Invariance Principles for Diophantine Approximation of Formal Laurent Series over A Finite Base Field

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Abstract

In a recent paper, the first and third author proved a central limit theorem for the number of coprime solutions of the diophantine approximation problem for formal Laurent series in the setting of the classical theorem of Khintchine. In this note, we consider a more general setting and show that even an invariance principle holds, thereby improving upon earlier work of the second author. Our result yields two consequences: (i) the functional central limit theorem and (ii) the functional law of the iterated logarithm. The latter is a refinement of Khintchine's theorem for formal Laurent series. Despite a lot of research efforts, the corresponding results for diophantine approximation of real numbers have not been established yet.

1 Introduction

The last few years have witnessed an increasing interest in the metric theory of diophantine approximation for formal Laurent series; for recent results concerning limit laws see Deligero and Nakada [1], Fuchs [3], [5], Inoue and Nakada [6]; for recent results concerning Hausdorff dimensions of exceptional sets see Kristensen [7], Niederreiter and Vielhaber [12], Wu [15].

In this short note, we are studying invariance principles for the number of coprime solutions of the diophantine approximation problem. In the classical case, invariance principles were obtained by Fuchs in [4]; see Fuchs [5] for corresponding results for formal Laurent series. The main difference to the previous

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line of research is a new approach that does not involve continued fraction expansion. Continued fraction expansion made necessary several restrictions on earlier results which will be shown to be superfluous in this paper. This new approach was devised by Deligero and Nakada in [1] and it is the paper's aim to further demonstrate its usefulness.

We give a short outline of the paper: in this section, we briefly recall metric diophantine approximation for formal Laurent series, state our new result and discuss some consequences. The proof of the main result which rests on blocking techniques and a general invariance principle obtained by Fuchs [4] will then be given in the final two sections.

Formal Laurent Series. Denote by $\mathbb{F}_q$ the finite field with q elements, where q is a power of p, p a prime. We consider the field of formal Laurent series

$$\mathbb{F}_q((T^{-1})) = \left\{ f = \sum_{n=n_0}^{\infty} a_n T^{-n} \Big| a_n \in \mathbb{F}_q, n_0 \in \mathbb{Z}, a_{n_0} \neq 0 \right\} \cup \{0\}$$

together with the valuation $|f| = q^{-n_0}$, $f \neq 0$ and $|0| = 0$. It is easy to see that $|\cdot|$ is non-Archimedean and that the polynomial ring $\mathbb{F}_q[T]$ and the field of rational functions $\mathbb{F}_q(T)$ are both contained in $\mathbb{F}_q((T^{-1}))$, where we have the chain of inclusions $\mathbb{F}_q[T] \subseteq \mathbb{F}_q(T) \subseteq \mathbb{F}_q((T^{-1}))$, a situation that closely resembles the corresponding chain $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

In order to consider metric diophantine approximation, we restrict to the set

$$\mathbb{L} = \left\{ f \in \mathbb{F}_q((T^{-1})) \big| |f| < 1 \right\}$$

as we restrict to the unit interval in the classical case. It is straightforward to prove that $\mathbb{L}$ together with the restriction of the valuation is a compact metric space. Hence, there exists a unique, translation-invariant probability measure on $(\mathbb{L}, \mathcal{L})$ ($\mathcal{L}$ denoting the set of all Borel sets) that we are going to denote by m.

Diophantine Approximation Problem and Three Sets. For f a formal Laurent series with $|f| < 1$, consider the diophantine approximation problem in unknowns $P, Q \in \mathbb{F}_q[T], Q \neq 0$,

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_n}}, \qquad \deg Q = n, \qquad (P, Q) = 1, \tag{1}$$

where (l_n) is a sequence of positive integers.

