AN ANALOGUE OF A THEOREM OF SZÜSZ FOR FORMAL LAURENT SERIES OVER FINITE FIELDS

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Abstract. About 40 years ago, Szüsz proved an extension of the well-known Gauss-Kuzmin theorem. This result played a crucial role in several subsequent papers (for instance papers due to Szüsz, Philipp, and the author). In this note, we provide an analogue in the field of formal Laurent series and outline applications to the metric theory of continued fractions and to the metric theory of diophantine approximation.

1. Introduction

In order to fix notations let $x = [a_0, a_1, a_2, \ldots]$ be the continued fraction expansion of $x \in [0, 1]$ and denote by

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k]$$

the k-th convergent. Furthermore put

$$\xi_k = [0, a_{k+1}, a_{k+2}, \ldots].$$

In [13], Szüsz proved the following generalization of the well-known Gauss-Kuzmin theorem

Theorem 1. *For $t \in [0, 1]$, $a, b, r \in \mathbb{N}$, and $r \geq 1$ define*

$$m_k(a, b, t) := \lambda\{x \in [0, 1] : \ q_{k-1} \equiv a \ (r), q_k \equiv b \ (r), \xi_k \leq t\}$$

where λ denotes Lebesque measure. Then, we have

$$m_k(a, b, t) = \begin{cases} \frac{1}{C(r)} \frac{\log(1+t)}{\log 2}(1 + \mathcal{O}(\rho^k)) & \text{if} \quad (a, b, r) = 1 \\ 0 & \text{if} \quad (a, b, r) \neq 1 \end{cases}$$

where $C(r) = r^2 \prod_{p|r} \left(1 - \frac{1}{p^2}\right)$, $\rho < 1$ is a constant, and the constant implied in the error term only depends on r.

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Furthermore, Szüsz used the method of proof of Theorem 1 to obtain asymptotic independence of certain events (see Satz 3.1 in [13]). As an application he was able to prove an extension of Khintchine's strong law of large numbers in the metric theory of continued fractions (see Theorem 1 in [11] resp. Satz 3.2 in [13]).

Another application of Szüsz's theorem was given by Philipp who improved and extended a central limit theorem in the metric theory of diophantine approximation proved by LeVeque (see [6], [7], and [9]). Recently, Philipp's theorem was improved by the author (see [3]) and we managed to extend and solve a problem raised by LeVeque (see [2] and [7]). In both papers Szüsz's theorem was one of the fundamental lemmas.

Since Szüsz's theorem has so many applications, it is worth it to ask for an analogue in the field of formal Laurent series over a finite base field. In this note, we provide such an analogue and outline applications to the metric theory of continued fractions and diophantine approximation.

Before we state the results, let us recall some definitions and basic properties. By $\mathbb{F}_q$, we denote the finite field with $q = p^t$ ($p \in \mathbb{P}, t \geq 1$) elements. Furthermore, we use the standard notation $\mathbb{F}_q((T^{-1}))$ for the field of formal Laurent series over $\mathbb{F}_q$ which is the set of all formal sums

$$\alpha = \sum_{k \leq n} a_k T^k, \quad a_k \in \mathbb{F}_q, a_n \neq 0.$$

Clearly, the ring of polynomials and the field of rational functions are contained in $\mathbb{F}_q((T^{-1}))$, whereas, we have the following chain of inclusions

$$\mathbb{F}_q[T] \subset \mathbb{F}_q(T) \subset \mathbb{F}_q((T^{-1})).$$

Throughout the paper, we write $\alpha, \beta, \ldots$ for the elements of $\mathbb{F}_q((T^{-1}))$ and $P, Q, \ldots$ for the elements of $\mathbb{F}_q[T]$. We define a valuation on $\mathbb{F}_q((T^{-1}))$ by $|\alpha| = q^n$ for $\alpha \neq 0$ resp. $|0| = 0$. It is easy to see that $\mathbb{F}_q((T^{-1}))$ endowed with this evaluation is an ultrametric space.

