergents p_k/q_k. Then, for almost all $s \in \mathbb{R}$,*

$$\|n\theta - s\| < \psi(n) \quad \textit{for infinitely many } n \in \mathbb{N}$$

if and only if

$$\sum_{k=0}^{\infty} \Big(\sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|) \Big) = \infty. \tag{1.3}$$

This result contains several previous results as special cases: the above-mentioned theorem of Kurzweil as well as its extensions given in his paper [14] and by Tseng [20]. Below we recall these results and show that our result implies them.

Analogues of all the above results were also obtained in the field of formal Laurent series over a finite base field; see [3], [11], [12], [13], [16]. Indeed, an analogue of our Theorem 1.2 also holds in this situation, again implying many previous results. This will be shown below as well.

We conclude the introduction by giving a short sketch of the paper. In the next section, we will prove our main result. In Section 3, we will show that our result implies the previous ones of Tseng and Kurzweil (which will also be recalled in this section). In Section 4, we will consider Khintchine sequences $\psi(n)$ and show that in this case (1.3) is equivalent to the condition from the main result of [10]. In Section 5, we will discuss an analogue of our result in the field of formal Laurent series over a finite field (whose definition will be recalled in this section), and show relations of this analogue to previous results. Finally, we will give a conclusion in Section 6.

2. Proof of the main theorem. We first fix some notation. Let $X = \mathbb{R}/\mathbb{Z}$. Let $B(x, r)$ be the open ball in X centered at x with radius r. We denote by μ the Lebesgue measure on the unit circle X. Let $\psi(n)$ be a positive, non-increasing sequence and θ be an irrational with principal convergents p_k/q_k.

2.1. Proof of the convergence part. This part of Theorem 1.2 is a consequence of the following lemma.

LEMMA 2.1. *If*

$$\sum_{k=0}^{\infty}\Big(\sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|)\Big) < \infty,$$

then, for almost all $s \in \mathbb{R}$,

$$\|n\theta - s\| < \psi(n) \quad \textit{for finitely many } n \in \mathbb{N}.$$

Proof. In the proof (and also below), we will use the following well-known facts about the sequence q_k:

$$1/2 \leq q_{k+1}\|q_k\theta\| \leq 1 \quad \text{and} \quad q_{k+1} \geq 2q_{k-1}.$$

We will consider two cases.

In the first case, we assume that $\psi(q_{k+1} - 1) \geq \|q_k\theta\|$ for infinitely many k. Then, for such k, we have

$$\psi(n) \geq \psi(q_{k+1} - 1) \geq \|q_k\theta\|$$

for all $q_{k-1} \leq n < q_{k+1}$. Hence,

$$\sum_{n=q_{k-1}}^{q_k-1}\min(\psi(n),\|q_{k-1}\theta\|)+\sum_{n=q_k}^{q_{k+1}-1}\min(\psi(n),\|q_k\theta\|)$$
$$\geq (q_{k+1}-q_{k-1})\|q_k\theta\| \geq \frac{q_{k+1}}{2}\|q_k\theta\| \geq 1/4.$$

Since this happens infinitely often, the convergence assumption is violated and thus this case will not happen.

Therefore, we may assume that $\psi(q_k-1) < \|q_{k-1}\theta\|$ for all k large enough. In order to prove our claim in this case, set

$$E_{k+1} = \bigcup_{q_k \leq n < q_{k+1}} B(n\theta,\psi(n)).$$

Then

$$\bigcap_{N\geq 1}\bigcup_{n\geq N} B(n\theta,\psi(n)) = \bigcap_{K\geq 1}\bigcup_{k\geq K} E_k.$$

Since $\|n\theta-(n-q_k)\theta\| = \|q_k\theta\|$ and $\psi(n)$ is non-increasing, we observe for each $q_k \leq n < q_{k+1}$ that

$$\mu\big(B(n\theta,\psi(n)) \setminus B((n-q_k)\theta,\psi(n-q_k))\big) \leq \|q_k\theta\|.$$

For each $q_k \leq n < q_{k+1}$ we also have

$$\mu\big(B(n\theta,\psi(n)) \setminus B((n-q_k)\theta,\psi(n-q_k))\big) \leq \mu(B(n\theta,\psi(n))) = 2\psi(n).$$

Thus,

$$\begin{aligned}\mu(E_{k+1}) &\leq \sum_{n=q_k}^{2q_k-1}\mu(B(n\theta,\psi(n)))\\ &\quad+\sum_{n=2q_k}^{q_{k+1}-1}\mu\big(B(n\theta,\psi(n)) \setminus B((n-q_k)\theta,\psi(n-q_k))\big)\\ &\leq 2q_k\psi(q_k)+\sum_{n=2q_k}^{q_{k+1}-1}\min(2\psi(n),\|q_k\theta\|).\end{aligned}$$