We are interested in studying the solution set. Results of different strengths made necessary different restrictions on the set of sequences (l_n). The sets which will be considered in this paper are as follows:

$\mathcal{A} = \left\{ (l_n)_{n \geq 0} \mid l_n > 0 \text{ and non-decreasing} \right\};$

$\mathcal{B} = \left\{ (l_n)_{n \geq 0} \mid l_n > 0 \text{ and either (C1) } \lim_{n \to \infty} l_n = l < \infty, \text{ or (C2) } \lim_{n \to \infty} l_n = \infty, \lim_{i \to \infty} \sum_{i < j \leq i+l_i} q^{-l_j} \text{ exists} \right\};$

$\mathcal{C} = \left\{ (l_n)_{n \geq 0} \mid l_n > 0 \right\}.$

Note that we have the following chain of proper inclusions $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$.

0-1 Laws. In [2], deMathan proved an analogue of Khintchine's theorem: for $(l_n) \in \mathcal{A}$ the solution set of the above inequality is either finite or infinite for almost all f, the latter holding if and only if $\sum_{n=0}^{\infty} q^{-l_n} = \infty$ (see Fuchs [3] for a different approach based on continued fraction expansions).

In a recent paper, Inoue and Nakada [6] showed that the monotonicity assumption is in fact superfluous (see Section 2 for a simplified proof of their result).

Theorem 1 (Inoue and Nakada [6]). *Let $(l_n) \in \mathcal{C}$. (1) has either finitely many or infinitely many solutions for almost all f; the latter holds if and only if*

$$\sum_{n=0}^{\infty} q^{-l_n} = \infty.$$

Central Limit Theorems. Define a sequence of random variables as

$$Z_N(f) := \# \left\{ P/Q : \langle P, Q \rangle \text{ is a solution of (1)}, \ \deg Q \leq N \right\}.$$

Assuming that $(l_n) \in \mathcal{A}$, $\sum_{n=0}^{\infty} q^{-l_n} = \infty$ and under some further technical conditions on (l_n), Fuchs [3] proved the central limit theorem for (Z_N). His approach was based on continued fraction expansions which made the additional conditions seemingly hard to drop.

A new approach, not relying on continued fraction expansions, was devised by Deligero and Nakada in [1]. With this approach they succeeded in dropping the additional conditions in Fuchs's result, thereby generalizing the central limit theorem to Khintchine's setting, i.e., to all sequences $(l_n) \in \mathcal{A}$ with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$. Note that a similar result for the real number field has not been proved yet; see LeVeque [9], [10] and Philipp [13] for similar but weaker results in the real case.

The Invariance Principle. In [5], Fuchs obtained the invariance principle for sequences $(l_n) \in \mathcal{A}$ that satisfy $\sum_{n=0}^{\infty} q^{-l_n} = \infty$ and some technical extra conditions. Here, we are going to explore further the approach of Deligero and Nakada in order to extend Fuchs's result to all sequences $(l_n) \in \mathcal{B}$ with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$.

In order to state the result we fix some notation. Set

$$F(N) := \begin{cases} q^{-2l-2} \left(q^{l+1}(q-1) - (2l+1)(q-1)^2 \right) N, & \text{if (C1);} \\ q^{-1}(q-1) \sum_{n \leq N} q^{-l_n}, & \text{if (C2),} \end{cases}$$

and

$$N_t := \begin{cases} \max\{n : F(n) \leq t\}, & \text{if } t \geq F(0); \\ 0, & \text{otherwise,} \end{cases}$$

for $t \geq 0$. Define on $(\mathbb{L}, \mathcal{L}, m) \times ([0,1], \bar{\mathcal{B}}, \lambda)$ the following stochastic process

$$Z(t) := Z(t; f, x) := Z_{N_t}(f) - \left(1 - \frac{1}{q} \right) \sum_{n=0}^{N} q^{-l_n},$$

where $\bar{\mathcal{B}}$ denotes the set of Borel sets on $[0,1]$ and λ is the Lebesgue measure. Note that the definition does not depend on the second variable. However, adjoining a uniformly distributed random variable is necessary to guaranteeing that the probability space is rich enough (see Remark 6 in Fuchs [4]).