There is a continued fractions theory in $\mathbb{F}_q((T^{-1}))$ with polynomials playing the role of integers (see [12] for example). As in the classical theory, we denote by $\alpha = [A_0, A_1, A_2, \ldots]$ the continued fraction expansion of $\alpha \in \mathbb{F}_q((T^{-1}))$. Furthermore, we use

$$\frac{P_k}{Q_k} = [A_0, A_1, \ldots, A_k]$$

for the k-th convergent and

$$\xi_k = [0, A_{k+1}, A_{k+2}, \ldots].$$

Since we are interested in the metric theory, we consider the following subset of $\mathbb{F}_q((T^{-1}))$

$$H = \{\alpha \in \mathbb{F}_q((T^{-1})) : |\alpha| < 1\}$$

which can be seen as an analogue of the interval $[0, 1]$. Observe, that H together with the restriction of the valuation on H is a compact abelian group. Let $\mathcal{H}$ denote its σ-algebra of Borel sets. Then, it is well known that there exists a unique, translation invariant probability measure on the measure space $(H, \mathcal{H})$ that we are going to denote by h. Because of the ultrametric structure of the probability space $(H, \mathcal{H}, h)$, the metric theory of continued fractions in the Laurent series case is much easier than in the classical case (compare with [4]).

We conclude the introduction by giving a short plan of the paper: in the next section, we state our results. Section 3 is devoted to the proof of the analogue of Szüsz's theorem and in Section 4, we outline applications to the metric theory of continued fractions. Finally, Section 5 and Section 6 contain applications to the metric theory of diophantine approximation.

2. Results

The main result is the following analogue of Szüsz's theorem

Theorem 2. *For $x \in [0,1]$, $S, T, R \in \mathbb{F}_q[T]$, and $\deg R \geq 0$ define*

$$m_k(S,T,x) := h(\{\alpha \in H: \ Q_{k-1} \equiv S \ (R), Q_k \equiv T \ (R), |\xi_k| < x\}).$$

Then, we have

$$m_k(S,T,x) = \begin{cases} \frac{1}{C(R)} q^{\lceil \log_q x \rceil}(1+\mathcal{O}(\rho^k)) & \text{if} \quad (S,T,R)=1 \\ 0 & \text{if} \quad (S,T,R) \neq 1 \end{cases}$$

where $C(R) = |R|^2 \prod_{P|R} \left(1 - \frac{1}{|P|^2}\right)$ (the product is extended over all monic, irreducible polynomials P dividing R), $\rho < 1$ is a constant, and the constant implied in the error term only depends on R.

Furthermore, we can use the proof technique of this theorem to obtain the following asymptotic independence result

Theorem 3. *For $x \in [0,1]$, $A_1, \ldots, A_l, S, T, R \in \mathbb{F}_q[T]$, $\deg R \geq 0$, and $1 \leq \deg A_i$, $1 \leq i \leq l$ define*

$$\begin{aligned} m_k(A_1, \ldots, A_l; S, T, x) := \{\alpha \in H: \ & A_1(\alpha) = A_1, \ldots, A_l(\alpha) = A_l, \\ & Q_{k-1} \equiv S \ (R), Q_k \equiv T \ (R), |\xi_k| < x\} \end{aligned}$$

and

$$m(A_1, \ldots, A_l) := \{\alpha \in H: \ A_1(\alpha) = A_1, \ldots, A_l(\alpha) = A_l\}$$

where $A_i(\alpha)$ is the i-th partial quotient in the continued fraction expansion of α. Then, we have

$$\frac{m_k(A_1, \ldots, A_l; S, T, x)}{m(A_1, \ldots, A_l)} = \begin{cases} \frac{1}{C(R)} q^{\lceil \log_q x \rceil}(1+\mathcal{O}(\rho^{k-l})) & \text{if} \quad (S,T,R)=1 \\ 0 & \text{if} \quad (S,T,R) \neq 1 \end{cases}$$

where $C(R)$, ρ are as in Theorem 2, and the implied constant again only depends on R.

As in the classical theory, these two results have several applications. Firstly, we concentrate on the metric theory of continued fraction expansion.