Now, from $\psi(q_k-1) < \|q_{k-1}\theta\|$,

$$\begin{aligned}q_k\psi(q_k) &\leq 2(q_k-q_{k-2})\psi(q_k-1) = 2(q_k-q_{k-2})\min(\psi(q_k-1),\|q_{k-1}\theta\|)\\ &\leq 2\Big(\sum_{n=q_{k-2}}^{q_{k-1}-1}\min(\psi(n),\|q_{k-2}\theta\|)+\sum_{n=q_{k-1}}^{q_k-1}\min(\psi(n),\|q_{k-1}\theta\|)\Big).\end{aligned}$$

Since this holds for all large k, we have

$$\sum_k \mu(E_{k+1}) < \infty.$$

Hence, the first Borel–Cantelli lemma completes the proof. ∎

2.2. Proof of the divergence part. Now, we prove the second half of Theorem 1.2. First, for each $n \in \mathbb{N}$ denote by $h(n)$ the non-increasing sequence

$$h(n) := \min(\psi(n), \|q_k\theta\|), \qquad q_k \le n < q_{k+1}.$$

Let, for $0 \le i < a_{k+1}$,

$$G_{k,i} := \bigcup_{q_{k+1}-(i+1)q_k < n \le q_{k+1}-iq_k} B\left(n\theta, \frac{h(q_{k+1}-iq_k)}{2}\right), \qquad G_k := \bigcup_{i=0}^{a_{k+1}-1} G_{k,i}.$$

The balls in G_k are disjoint since any two points in $\{n\theta : 1 \le n \le q_{k+1}\}$ are separated by at least $\|q_k\theta\|$.

Lemma 2.2. *If*

$$\sum_{k=0}^{\infty} \mu(G_k) = \infty,$$

then

$$\mu\Big(\bigcap_{K\ge 1} \bigcup_{k\ge K} G_k\Big) = 1.$$

Proof. We estimate $\mu(G_\ell \cap G_k)$, $\ell < k$, by the Denjoy–Koksma inequality (see, e.g., [7]). Let T be an irrational rotation by θ and f be a real-valued function of bounded variation on the unit interval. Then, for any x, we have

$$\Big|\sum_{n=0}^{q_k-1} f(T^n x) - q_k \int f\, d\mu\Big| \le \mathrm{var}(f). \tag{2.1}$$

For a given interval I, by the Denjoy–Koksma inequality (2.1), we have

$$\#\{0 \le n < q_k : n\theta \in I\} = \sum_{n=0}^{q_k-1} 1_I(T^n x) \le q_k\mu(I) + 2.$$

Since $G_{k,i}$ consists of the disjoint balls centered at q_k orbital points with radius $r := h(q_{k+1} - iq_k)/2$, for each i we have

$$\begin{aligned}\mu(G_{k,i} \cap I) &\le \#\{0 \le n < q_k : n\theta \in I\} \cdot 2r + 2r\\ &\le (q_k\mu(I) + 3) \cdot 2r = \mu(G_{k,i})\mu(I) + \frac{3}{q_k}\mu(G_{k,i}).\end{aligned}$$

Note that G_ℓ consists of at most $q_{\ell+1}$ intervals.

Therefore, for $k > \ell$ we have

$$\mu(G_{k,i} \cap G_\ell) \le \mu(G_{k,i})\mu(G_\ell) + \frac{3q_{\ell+1}}{q_k}\mu(G_{k,i}).$$

Since $G_k = \bigcup G_{k,i}$ is a disjoint union, we have

$$\begin{aligned}\mu(G_k \cap G_\ell) &\le \mu(G_k)\mu(G_\ell) + \frac{3q_{\ell+1}}{q_k}\mu(G_k)\\ &\le \mu(G_k)\mu(G_\ell) + 3\left(\frac{1}{2}\right)^{\lfloor (k-\ell-1)/2 \rfloor}\mu(G_k)\\ &\le \mu(G_k)\mu(G_\ell) + \frac{6}{2^{(k-\ell)/2}}\mu(G_k).\end{aligned}$$

We need a version of the Borel–Cantelli lemma (e.g. [6, 18]) to go further:

LEMMA 2.3. *Let (Ω, μ) be a measure space, let $f_k(\omega)$ $(k = 1, 2, \dots)$ be a sequence of non-negative μ-measurable functions, and let φ_k be a sequence of real numbers such that $0 \le \varphi_k \le 1$ $(k = 1, 2, \dots)$. Suppose that*

$$\int_\Omega \Big(\sum_{m<k\le n} f_k(\omega) - \sum_{m<k\le n} \varphi_k\Big)^2 d\mu \le C \sum_{m<k\le n} \varphi_k$$

for arbitrary integers $m < n$. Then

$$\sum_{1\le k\le n} f_k(\omega) = \Phi(n) + \mathcal{O}(\Phi^{1/2}(n) \ln^{3/2+\varepsilon} \Phi(n))$$

for almost all $\omega \in \Omega$, where $\varepsilon > 0$ is arbitrary and $\Phi(n) = \sum_{1\le k\le n} \varphi_k$.