Theorem 2. *There exists a sequence $(Y_n)_{n \geq 0}$ of independent, standard normal random variables on $(\mathbb{L}, \mathcal{L}, m) \times ([0,1], \bar{\mathcal{B}}, \lambda)$ such that, as $N \to \infty$,*

$$\left| Z(N) - \sum_{n \leq N} Y_n \right| = o\left((N \log \log N)^{1/2} \right), \quad a.s.$$

and

$$(m \times \lambda) \left[\frac{1}{\sqrt{N}} \max_{n \leq N} \left| Z(n) - \sum_{k \leq n} Y_k \right| \geq \epsilon \right] \to 0$$

for all $\epsilon > 0$.

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Consequences. The above result implies the functional central limit theorem which generalizes the result of Deligero and Nakada [1].

Corollary 1. *As $N \to \infty$,*

$$\left\{ \frac{Z(F(N)t)}{\sqrt{F(N)}}, 0 \leq t \leq 1 \right\} \longrightarrow \{W(t), 0 \leq t \leq 1\},$$

where $W(t)$ denotes the standard Brownian motion.

Moreover, we have the functional law of the iterated logarithm.

Corollary 2. *The sequence of functions*

$$\left\{ \frac{Z(F(N)t)}{(2F(N)\log\log F(N))^{1/2}}, \ 0 \leq t \leq 1 \right\}_{N \geq 0}$$

is a.s. relatively compact in the topology of uniform convergence and has Strassen's set as its set of limit points.

Since our set of sequences (l_n) contains the sequences of Khintchine's theorem, we note the following consequence of the latter result which is a refinement of Khintchine's theorem for formal Laurent series.

Corollary 3 (Law of the iterated logarithm for Khintchine's setting). *Assume that $(l_n) \in \mathcal{A}$ and $\sum_{n=0}^{\infty} q^{-l_n} = \infty$. Then, for almost all f,*

$$\limsup_{n \to \infty} \frac{\left| Z_N(f) - (1 - q^{-1}) \sum_{n \leq N} q^{-l_n} \right|}{\sqrt{2F(N)\log\log F(N)}} = 1.$$

Note that a similar result for the real number field has so far not been established; see Philipp [13] and Fuchs [4] for similar but weaker results in the real case. Moreover note that the above result also gives the optimal bound in the law of large numbers:

Let $(l_n) \in \mathcal{A}$. Then, for almost all f,

$$Z_N(f) = \left(1 - q^{-1}\right) \sum_{n \leq N} q^{-l_n} + \mathcal{O}\left((F(N)\log\log F(N))^{1/2} \right).$$

The previous best bound was of order $F(N)^{1/2}(\log F(N))^{3/2+\epsilon}, \epsilon > 0$ which more generally even holds for all $(l_n) \in \mathcal{C}$; see a remark by Inoue and Nakada [6].

2 Blocking

Define a sequence of sets as

$$F_n := \{ f \in \mathbb{L} : \exists \langle P, Q \rangle \text{ such that (1) holds} \}.$$

The measure of these sets was computed by Inoue and Nakada [6],

$$m(F_n) = q^{-l_n} \left(1 - \frac{1}{q} \right). \tag{2}$$

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Moreover, as was proved by Inoue and Nakada [6] as well, two distinct sets F_i and F_j are either independent or have empty intersection, the first case occurring if and only if $i + l_i < j$.

Note that the latter implies

$$m(F_i \cap F_j) \leq m(F_i)m(F_j) \qquad (i \neq j). \tag{3}$$

Sequences of sets satisfying this condition are called negative quadrant dependent (see Lehmann [8]). This gives a simplified proof of Theorem 1.

Proof of Theorem 1. Since $\sum_{n \leq N} m(F_n) = (1 - q^{-1}) \sum_{n \leq N} q^{-l_n}$ the result follows from the Borel-Cantelli lemma for negative quadrant dependent sequences of sets (see Matula [11] or Rényi [14]). ∎

In the sequel, we use the notation

$$X_n := \mathbf{1}_{F_n} - m(F_n),$$

where $\mathbf{1}_A$ denotes the indicator function of the set A. Furthermore, we set $\lim_{n \to \infty} l_n = l$ regardless whether we have (C1) or (C2). Subsequently, we shall interpret all expressions in terms of l for (C2) as the corresponding value obtained by taking the limit, e.g. $q^{-\infty} = 0$. Finally, the constant c is defined in the following lemma.