We give a common extension of a result due to Szüsz (or more precisely of the analogue of this result in the Laurent series case which is straightforward to prove) and a recent result of Harman and Wong (see Theorem 1 in [5] and Satz 3.2 in [13]). Therefore, we define

Definition 1. *Let* $B_1, \ldots, B_l, R \in \mathbb{F}_q[T]$ *with* $\deg R \geq 0$ *and* $\deg B_i < \deg R,\ 1 \leq i \leq l$ *be given. We call the* l*-tuple* $B_1, \ldots, B_l$ *acceptable mod* R *if*

$$(B_i, B_{i+1}) = 1, \quad 1 \leq i < l$$

and

$$(R, B_{i-1}) | (B_i - B_{i-2}), \quad 3 \leq i \leq l.$$

If the l*-tuple* $B_1, \ldots, B_l$ *is not acceptable mod* R*, we call it prohibited mod* R*.*

Using this notation, we can prove the following theorem

Theorem 4. *Let* $f(C_1, \ldots, C_l, A)$ *be a function where* $C_1, \ldots, C_l, A \in \mathbb{F}_q[T],\ \deg A \geq 1,$ *and* $l \geq 2$*. Furthermore, assume that* f *is periodic mod* R *in the first* l *coordinates and satisfies*

$$|f(C_1, \ldots, C_l, A)| \ll |A|^{1/2-\delta}$$

where $\delta > 0$ *is a real constant. Denote by* $(k_i)_{i \geq 1}$ *an increasing sequence of positive integers. Then, we have*

$$\sum_{i \leq n} f(Q_{k_i}, \ldots, Q_{k_i+l-1}, A_{k_i+l}) = C(f)n + \mathcal{O}\left(n^{1/2}(\log n)^{3/2+\epsilon}\right) \quad a.s.$$

for all $\epsilon > 0$*. Here,*

$$C(f) = \sum_{\substack{A \in \mathbb{F}_q[T],\ \deg A \geq 1 \\ B_1, \ldots, B_l \ acceptable}} f(B_1, \ldots, B_l, A) C(B_1, \ldots, B_l) q^{-2 \deg A}$$

and

$$C(B_1, \ldots, B_l) = \frac{1}{C(R)} \prod_{j=3}^{l} \sum_{C \in L_j} q^{-2 \deg C}$$

where

$$L_j = \{A \in \mathbb{F}_q[T] :\ AB_{j-1} \equiv B_j - B_{j-2}\ (R)\}.$$

This result entails the following interesting consequence

Corollary 1. *Let* $S, R \in \mathbb{F}_q[T],\ \deg R \geq 0$ *be given. Assume that* $(k_i)_{i \geq 1}$ *is an increasing sequence of positive integers. Then, we have a.s.*

$$Q_{k_i} \equiv S\ (R)$$

for infinitely many i*.*

Remark 1. This consequence has to be compared with a remark in [5] where the special case that $(k_i)_{i\geq 1}$ is an arithmetic progression was considered. Indeed, a similar result is also true in the classical case (the method of the proof can be used without difficulties).

Next, we give applications to the metric theory of diophantine approximation. Therefore, let f be a positive, real-valued function defined on the non-negative real numbers satisfying the following conditions

$$f\downarrow 0, \quad \textstyle\sum_{k=0}^{\infty} f(k)=\infty, \tag{1}$$

$$\textstyle\sum_{k=0}^{n} f(k)k^{-\delta} \ll \left(\sum_{k=0}^{n} f(k)\right)^{1/2} \tag{2}$$

with $0<\delta<1/2$. We are concerned with the diophantine approximation problem

$$\left|\alpha-\frac{P}{Q}\right|<\frac{f(\deg Q)}{|Q|^2} \tag{3}$$

where $\alpha\in H$. Fix polynomials S, R with $\deg R\geq 0$ and define the following sequence of random variables

$$X_n(\alpha):=\#\{\langle P,Q\rangle:\ 0\leq \deg Q\leq n, Q\equiv S\ (R), P/Q \text{ is a solution of (3)}\}.$$

In the classical case, LeVeque conjectured a central limit theorem for the above sequence of random variables (actually, LeVeque considered a more restrictive class of functions f and defined the sequence of random variables without the restriction that the denominators have to be in an arithmetic progression; see [7]) and this conjecture was settled by the author in [2].

A similar result is true in the Laurent series case (compare with Theorem 1 in [2]).

Theorem 5. *Set*

$$F(n):=\sum_{k=0}^{n} q^{\lceil \log_q f(k)\rceil}.$$

Then, we have

$$\lim_{n\longrightarrow\infty} h\left[X_n\leq \sigma_1 F(n)+\omega(\sigma_2 F(n)\log F(n))^{1/2}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega} e^{-u^2/2}du.$$

Here,

$$\sigma_1=\frac{q-1}{|R|}, \quad \sigma_2=\frac{(q+1)(q-1)^2|(S,R)|\varphi(R)}{2q\log q|R|C(S,R)}$$

and

$$C(S,R)=|R|^2\prod_{P|R}\left(1-\frac{1}{|P|}\right)\prod_{P|\frac{R}{(S,R)}}\left(1+\frac{1}{|P|}\right)$$

where the products are extended over all monic, irreducible polynomials satisfying the desired properties.