Let $\varphi_k = \mu(G_k)$ and $f_k(\omega) = 1_{G_k}(\omega)$ in Lemma 2.3. Then, for any $m < n$, we have

$$\begin{aligned}&\int \Big(\sum_{m<k\le n} f_k(\omega) - \sum_{m<k\le n} \varphi_k\Big)^2 d\mu\\ &\qquad \le 2 \sum_{m<\ell<k\le n} \big(\mu(G_k \cap G_\ell) - \mu(G_k)\mu(G_\ell)\big) + \sum_{m<k\le n} \mu(G_k)\\ &\qquad \le 2 \sum_{m<k\le n}\sum_{m<\ell<k} \frac{6}{2^{(k-\ell)/2}}\mu(G_k) + \sum_{m<k\le n} \mu(G_k)\\ &\qquad \le \left(\frac{12}{\sqrt{2}-1} + 1\right) \sum_{m<k\le n} \mu(G_k).\end{aligned}$$

Therefore, by Lemma 2.3, if

$$\sum_k \mu(G_k) = \infty,$$

then, for almost every ω,

$$\sum_{k=1}^{\infty} 1_{G_k}(\omega) = \infty,$$

i.e., $\omega \in G_k$ for infinitely many k's. ∎

LEMMA 2.4. *If* $\sum_{n=1}^{\infty} h(n) = \infty$, *then*

$$\sum_{k=0}^{\infty} \mu(G_k) = \infty.$$

Proof. For $k \geq 0$ we have

$$\begin{aligned}
\sum_{n=q_k}^{q_{k+1}-1} h(n) &= \sum_{n=q_k}^{q_k+q_{k-1}-1} h(n) + \sum_{i=1}^{a_{k+1}-1} \Big(\sum_{n=iq_k+q_{k-1}}^{(i+1)q_k+q_{k-1}-1} h(n) \Big) \\
&\leq q_{k-1} h(q_k) + \sum_{i=1}^{a_{k+1}-1} q_k h(iq_k + q_{k-1}) \\
&= q_{k-1} h(q_k) + \sum_{i=1}^{a_{k+1}} q_k h(iq_k + q_{k-1}) - q_k h(q_{k+1}) \\
&= q_{k-1} h(q_k) + \mu(G_k) - q_k h(q_{k+1}),
\end{aligned}$$

where $q_{-1} = 0$; therefore

$$\sum_{k=0}^{K} \sum_{n=q_k}^{q_{k+1}-1} h(n) + q_K h(q_{K+1}) \leq \sum_{k=0}^{K} \mu(G_k).$$

From this the claim follows. ■

Since

$$\begin{aligned}
G_k &= \bigcup_{i=0}^{a_{k+1}-1} \Big(\bigcup_{q_{k+1}-(i+1)q_k < n \leq q_{k+1} - iq_k} B\Big(n\theta, \frac{f(q_{k+1} - iq_k)}{2}\Big)\Big) \\
&\subseteq \bigcup_{q_{k-1} < n \leq q_{k+1}} B(n\theta, \psi(n)),
\end{aligned}$$

we have

$$\bigcap_{K \geq 0} \bigcup_{k \geq K} G_k \subseteq \bigcap_{N \geq 1} \bigcup_{n \geq N} B(n\theta, \psi(n)).$$

Therefore, $\sum_{n=1}^{\infty} f(n) = \infty$ implies that

$$\mu\Big(\bigcap_{N \geq 1} \bigcup_{n \geq N} B(n\theta, \psi(n))\Big) = 1.$$

This concludes the proof of the divergence part.

3. The theorems of Tseng and Kurzweil. In this section, we will give several consequences of Theorem 1.2. More precisely, we will show that our result contains three previous theorems. One of them is Kurzweil's theorem mentioned in the introduction. The other two are generalizations of

Kurzweil's result: the first is due to Tseng [20] and the second is due to Kurzweil himself [14]. We start by introducing these two results.

First, we explain Tseng's theorem. We need the following notation:

$$\Omega^{(\tau)} := \{\theta \in \mathbb{R} : \text{there exists } c > 0 \text{ with } \|n\theta\| \geq c/n^{\tau} \text{ for all } n \geq 1\}.$$

Note that this definition slightly differs from [20], where τ was replaced by $\tau - 1$. Also, note that $\tau = 1$ is by definition the set of badly approximable numbers. Moreover, we let

$$\Theta^{(\tau)} := \Big\{\theta \in \mathbb{R} : (1.2) \text{ holds for all } \psi(n) \text{ with } \sum_{n\geq 1} \psi(n)^{\tau} = \infty\Big\}.$$

Now, we can state Tseng's theorem.