Lemma 1. *With the assumptions from the introduction,*

$$c := \lim_{i \to \infty} \sum_{i < j \leq i + l_i} q^{-l_j} = lq^{-l}.$$

Proof. If we assume (C1), then the assertion follows from the fact that $l_n = l, n \geq N$ for a sufficiently large N. For (C2), since the limit is assumed to exist, it suffices to prove that

$$\liminf_{i \to \infty} \sum_{i < j \leq i + l_i} q^{-l_j} = 0.$$

Assume that this is wrong. Then there is an $\epsilon > 0$ such that for all $i \geq i(\epsilon)$,

$$\sum_{i < j \leq i + l_i} q^{-l_j} \geq \epsilon.$$

If $l_i \leq l_{i+1} \leq \cdots \leq l_{i+l_i}$ then

$$\sum_{i < j \leq i + l_i} q^{-l_j} \leq l_i q^{-l_i}.$$

Since $l_n \to \infty$, the above chain of inequalities cannot hold if $i(\epsilon)$ is chosen large enough. Hence, starting with any fixed $i_0 \geq i(\epsilon)$, we can find an $i_1 > i_0$ such that $l_{i_0} > l_{i_1}$ etc. This gives a contradiction. ∎

Blocking I: 2-dependent process. Define the sequence τ_n recursively as $\tau_0 = 0$ and

$$\tau_{n+1} := \max_{\tau_n \leq j \leq \tau_n + l_{\tau_n}} \{ j : j + l_j \geq i + l_i \text{ for all } \tau_n \leq i \leq \tau_n + l_{\tau_n} \}.$$

Furthermore, denote by

$$Y_n := \sum_{j=\tau_n}^{\tau_{n+1}-1} X_j, \qquad (n \geq 0).$$

We gather some properties of the sequence (Y_n).

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Lemma 2. *(i)* $(Y_n)_{n \geq 0}$ *is a 2-dependent process.*

(ii)
$$\mathbb{V}\left(\sum_{n \leq N} Y_n\right) \sim F\left(\tau_{N+1} - 1\right). \tag{4}$$

Proof. Due to the properties of the sets F_n, the first part follows from

$$\max_{\tau_n \leq j < \tau_{n+1}} (j + l_j) < \tau_{n+3}.$$

In order to prove the latter, observe that the left hand side is bounded by $\tau_{n+1} + l_{\tau_{n+1}}$. Moreover, we have

$$\tau_{n+2} + l_{\tau_{n+2}} < \tau_{n+3} + l_{\tau_{n+3}}. \tag{5}$$

Assuming that $\tau_{n+3} \leq \tau_{n+1} + l_{\tau_{n+1}}$ would now imply that

$$\tau_{n+2} + l_{\tau_{n+2}} \geq \tau_{n+3} + l_{\tau_{n+3}}$$

which however contradicts (5). Hence, we have proved the first part of the lemma.

For the second part, we first observe that

$$\mathbb{V}\left(\sum_{n \leq N} Y_n\right) = \sum_{n < \tau_{N+1}} m(F_n) - \sum_{n < \tau_{N+1}} m(F_n)^2 + 2 \sum_{i < j < \tau_{N+1}} \left(m(F_i \cap F_j) - m(F_i)m(F_j)\right).$$

From the assumptions on (l_n) and (2),

$$\sum_{n < \tau_{N+1}} m(F_n)^2 \sim q^{-l}\left(1 - \frac{1}{q}\right)^2 \sum_{n < \tau_{N+1}} q^{-l_n}.$$

Moreover, from the property of the sequence F_n mentioned in the paragraph preceding (3),

$$\sum_{i < j < \tau_{N+1}} \left(m(F_i \cap F_j) - m(F_i)m(F_j)\right) = - \sum_{i < \tau_{N+1}} m(F_i) \sum_{i < j \leq \min\{i + l_i, \tau_{N+1} - 1\}} m(F_j)$$

$$\sim -c\left(1 - \frac{1}{q}\right)^2 \sum_{i < \tau_{N+1}} q^{-l_i},$$

the last step following from the assumptions on (l_n), Lemma 1, and (2).