The situation is getting much easier if we use instead of (2)

$$\sum_{k=0}^{n} f(k)k^{-\delta_1} \ll \left(\sum_{k=0}^{n} f(k)\right)^{1/2-\delta_2} \tag{4}$$

with $0 < \delta_1, \delta_2 < 1/2$ and if we replace X_n by the following sequence of random variables

$$\begin{aligned} Y_n(\alpha) := \#\{\langle P, Q\rangle :\ & 0 \leq \deg Q \leq n, \deg(P,Q) \leq d, Q \equiv S\ (R), \\ & P/Q \text{ is a solution of (3)}\} \end{aligned} \tag{5}$$

where $d \geq 1$ is a fixed integer.

In the classical case, the statistical behaviour of (5) was already treated by LeVeque (see [6]). LeVeque's results were improved by Philipp a few years later (see [9]). In [3], we in turn improved Philipp's results by providing almost sure and distribution type invariance principles.

These results carry over in the Laurent series case. Therefore put

$$F(n) = \sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil}$$

and

$$\sigma = \sigma_d = \frac{q-1}{q} \sum_{\deg D < \deg R} \sum_{\substack{0 \leq \deg C \leq d \\ CD \equiv S\ (R)}} \frac{K(D)}{|C|^2},$$

$$\tau^2 = \tau_d^2 = \frac{q-1}{q} \sum_{\deg D < \deg R} \sum_{0 \leq l \leq d} k_{l,D} q^{-2l}$$

where $k_{l,D}$ is the number of pairs $\langle C_1, C_2\rangle$ with C_1, C_2 solutions of $CD \equiv S\ (R)$, $\deg C_1, \deg C_2 \leq l$ and either $\deg C_1 = l$ or $\deg C_2 = l$. Furthermore, set

$$n_t = \begin{cases} \max\{n :\ \tau^2 F(n) \leq t\} & \text{if } t \geq \tau^2 F(0) \\ 0 & \text{otherwise} \end{cases}$$

where $t \geq 0$.

Now, consider the product probability space $(H, \mathcal{H}, h) \times ([0,1], \mathcal{B}, \lambda)$ where $\mathcal{B}$ denotes the σ-algebra of Borel sets on $[0,1]$ and λ is the Lebesgue measure. Using the above notation, we define on this probability space a stochastic process Y by setting for $\alpha \in H$ and $x \in [0,1]$

$$Y(t) = Y(t; \alpha, x) = Y_{n_t}(\alpha) - \sigma F(n_t)$$

Adjoining a uniformly distributed random variable independent of the entire sequence $(Y_n(\alpha))_{n \geq 1}$ guarantees that the probability space is rich enough.

Theorem 6. *There exists a sequence $(Z_i(\alpha, x))_{i\geq 1}$ of independent, standard normal $\mathcal{N}(0,1)$ random variables defined on the above probability space such that, as $n \longrightarrow \infty$,*

$$\Big|Y(n) - \sum_{i\leq n} Z_i\Big| = o((n\log\log n)^{1/2}) \ a.s. \tag{6}$$

and

$$(h \times \lambda)\left[\frac{1}{\sqrt{n}}\max_{k\leq n}\Big|Y(k) - \sum_{i\leq k} Z_i\Big| \geq \epsilon\right] \longrightarrow 0 \tag{7}$$

for all $\epsilon > 0$.

As in [3], we can even prove more if f satisfies an additional assumption.

Theorem 7. *Let f satisfy (1), (4) and the following additional condition*

$$\sum_{k=0}^{n} f^2(k) \ll \left(\sum_{k=0}^{n} f(k)\right)^{1-\delta_3} \tag{8}$$

where $0 < \delta_3 < 1$. Then, there exists a sequence $(Z_i(\alpha, x))_{i\geq 1}$ of independent, standard normal $\mathcal{N}(0,1)$ random variables defined on the above probability space such that, as $n \longrightarrow \infty$,

$$Y(n) - \sum_{i\leq n} Z_i \ll n^{1/2-\lambda} \quad a.s. \tag{9}$$

where $0 < \lambda < 1/2$ is a real constant.