Theorem 3.1 (Tseng [20]). *For $\tau \geq 1$, we have*

$$\Omega^{(\tau)} = \Theta^{(\tau)}.$$

Note that for $\tau = 1$ this is Kurzweil's theorem. In [14], Kurzweil himself gave a generalization of his theorem. To state it, again we need some notation. First, consider a sequence $\varphi(n)$ with

$$n\varphi(n) \text{ non-increasing,} \tag{3.1}$$

$$0 < n^2\varphi(n) \leq 1 \text{ for } n \geq 1. \tag{3.2}$$

For such a sequence, we define

$$\Omega^{(\varphi)} := \{\theta \in \mathbb{R} : \text{there exists } c > 0 \text{ with } \|n\theta\| \geq n\varphi(cn) \text{ for all } n \geq 1\}.$$

Moreover, we consider positive, non-increasing sequences $\psi(n)$ such that there exists an increasing sequence t_i and a non-decreasing function $\delta(n) \geq 1$ which tends to infinity as n tends to infinity with

$$t_{i+1} \geq \frac{1}{t_i\varphi(t_i\delta(t_i))} \tag{3.3}$$

and

$$\sum_{i\geq 1} t_i\psi\left(\left\lfloor \frac{1}{t_i\varphi(t_i\delta(t_i))}\right\rfloor\right) = \infty. \tag{3.4}$$

For such sequences, we define

$$\Xi^{(\varphi)} := \{\theta \in \mathbb{R} : (1.2) \text{ holds for all } \psi(n) \text{ with the above properties}\}.$$

Kurzweil's result in [14] reads as follows.

Theorem 3.2 (Kurzweil [14]). *We have*

$$\Omega^{(\varphi)} = \Xi^{(\varphi)}.$$

Note that $\Omega^{(1/n^{\tau+1})} = \Omega^{(\tau)}$. However, it was shown in [20] that the sets of $\psi(n)$ involved in the definition of $\Theta^{(\tau)}$ and $\Xi^{(1/n^{\tau+1})}$ are different except in the case $\tau = 1$, making Tseng's theorem more than just a mere special case of Kurzweil's Theorem 3.2.

Both of the above result are consequences of Theorem 1.2, as will be shown next.

3.1. $\theta \in \Omega^{(*)}$. Here, we show that if $\theta \in \Omega^{(\tau)}$ or $\Omega^{(\varphi)}$, then $\theta \in \Theta^{(\tau)}$ or $\Xi^{(\varphi)}$, respectively.

We first consider Tseng's theorem, and as a warm-up we deal with the case $\tau = 1$ (Kurzweil's theorem).

LEMMA 3.3. *We have*

$$\Omega^{(1)} \subseteq \Theta^{(1)}.$$

Proof. First recall that $\theta \in \Omega^{(1)}$ means that there exists a $c > 0$ such that $\|n\theta\| \geq c/n$ for all $n \geq 1$. Thus, for $q_k \leq n < q_{k+1}$,

$$\|q_k\theta\| \geq c/q_k \geq c/n.$$

This implies (1.3) provided that

$$\sum_{n\geq 1} \min(\psi(n), c/n) = \infty.$$

By Cauchy's condensation principle, the latter is equivalent to showing that

$$\sum_{n\geq 0} \min(2^n\psi(2^n), c) = \infty.$$

This in turn follows from $\sum_{n\geq 0} 2^n\psi(2^n) = \infty$, which again by Cauchy's condensation principle is equivalent to the assumption. ∎

We now generalize this to general τ.

LEMMA 3.4. *For $\tau \geq 1$, we have*

$$\Omega^{(\tau)} \subseteq \Theta^{(\tau)}.$$

Proof. First, note that with the same argument as in the proof of Lemma 2.1, the claim holds when $\psi(q_{k+1} - 1) \geq \|q_k\theta\|$ for infinitely many k. Thus, in what follows, we may assume that $\psi(q_k - 1) < \|q_{k-1}\theta\|$ for all $k \geq k_0 \geq 1$. Fix $q_k \leq n < q_{k+1}$. Then

$$\psi(n)^\tau \leq \psi(q_k - 1)^\tau < \|q_{k-1}\theta\|^\tau \leq 1/q_k^\tau \leq \|q_k\theta\|/c,$$

where $c > 0$ is such that $\|n\theta\| \geq c/n^\tau$ for all $n \geq 1$. Hence,

$$\sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|) \geq \min(c, 1) \cdot \sum_{n=q_k}^{q_{k+1}-1} \psi(n)^\tau.$$

Summing over $k \geq k_0$ gives

$$\sum_{k\geq k_0}\sum_{n=q_k}^{q_{k+1}-1}\min(\psi(n),\|q_k\theta\|)\geq \min(c,1)\cdot\sum_{n\geq q_{k_0}}\psi(n)^{\tau}=\infty,$$

which proves the claim also in this case. ∎

We next show that Theorem 1.2 implies one direction of Theorem 3.2.