Putting everything together yields the claimed result. ∎

Blocking II: Linear variance. For any positive integer n define the integer j_n by

$$F\left(\tau_{j_n+1} - 1\right) \leq n < F\left(\tau_{j_n+2} - 1\right)$$

and set $j_0 = -1$. Note that

$$F\left(\tau_{n+2} - 1\right) - F\left(\tau_{n+1} - 1\right)$$

$$\leq \left(\left(1 - \frac{1}{q}\right) - (2c + q^{-l})\left(1 - \frac{1}{q}\right)^2\right) \sum_{\tau_{n+1} \leq j \leq \tau_{n+1} + l_{\tau_{n+1}}} q^{-l_j}$$

$$< 1,$$

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where the last line holds if n is chosen large enough. Hence, the above definition makes sense.

Now, we define

$$\xi_n := \sum_{j=j_n+1}^{j_{n+1}} Y_j, \qquad (n \geq 0).$$

Some properties of (ξ_n) are summarized in the next lemma.

Lemma 3. *We have*

(i) $(\xi_n)_{n\geq 0}$ *is a 2-dependent process.*

(ii)
$$\mathbb{E}|\xi_n|^3 \ll 1.$$

(iii)
$$\mathbb{V}\left(\sum_{n\leq N} \xi_n\right) \sim N.$$

Proof. Property (i) is clear. For the proof of (ii), we first apply the multinomial theorem,

$$\mathbb{E}|\xi_n|^3 \leq \mathbb{E}\left(\sum_{j=\tau_{jn+1}}^{\tau_{j_{n+1}+1}-1} |X_j|\right)^3$$

$$= \sum_{e_{\tau_{jn+1}}+\cdots+e_{\tau_{j_{n+1}+1}-1}=3} \binom{3}{e_{\tau_{jn+1}},\ldots,e_{\tau_{j_{n+1}+1}-1}} \mathbb{E}|X_{\tau_{jn+1}}|^{e_{\tau_{jn+1}}} \cdots |X_{\tau_{j_{n+1}+1}-1}|^{e_{\tau_{j_{n+1}+1}-1}}. \quad (6)$$

In order to estimate the right hand side, we use property (3), a property that more generally holds for any finite number of pairwise distinct F_i's as was proved by Deligero and Nakada [1].

Now, observe

$$\sum_{j=\tau_{jn+1}}^{\tau_{j_{n+1}+1}-1} \mathbb{E}|X_j|^3 \ll \sum_{j=\tau_{jn+1}}^{\tau_{j_{n+1}+1}-1} m(F_j) \ll 1,$$

where the last estimate follows by the definition of j_n.

Next, we treat the following sum

$$\sum_{\tau_{jn+1}\leq i<j\leq\tau_{j_{n+1}+1}-1} \mathbb{E}|X_i|^2|X_j| \ll \sum_{\tau_{jn+1}\leq i<j\leq\tau_{j_{n+1}+1}-1} m(F_i)m(F_j) \ll \left(\sum_{j=\tau_{jn+1}}^{\tau_{j_{n+1}+1}-1} m(F_j)\right)^2 \ll 1.$$

Similarly, we have

$$\sum_{\tau_{jn+1}\leq i<j\leq\tau_{j_{n+1}+1}-1} \mathbb{E}|X_i||X_j|^2 \ll 1.$$

Hence, we are left with

$$\sum_{\tau_{jn+1}\leq i<j<l\leq\tau_{j_{n+1}+1}-1} \mathbb{E}|X_i||X_j||X_l| \ll \left(\sum_{j=\tau_{jn+1}}^{\tau_{j_{n+1}+1}-1} m(F_j)\right)^3 \ll 1.$$

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Plugging the last three estimates into (6) gives property (ii).