Remark 2. As it was pointed out in [3] these theorems entail several consequences, for instance, a functional central limit theorem and a Strassen's type version of the iterated logarithm law. Under the stronger assumptions of the second theorem, we even obtain a functional iterated logarithm law for the maximum.

Remark 3. Theorem 6 as well as Theorem 7 contain the main result of [4] as a special case. Thereby notice that from a probabilistic point of view the situation in the cited paper is totally different from the situation here; more specifically in [4], we had to deal with independent sequences of random variables whereas here the involved random variables will just satisfy some mixing conditions.

3. Proof of Theorem 2 and Theorem 3

We need a few lemmas before we can start with the proofs of the theorems. The first two lemmas collect well-known properties of the continued fraction expansion in the Laurent series case.

Lemma 1. *Let $A \in \mathbb{F}_q[T]$ be a given polynomial with degree at least 1. We have*

$$h(\{\alpha \in H : \ A_k(\alpha) = A\}) = q^{-2\deg A}.$$

Proof. This is an easy calculation. □

Lemma 2. *Let f be a real-valued function defined on the set of all polynomials with degree at least* 1*. Then the sequence of functions $f(A_k)$ defined on H is an independent and identically distributed sequence of random variables.*

Proof. Lemma 4 in [8]. □

For the third and final lemma, we need a little bit notation. Define, for given polynomials $S, T, R, C_1, C_2, C_3, C_4 \in \mathbb{F}_q[T]$ with $\deg R \geq 0$, consecutively

$$\begin{aligned} U_1(C_1) &\equiv T - C_1 S & (R),\\ U_2(C_2) &\equiv S - C_2 U_1(C_1) & (R),\\ U_3(C_3) &\equiv U_1(C_1) - C_3 U_2(C_2) & (R),\\ U_4(C_4) &\equiv U_2(C_2) - C_4 U_3(C_3) & (R). \end{aligned}$$

As in [13], we can prove the following elementary lemma

Lemma 3. *Let $S, T, \bar{S}, \bar{T}, R \in \mathbb{F}_q[T]$ be given polynomials with $\deg R \geq 0$ and $(S, T, R) = 1, (\bar{S}, \bar{T}, R) = 1$. There are polynomials $C_1, C_2, C_3, C_4 \in \mathbb{F}_q[T]$ such that*

$$\begin{aligned} U_3(C_3) &\equiv \bar{T}\ (R),\\ U_4(C_4) &\equiv \bar{S}\ (R). \end{aligned}$$

Now, we can start proving the theorems.

Proof of Theorem 2. Because of the fact that denominators of two consecutive convergents have to be coprime, it is clear that $m_k(S, T, x) = 0$ for all polynomials with $(S, T, R) \neq 1$. Therefore, we can concentrate on polynomials S, T with $(S, T, R) = 1$.

Since

$$|\xi_k| = \frac{1}{|A_{k+1}|}$$

Lemma 1 and Lemma 2 implies

$$\begin{aligned} m_k(S, T, x) &= h(\{\alpha \in H : \ Q_{k-1} \equiv S\ (R), Q_k \equiv T\ (R)\}) \cdot \\ &\qquad \cdot h(\{\alpha \in H : \ |A_{k+1}| > 1/x\}) \\ &= h(\{\alpha \in H : \ Q_{k-1} \equiv S\ (R), Q_k \equiv T\ (R)\}) q^{\lceil \log_q x \rceil} \end{aligned}$$

and so, it is sufficient to consider

$$m_k(S, T) := h(\{\alpha \in H : \ Q_{k-1} \equiv S\ (R), Q_k \equiv T\ (R)\}).$$

Using once more Lemma 1 and Lemma 2, we obtain

$$\begin{aligned} m_k(S, T) &= \sum_{C_1 \in \mathbb{F}_q[T],\ \deg C_1 \geq 1} h(\{\alpha \in H : Q_{k-2} \equiv U_1(C_1)\ (R), \\ &\qquad Q_{k-1} \equiv S\ (R), A_k = C_1\}) \qquad (10) \\ &= \sum_{C_1 \in \mathbb{F}_q[T],\ \deg C_1 \geq 1} m_{k-1}(U_1(C_1), S) q^{-2 \deg C_1} \end{aligned}$$

and by applying the last equality three times, we get

$$m_k(S,T) = \sum_{C_1,C_2,C_3,C_4} m_{k-4}(U_4(C_4),U_3(C_3))q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)}$$

where the sum runs over all polynomials of degree at least 1.