LEMMA 3.5. *We have*

$$\Omega^{(\varphi)}\subseteq \Xi^{(\varphi)}.$$

Proof. First note that, as above, we can assume $\psi(q_k-1)<\|q_{k-1}\theta\|$ for all k large enough.

Now, consider $t_{i-1}\leq n<t_i$. Observe that by (3.2) and the assumption on $\delta(n)$, we have

$$\frac{1}{t_i\varphi(t_i\delta(t_i))}\geq t_i\delta^2(t_i)\geq t_i. \tag{3.5}$$

Thus,

$$\psi(n)\geq \psi\left(\left\lfloor\frac{1}{t_i\varphi(t_i\delta(t_i))}\right\rfloor\right). \tag{3.6}$$

Next, define i_s such that

$$q_{i_s-1}<t_i\leq q_{i_s}.$$

Note that from the assumptions on $\varphi(n)$ and the properties of principal convergents stated at the beginning of the proof of Lemma 2.1, we have

$$q_{i_s-1}\varphi(cq_{i_s-1})\leq \|q_{i_s-1}\theta\|\leq 1/q_{i_s}.$$

From this and (3.1), we obtain

$$q_{i_s}\leq \frac{1}{q_{i_s-1}\varphi(cq_{i_s-1})}\leq \frac{c}{t_i\delta(t_i)\varphi(t_i\delta(t_i))}\leq \frac{1}{t_i\varphi(t_i\delta(t_i))}$$

for i large enough. Thus, for $q_k\leq n<q_{k+1}$, we have

$$\|q_k\theta\|\geq \|q_{i_s-1}\theta\|>\psi(q_{i_s}-1)\geq\psi(q_{i_s})\geq \psi\left(\left\lfloor\frac{1}{t_i\varphi(t_i\delta(t_i))}\right\rfloor\right).$$

Combining the latter with (3.6) yields, for n with $t_{i-1}\leq n<t_i$ in the series of (1.3), the lower bound

$$(t_i-t_{i-1})\psi\left(\left\lfloor\frac{1}{t_i\varphi(t_i\delta(t_i))}\right\rfloor\right)$$

for i large enough. Since, from (3.3) and (3.5), we have

$$t_i \geq t_{i-1}\delta^2(t_{i-1}) \geq 2t_{i-1}$$

for i large enough, we see that a remainder of the series in (1.3) has a remainder of the series in (3.4) as lower bound, which proves the desired result. ∎

3.2. $\theta \notin \Omega^{(*)}$**.** Here, we have to show that there exists a positive, non-increasing sequence $\psi(n)$ satisfying $\sum_{n\geq 1}\psi(n)^\tau = \infty$ in the case of Tseng's theorem, and the condition above Theorem 3.2 in the case of that theorem, such that (1.2) does not hold. Such sequences have already been constructed by Tseng and Kurzweil in the proof of their results. One only has to check that these sequences do not satisfy (1.3). Since the check is the same for all of them, we only give details for Tseng's construction which we recall next.

First, since $\theta \notin \Omega^{(\tau)}$, there exists a sequence of positive integers v_ℓ with $v_{\ell+1} \geq 2v_\ell$ and

$$\|v_\ell\theta\| \leq \frac{1}{2\ell^{2\tau+2}v_\ell^\tau}. \tag{3.7}$$

Now, set $u_\ell = \lfloor \ell^{2\tau}v_\ell^\tau \rfloor$ and, for $u_\ell \leq n < u_{\ell+1}$,

$$\psi(n) = 2^{-1}(\ell+1)^{-2}v_{\ell+1}^{-1}.$$

Obviously,

$$\sum_{n=u_\ell}^{u_{\ell+1}-1}\psi(n)^\tau \geq c$$

for some constant c, and hence $\sum_{n\geq 1}\psi(n)^\tau = \infty$. Next, in order to show that (1.3) does not hold, for $q_k \leq n < q_{k+1}$ we set

$$h(n) = \min\{\psi(n), \|q_k\theta\|\}.$$

Thus,

$$\begin{aligned}\sum_{n=u_\ell}^{u_{\ell+1}-1}h(n) &= \sum_{n=u_\ell}^{v_{\ell+1}-1}h(n) + \sum_{n=v_{\ell+1}}^{u_{\ell+1}-1}h(n) \leq v_{\ell+1}\psi(u_\ell) + u_{\ell+1}\|v_{\ell+1}\theta\| \\ &\leq \frac{v_{\ell+1}}{2(\ell+1)^2v_{\ell+1}} + \frac{u_{\ell+1}}{2(\ell+1)^{2\tau+2}v_{\ell+1}^\tau} \leq \frac{1}{(\ell+1)^2};\end{aligned}$$

we used (3.7) in the above estimate. Summing over ℓ shows that (1.3) does not hold, as required.

