On Maxima in Geometric Words that satisfy a Generalized Restricted Growth Property

(joint work with Mehri Javanian)

Michael Fuchs

Institute of Applied Mathematics
National Chiao Tung University

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Geometric Words

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Studied because related to:

- Approximate counting;
- Digital trees

(Some) Previous Work

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  Archibald & Knopfmacher (2007, 2009); Bai & Hwang & Liang (1998); Brennan & Knopfmacher & Mansour & Wagner (2011); Knopfmacher & Prodinger (2001); Oliver & Prodinger (2012); Prodinger (1993, 1996, 2002, 2006, 2012)

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Other parameters: **# of different letters, missing letters, gaps, inversion, ascends and descends, runs, etc.**

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Theorem (Prodinger; 1996)

We have,

$$\mathbb{E}(L_n) \sim p \log_Q n + \Phi_1(\log_Q n) \quad \textit{and} \quad \mathrm{Var}(L_n) \sim pq \log_Q n + \Phi_2(\log_Q n),$$

where Φ_1, Φ_2 *are* 1*-periodic functions.*

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Goal: find asymptotics of **moment generating function**

$$\mathbb{E}\left(e^{L_n^{(d)} t}\right) = \sum_k \mathbb{P}\left(L_n^{(d)} = k\right) e^{kt}$$

in a complex neighbourhood of 0.

Asymptotics of Moment Generating Function (i)

Set

$$\tilde{L}(z,t) := e^{-z}\sum_{n}\sum_{k} p_{n,k}e^{kt}\frac{z^n}{n!}.$$

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Can be solved with the **Mellin transform**:

$$\mathscr{M}[\tilde{f}(z);\omega] := \int_0^\infty \tilde{f}(z) z^{\omega-1} \mathrm{d}z$$

because of

$$\mathscr{M}[\tilde{f}(az);\omega] = a^{-\omega} \mathscr{M}[\tilde{f}(z);\omega].$$

Converse Mapping Theorem

Theorem (Flajolet, Gourdon, Dumas; 1995)

Let the Mellin transform of $\tilde{f}(z)$ *exist in the strip* $\langle \alpha, \beta \rangle$.

Assume that $\mathscr{M}[\tilde{f}(z); s]$ *can be continued to a meromorphic function on* $\langle \alpha, \gamma \rangle$ *with* $\beta < \gamma$ *with simple poles at* $s_1, \cdots, s_k$.

Then, under some technical conditions,

$$\tilde{f}(z) = -\sum_{j=1}^{k} \mathrm{Res}(\mathscr{M}[\tilde{f}(z); s], s = s_j) z^{-s_j} + \mathcal{O}\left(z^{-\gamma}\right)$$

as $z \to \infty$.

Asymptotics of Moment Generating Function (ii)

Applying the converse mapping theorem gives:

$$\tilde{L}(z,t) \sim -\frac{P_t(1)\Omega_t(1)}{\log(Q)\rho_t P_t'(\rho_t)\Omega_t(\rho_t)} z^{-\log_Q \rho_t} \sum_k \Gamma(\log_Q \rho_t + \chi_k) z^{-\chi_k},$$

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where $\chi_k = 2k\pi i/\log(Q)$ and

$$P_t(z) = 1 - pe^t \sum_{\ell=1}^{d} q^{\ell-1} z^{\ell}, \qquad \Omega_t(s) = \prod_{\ell \geq 1} P_t(q^{\ell} s)$$

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Finally, note that

$$\tilde{L}(n,t) = e^{-n} \sum_m \sum_k p_{m,k} e^{kt} \frac{n^m}{m!} \sim \sum_k p_{n,k} e^{kt} \quad \ldots \textbf{ Poisson heuristic!}$$

Asymptotic of Moment Generating Function (iii)

Proposition

Uniformly in a neighbourhood of 0,

$$\mathbb{E}\left(e^{L_n^{(d)}t}\right) \sim \frac{P_t(1)\Omega_t(1)\rho_0 P_0'(\rho_0)\Omega_0(\rho_0)}{q^d\Omega_0(1)\rho_t P_t'(\rho_t)\Omega_t(\rho_t)} n^{-\log_Q(\rho_t/\rho_0)} \times \frac{\sum_k \Gamma(\log_Q \rho_t + \chi_k)n^{-\chi_k}}{\sum_k \Gamma(\log_Q \rho_0 + \chi_k)n^{-\chi_k}}.$$

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Corollary

For $m \geq 1$,

$$\mathbb{E}(L_n^{(1)} - \log_Q n)^m \sim \Phi_m^{(1)}(\log_Q n),$$

where $\Phi_m^{(1)}$ *are* 1*-periodic functions.*

Limit Law of $L_n^{(d)}$

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We have,

$$\frac{L_n^{(d)} + \log_Q n/(\rho_0 P_0'(\rho_0))}{\sqrt{\log_Q n}} \xrightarrow{d} N(0, \sigma_d^2),$$

for a constant σ_d^2 *which is* > 0 *iff* $d \geq 2$*.*

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for a constant σ_d^2 which is > 0 iff $d \geq 2$.

Thus, the limit law of $L_n^{(d)}$ undergoes a **phase change** from non-existence for $d = 1$ to normal for $d \geq 2$!

Maximum Value (i)

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Answer to above question is **NO**!

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With the same method as before:

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Thus, N_n *does not converge to a limit law.*

Answer to above question is again **NO!**

Summary

Seventh Cross-straight Conference on Combinatorics and Graph Theory:

$$\mathbb{E}(L_n^{(1)}) = \mathbb{E}(M_n^{(1)}) \sim \log_Q n + \Phi_1^{(1)}(\log_Q n).$$

I listed results for higher moments and limit laws as open problem.

Summary

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I listed results for higher moments and limit laws as open problem.

These open problem were solved by F. & Javanian (2015):

parameter	m-th central moments	limit law
$L_n^{(d)}$	$\begin{cases} d = 1 : \text{periodic} \\ d \geq 2 : \Theta\left((\log n)^{m/2}\right) \end{cases}$	$\begin{cases} d = 1 : \text{does not exist} \\ d \geq 2 : \text{normal} \end{cases}$
$M_n^{(d)}$	periodic for all $d \geq 1$	does not exist for all $d \geq 1$
N_n	periodic	does not exist