Next, define

$$\begin{aligned} m_k &= \min_{(S,T,R)=1,\ \deg S,\ \deg T<\deg R} m_k(S,T), \\ M_k &= \max_{(S,T,R)=1,\ \deg S,\ \deg T<\deg R} m_k(S,T), \end{aligned}$$

and observe

$$\begin{aligned} m_k(S,T) &= \sum_{C_1\in\mathbb{F}_q[T],\ \deg C_1\geq 1} m_{k-1}(U_1(C_1),S)q^{-2\deg C_1} \\ &\geq m_{k-1} \sum_{C_1\in\mathbb{F}_q[T],\ \deg C_1\geq 1} q^{-2\deg C_1} = m_{k-1} \end{aligned}$$

for all polynomials S,T with $(S,T,R)=1$. Consequently, the sequence m_k is a non-decreasing sequence of real numbers.

Since

$$\sum_{C_1,C_2,C_3,C_4} q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)} = 1$$

where the sum runs over all polynomials with degree at least 1, we have

$$\begin{aligned} m_k(S,T) - m_k \leq \sum_{C_1,C_2,C_3,C_4} &(m_{k-4}(U_4(C_4),U_3(C_3)) - m_{k-4})\cdot \\ &\cdot q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)}. \end{aligned}$$

Let $\bar{S},\bar{T}$ be polynomials with $(\bar{S},\bar{T},R)=1,\ \deg\bar{S},\ \deg\bar{T}<\deg R$ satisfying $m_{k-4}=m_k(\bar{S},\bar{T})$. According to Lemma 3, there are polynomials D_1,D_2,D_3,D_4 such that $U_3(D_3)=\bar{T},U_4(D_4)=\bar{S}$. Hence

$$\begin{aligned} m_k(S,T) - m_k \leq {\sum}^*(m_{k-4}(U_4(C_4),&U_3(C_3)) - m_{k-4})\cdot \\ &\cdot q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)} \end{aligned}$$

where the sum runs over all polynomials C_1,C_2,C_3,C_4 with degree at least 1 and there exists an index $i\in\{1,2,3,4\}$ such that $C_i\not\equiv D_i\ (R)$. Thus,

$$m_k(S,T) - m_k \leq (M_{k-4}-m_{k-4}){\sum}^* q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)}$$

and an easy calculation gives

$${\sum}^* q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)} \leq \bar{\rho}$$

where $\bar{\rho}<1$ is a constant that only depends on R. Therefore

$$M_k - m_k \leq (M_{k-4}-m_{k-4})\bar{\rho}$$

which implies $M_k - m_k = \mathcal{O}(\rho^k)$ for a suitable constant ρ.

Consequently, we have for fixed polynomials S, T with $(S, T, R) = 1$

$$m_k(\bar{S}, \bar{T}) = m_k(S, T) + \mathcal{O}(\rho^k)$$

for all polynomials $\bar{S}, \bar{T}$ with $(\bar{S}, \bar{T}, R) = 1$. This implies

$$\begin{aligned} 1 &= \sum_{(\bar{S},\bar{T},R)=1,\ \deg \bar{S},\ \deg \bar{T} < \deg R} m_k(\bar{S}, \bar{T}) \\ &= (m_k(S, T) + \mathcal{O}(\rho^k)) \sum_{(\bar{S},\bar{T},R)=1,\ \deg \bar{S},\ \deg \bar{T} < \deg R} 1. \end{aligned}$$

and since

$$\sum_{(\bar{S},\bar{T},R)=1,\ \deg \bar{S},\ \deg \bar{T} < \deg R} 1 = C(R),$$

we get

$$m_k(S, T) = \frac{1}{C(R)} + \mathcal{O}(\rho^k)$$

which gives the desired result. □

Proof of Theorem 3. Define

$$\begin{aligned} m_k(A_1, \dots, A_l; S, T) := h(\{\alpha \in H :\ & A_1(\alpha) = A_1, \dots, A_l(\alpha) = A_l, \\ & Q_{k-1} \equiv S\ (R), Q_k \equiv T\ (R)\}). \end{aligned}$$