4. Khintchine sequences. In this section, we assume that $\psi(n)$ is a Khintchine sequence, i.e., $\psi(n) = 1/(n\phi(n))$ with $\phi(n)$ non-decreasing. For this special case, the second author proved in [10] the following result.

THEOREM 4.1 (Kim [10]). *Let $\phi(n)$ be a positive, non-decreasing sequence which tends to infinity, and θ be an irrational number with principal convergents p_k/q_k. Then, for almost all $s \in \mathbb{R}$,*

$$\|n\theta - s\| < \frac{1}{n\phi(n)} \qquad \text{for infinitely many } n \in \mathbb{N}$$

if and only if

$$\sum_{k=0}^{\infty} \frac{\log\big(\min(\phi(q_k), q_{k+1}/q_k)\big)}{\phi(q_k)} = \infty.$$

REMARK 4.2. By using the main result of [9] and replacing $\log x$ by $\operatorname{Log} x := \max\{\log x, 0\}$, the assumption that $\phi(n)$ tends to infinity can be dropped (if $\phi(n)$ is bounded, then there are always an infinite number of solutions). Moreover, this can also be obtained by Minkowski's inhomogeneous approximation theorem (see, e.g., [2, p. 48]) and Cassels' lemma [6, Lemma 2.1]. Note that this situation is also covered by our main result. More precisely, if $\phi(n) \le c$ for some $c \ge 2$, then we have

$$\begin{aligned}\sum_{k=0}^{\infty}\Bigg(\sum_{n=q_k}^{q_{k+1}-1} \min\Big(\frac{1}{n\phi(n)}, \|q_k\theta\|\Big)\Bigg) &\ge \sum_{k=0}^{\infty}\Bigg(\sum_{n=q_{2k}}^{q_{2k+2}-1} \min\Big(\frac{1}{cn}, \|q_{2k+1}\theta\|\Big)\Bigg)\\ &\ge \sum_{k=0}^{\infty}\Bigg(\sum_{n=q_{2k}}^{q_{2k+2}-1} \frac{1}{cq_{2k+2}}\Bigg) = \sum_{k=0}^{\infty} \frac{q_{2k+2}-q_{2k}}{cq_{2k+2}} \ge \sum_{k=0}^{\infty} \frac{1}{2c} = \infty.\end{aligned}$$

Theorem 4.1 is indeed a special case of our Theorem 1.2 since the following proposition holds.

PROPOSITION 4.3. *Under the assumptions of the above theorem,*

$$\sum_{k=0}^{\infty} \frac{\log\big(\min(\phi(q_k), q_{k+1}/q_k)\big)}{\phi(q_k)} = \infty$$

if and only if

$$\sum_{k=0}^{\infty}\Bigg(\sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|)\Bigg) = \infty,$$

where $\psi(n) = 1/(n\phi(n))$.

Proof. Since we assume that $\phi(n)$ tends to infinity, $\psi(q_{k+1}-1) < \|q_k\theta\|$ for large enough k as before. For such a large k let

$$q_k^* = \min\{q_k \le n < q_{k+1} : \psi(n) < \|q_k\theta\|\}.$$

Then

$$\frac{1}{q_k^*} < \phi(q_k^*)\|q_k\theta\| \le \frac{\phi(q_k^*)}{q_{k+1}},$$

and if $q_k^* \geq q_k + 1$, then

$$\frac{2}{q_k^*} \geq \frac{1}{q_k^* - 1} \geq \phi(q_k^* - 1)\|q_k\theta\| \geq \phi(q_k)\|q_k\theta\| \geq \frac{\phi(q_k)}{2q_{k+1}}.$$

Therefore,

$$\min\left\{\frac{q_{k+1}}{q_k}, \frac{\phi(q_k)}{4}\right\} \leq \frac{q_{k+1}}{q_k^*} \leq \min\left\{\frac{q_{k+1}}{q_k}, \phi(q_k^*)\right\},$$

and if $q_k^* \geq q_k + 1$ and $\phi(q_k) \geq 2e$, then

$$\begin{aligned}(q_k^* - q_k)\|q_k\theta\| \leq (q_k^* - 1)\|q_k\theta\| &\leq \frac{1}{\phi(q_k)} \leq \frac{\log(\phi(q_k)/2)}{\phi(q_k)} \\ &\leq \frac{\log(q_{k+1}\|q_k\theta\|\phi(q_k))}{\phi(q_k)} \leq \frac{\log(q_{k+1}/q_k)}{\phi(q_k)}.\end{aligned}$$