For property (iii), observe that by (4),

$$\mathbb{V}\left(\sum_{n \leq N} \xi_n\right) = \mathbb{V}\left(\sum_{n \leq j_{N+1}} Y_n\right) = F\left(\tau_{j_{N+1}+1} - 1\right).$$

Moreover, by the definition of j_n and the remark succeeding the definition, we have

$$N < F\left(\tau_{j_{N+1}+2} - 1\right) + \left(F\left(\tau_{j_{N+1}+1} - 1\right) - F\left(\tau_{j_{N+1}+2} - 1\right)\right) = F\left(\tau_{j_{N+1}+1} - 1\right) \leq N + 1.$$

This yields the desired result. ∎

3   Proof of the invariance principle

The proof of Theorem 2 will rest on the following extension of a theorem of Philipp and Stout (see Fuchs [4]). We state the result in a simplified form that will be sufficient for our purpose.

Proposition 1. *Let ξ_n denote a 2-dependent process of centered random variables on the probability space $(\Omega, \mathcal{A}, P)$ and suppose that*

$$\mathbb{E}|\xi_n|^3 \ll 1$$

and

$$\mathbb{V}\left(\sum_{n \leq N} \xi_n\right) \sim N.$$

Define a stochastic process $\xi(t)$ on $(\Omega, \mathcal{A}, P) \times ([0,1], \bar{\mathcal{B}}, \lambda)$ by

$$\xi(t) = \sum_{n \leq t} \xi_n.$$

Then, as $t \to \infty$,

$$\xi(t) - W(t) = o\left((t \log \log t)^{1/2}\right), \quad \text{a.s.}$$

and

$$(P \times \lambda)\left[\frac{1}{\sqrt{t}} \sup_{s \leq t} |\xi(s) - W(s)| \geq \epsilon\right] \longrightarrow 0$$

for all $\epsilon > 0$.

Due to the Lemma 3, the sequence ξ_n of the previous section satisfies all the assumptions of the above proposition. Therefore, we obtain, as $t \to \infty$,

$$\xi(t) - W(t) = o\left((t \log \log t)^{1/2}\right), \quad \text{a.s.}$$

and

$$(m \times \lambda)\left[\frac{1}{\sqrt{t}} \sup_{s \leq t} |\xi(s) - W(s)| \geq \epsilon\right] \longrightarrow 0$$

for all $\epsilon > 0$, where $\xi(t) = \sum_{n \leq t} \xi_n$.

The invariance principle for $Z(t)$. We prove the following lemma.

Lemma 4. *As $t \to \infty$,*

$$Z(t) - \xi(t) \ll t^{1/2 - \epsilon}, \quad \text{a.s.}$$

for all $0 < \epsilon < 1/6$.

Proof. We have

$$m \left[\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} |X_j| \geq n^{1/2-\epsilon} \right] \leq n^{-3/2+3\epsilon} \mathbb{E} \left(\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} |X_j| \right)^3 \ll n^{-3/2+3\epsilon}.$$

Consequently, by the Borel-Cantelli lemma,

$$\sum_{j=\tau_{j_n+1}}^{\tau_{j_{n+1}+1}-1} |X_j| \ll n^{1/2-\epsilon}, \qquad \text{a.s..} \tag{7}$$

Now, observe

$$|Z(t) - \xi(t)| = \left| \sum_{n \leq N_t} X_n - \sum_{n \leq \tau_{j_{[t]+1}+1}-1} X_n \right| \leq \sum_{j=\tau_{j_{[t]}+1}}^{\tau_{j_{[t]+1}+1}-1} |X_j|$$

and combining with (7) concludes the proof of the desired result. ▮

The above lemma yields, as $t \to \infty$,

$$Z(t) - W(t) = o\left((t \log \log t)^{1/2} \right), \quad \text{a.s.}$$

and

$$(m \times \lambda) \left[\frac{1}{\sqrt{t}} \sup_{s \leq t} |Z(s) - W(s)| \geq \epsilon \right] \longrightarrow 0$$

for all $\epsilon > 0$. Reformulation gives Theorem 2. ▮

Acknowledgement

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