If we consider ϕ as a function on $\mathbb{R}$, then for large k such that $\phi(q_k) \geq 16$, we have

$$\begin{aligned}\sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|) &\geq \sum_{n=q_k^*}^{q_{k+1}-1} \frac{1}{n\phi(n)} \\ &\geq \int_{q_k^*}^{q_{k+1}} \frac{dx}{x\phi(x)} = \int_{\log q_k^*}^{\log q_{k+1}} \frac{dt}{\phi(e^t)} \geq \frac{\log(q_{k+1}/q_k^*)}{\phi(q_{k+1})} \\ &\geq \frac{\log\big(\min(\phi(q_k)/4, q_{k+1}/q_k)\big)}{\phi(q_{k+1})} \geq \frac{\log\big(\min(\phi(q_k), q_{k+1}/q_k)\big)}{2\phi(q_{k+1})},\end{aligned}$$

and for $\phi(q_k) \geq 2e$,

$$\begin{aligned}\sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|) &= (q_k^* - q_k)\|q_k\theta\| + \frac{1}{q_k^*\phi(q_k^*)} + \sum_{n=q_k^*+1}^{q_{k+1}-1} \frac{1}{n\phi(n)} \\ &\leq \frac{\log\big(\min(\phi(q_k), q_{k+1}/q_k)\big)}{\phi(q_k)} + \frac{1}{q_k\phi(q_k)} + \int_{q_k^*}^{q_{k+1}} \frac{dx}{x\phi(x)} \\ &\leq \frac{\log\big(\min(\phi(q_k), q_{k+1}/q_k)\big)}{\phi(q_k)} + \int_{q_{k-1}^*}^{q_k} \frac{dx}{x\phi(x)} + \frac{\log(q_{k+1}/q_k^*)}{\phi(q_k^*)} \\ &\leq \frac{\log\big(\min(\phi(q_k), q_{k+1}/q_k)\big)}{\phi(q_k)} + \frac{\log(q_k/q_{k-1}^*)}{\phi(q_{k-1}^*)} + \frac{\log\big(\min(\phi(q_k^*), q_{k+1}/q_k)\big)}{\phi(q_k^*)} \\ &\leq \frac{2\log\big(\min(\phi(q_k), q_{k+1}/q_k)\big)}{\phi(q_k)} + \frac{\log\big(\min(\phi(q_{k-1}), q_k/q_{k-1})\big)}{\phi(q_{k-1})}.\end{aligned}$$

Therefore, for some $k_0 \geq 1$, we have

$$\sum_{k>k_0} \frac{\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{2\phi(q_{k+1})} \leq \sum_{k>k_0} \sum_{n=q_k}^{q_{k+1}-1} \min(\psi(n), \|q_k\theta\|) \leq \sum_{k\geq k_0}^{\infty} \frac{3\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{\phi(q_k)}.$$

Let $\Lambda = \{k \geq 1 : \phi(q_{k+1}) \leq 2\phi(q_k)\}$. Then

$$\sum_{k\in\Lambda} \frac{\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{\phi(q_{k+1})} \leq \sum_{k\in\Lambda} \frac{\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{\phi(q_k)} \leq 2\sum_{k\in\Lambda} \frac{\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{\phi(q_{k+1})}$$

and

$$\sum_{k\in\Lambda^c} \frac{\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{\phi(q_{k+1})} \leq \sum_{k\in\Lambda^c} \frac{\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{\phi(q_k)} \leq \sum_{k\in\Lambda^c} \frac{\log\phi(q_k)}{\phi(q_k)} < \infty.$$

Hence,

$$\sum_{k=0}^{\infty} \frac{\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{\phi(q_{k+1})} = \infty$$

if and only if

$$\sum_{k=0}^{\infty} \frac{\log\bigl(\min(\phi(q_k), q_{k+1}/q_k)\bigr)}{\phi(q_k)} = \infty,$$

which completes the proof. ■

5. An analogue in the field of formal Laurent series. In this section, we will briefly discuss an analogue of our Theorem 1.2 in the field of formal Laurent series. As in the real case, this analogue will imply the analogues of Kurzweil's theorem and its extensions as well as the analogue of Kim's theorem [10] which have all been established in the field of formal Laurent series.

We start by recalling the definition of the field of formal Laurent series; for further details see [5]. First, denote by $\mathbb{F}_q$ the finite field of q elements, where q is a prime power. Moreover, let $\mathbb{F}_q[X]$ be the polynomial ring over $\mathbb{F}_q$ and denote by $\mathbb{F}_q(X)$ its quotient field. The field of formal Laurent series is defined by

$$\mathbb{F}_q((X^{-1})) := \{f = a_nX^n + a_{n-1}X^{n-1} + \cdots,\ a_i \in \mathbb{F}_q,\ a_n \neq 0,\ n \in \mathbb{Z}\} \cup \{0\}$$

with addition and multiplication defined as for polynomials. We set $\{f\} = a_{-1}X^{-1} + \cdots$, which is called the *fractional part* of f. Moreover, we define a norm by setting $|f| = q^{\deg(f)}$, where $\deg(f)$ is the generalized degree function (by definition $|0| := 0$). This norm is non-Archimedean. Next, set

$$\mathbb{L} := \{f \in \mathbb{F}_q((X^{-1})) : |f| < 1\}.$$

Restricting the norm to $\mathbb{L}$ gives a compact topological group. Thus, there exists a unique, translation-invariant probability measure (the Haar measure).

Metric Diophantine approximation is now done in $\mathbb{L}$ equipped with the above measure with integers replaced by elements of $\mathbb{F}_q[X]$ and real numbers replaced by elements in $\mathbb{L}$. In particular, the inhomogeneous Diophantine approximation problem (1.1) in this setting becomes

$$|\{Qf\} - g| < \frac{1}{q^{l_n}}, \qquad \deg(Q) = n,$$

where $f, g \in \mathbb{L}$ and solutions are sought in $Q \in \mathbb{F}_q[X]$. Here, l_n is a non-negative sequence of integers which plays the role of the approximation sequence.

In this setting, our Theorem 1.2 reads as follows.

THEOREM 5.1. *Let l_n be a non-decreasing sequence and $f \in \mathbb{L}$ be irrational with principal convergents P_k/Q_k. Then, for almost all $g \in \mathbb{L}$,*

$$|\{Qf\} - g| < 1/q^{l_n}, \quad \deg(Q) = n \quad \textit{for infinitely many } Q \in \mathbb{F}_q[X]$$

if and only if

$$\sum_{k=0}^{\infty} \Big(\sum_{n=n_k}^{n_{k+1}-1} q^{n - \max\{n_{k+1}, l_n\}} \Big) = \infty,$$

where $n_k := \deg(Q_k)$.

We remark that under the stronger assumption that $l_n - n$ is non-decreasing this result was already proved in [12] (this corresponds to the case of a Khintchine sequence). Also, it was shown in [16] that the divergence part holds without the monotonicity assumption. The convergence part (which was conjectured in [16]) can be proved with a similar reasoning to the one above. It might be possible to remove also in this case the monotonicity assumption (as is frequently done for metric Diophantine approximation in the field of formal Laurent series), but we will not pursue this here.

We finish this section by pointing out that the above theorem implies the analogue of Kurzweil's theorem in the field of formal Laurent series which was proved in [3] and [11]. Moreover, the analogue of Tseng's theorem (proved in [13]) and the analogue of Theorem 3.2 (proved in [16]) are also deduced from the above theorem similarly to Section 3. Finally, our result again extends the analogue of Kim's theorem for Khintchine sequences which was obtained in [12]. This was already proved in [12] and the reader is referred to that paper for details.

6. Conclusion. The main goal of this paper was to prove a dual result to a classical result of Khintchine [8] and its extensions of Szüsz [19] and Schmidt [17]. Our result contains several previous results as special cases. Khintchine's theorem sparked a long line of research in metric Diophantine approximation. One could now ask: do some of the subsequent results have a "dual version" in the spirit of this paper, too?

For instance, for the convergence part ((1.2) holds for a set of s of Lebesgue measure zero), is it possible to compute the Hausdorff dimension of the set of s (similar to, e.g., [15])? By using the mass transference principle of [1], one seems to get a lower bound. In the opposite direction, for the divergence part ((1.2) holds for a set of s of Lebesgue measure one), is there an asymptotic formula for the number of solutions (in the style of, e.g., [17])? We might come back to these questions in a future work.

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Abstract (will appear on the journal's web site only)

We give a necessary and sufficient condition such that, for almost all $s \in \mathbb{R}$,

$$\|n\theta - s\| < \psi(n) \quad \text{for infinitely many } n \in \mathbb{N},$$

where θ is fixed and $\psi(n)$ is a positive, non-increasing sequence. This can be seen as a dual result to classical theorems of Khintchine and Szüsz which dealt with the situation where s is fixed and θ is random. Moreover, our result contains several earlier ones as special cases: two old theorems of Kurzweil, a theorem of Tseng and a recent result of the second author. We also discuss a similar result (with the same consequences) in the field of formal Laurent series.