As in the proof of Theorem 2 it suffices to concentrate on the above quantity on the one hand and to consider only polynomials S, R with $(S, T, R) = 1$ on the other hand. Next, observe that

$$\frac{m_k(A_1, \dots, A_l; S, T)}{m(A_1, \dots, A_l)}$$

satisfies recurrence (10). Furthermore, we have the right norming. Therefore, we get, as in the proof of Theorem 2,

$$\frac{m_k(A_1, \dots, A_l; S, T)}{m(A_1, \dots, A_l)} = \frac{1}{C(R)}(1 + \mathcal{O}(\rho^{k-l}))$$

(especially, notice that the implied constant does not depend on $A_1, \dots, A_l$) which immediately gives the result. □

4. Proof of Theorem 4

First, we need a little bit notation.

Definition 2. *Let $B_1, \dots, B_l$ be an acceptable l-tuple mod R and $A \in \mathbb{F}_q[T]$ a polynomial with* $\deg A \geq 1$. *Then, we put*

$$\begin{aligned} T_i(B_1, \dots, B_l; A) := \{\alpha \in H :\ & Q_i \equiv B_1\ (R), \dots, Q_{i+l-1} \equiv B_l\ (R), \\ & A_{i+l} = A\}. \end{aligned}$$

Next, we collect a few simple lemmas.

Lemma 4. *We have*

$$h(T_i(B_1,\ldots,B_l;A)) = C(B_1,\ldots,B_l)q^{-2\deg A}(1+\mathcal{O}(\rho^i)).$$

Here,

$$C(B_1,\ldots,B_l) = \frac{1}{C(R)}\prod_{j=3}^{l}\sum_{C\in L_j} q^{-2\deg C}$$

where

$$L_j = \{A\in\mathbb{F}_q[T]:\ AB_{j-1}\equiv B_j - B_{j-2}\ (R)\}.$$

Proof. Observe that

$$T_i(B_1,\ldots,B_l;A) = \bigcup_{C_3\in L_3,\ldots,C_l\in L_l}\{\alpha\in H:\ Q_i\equiv B_1\ (R), Q_{i+1}\equiv B_2\ (R), \\ A_{i+2}=C_3,\ldots,A_{i+l-1}=C_l, A_l = A\}$$

and the result is easily obtained by using Theorem 2, Lemma 1, and Lemma 2. □

Lemma 5. *We have*

$$h(T_i(B_1,\ldots,B_l;A_1)\cap T_j(C_1,\ldots,C_l;A_2)) \\ = h(T_i(B_1,\ldots,B_l;A_1))h(T_j(C_1,\ldots,C_l;A_2))(1+\mathcal{O}(\rho^{j-i}))$$

for $j\geq i+l-1$.

Proof. Notice that

$$T_i(B_1,\ldots,B_l;A_1)\cap T_j(C_1,\ldots,C_l;A_2) = \\ \bigcup_{E_1,\ldots,E_{i+l-1}}\ \bigcup_{D_3,\ldots,D_l}\{\alpha\in H:\ A_1=E_1,\ldots,A_{i+l-1}=E_{i+l-1}, A_{i+l}=A_1, \\ Q_j\equiv C_1\ (R), Q_{j+1}\equiv C_2\ (R), A_{j+2}=D_3,\ldots,A_{j+l-1}=D_l, \\ A_{j+l}=A_2\}$$

where the first joint runs over all polynomials $E_1,\ldots,E_{i+l-1}$ with degree at least 1 such that the denominators of the convergents of $[0,E_1,\ldots,E_{i+l-1}]$ satisfy $Q_i\equiv B_1\ (R),\ldots,Q_{i+l-1}\equiv B_l\ (R)$ and the second joint runs over all polynomials $D_3,\ldots,D_l$ with $D_i\in L_j$ where

$$L_j = \{A\in\mathbb{F}_q[T]:\ AC_{j-1}\equiv C_j - C_{j-2}\ (R)\}.$$

Applying Theorem 3, Lemma 1, Lemma 2, and Lemma 5 immediately gives the result. □

The proof of Theorem 4 follows from these two lemmas.

Proof of Theorem 4. Set $X_i := f(Q_{k_i}, \dots, Q_{k_{i+l-1}}, A_{k_{i+l}})$ and consider

$$\begin{aligned}\mathbf{E}X_i &= \sum_{\substack{A\in\mathbb{F}_q[T],\ \deg A\geq 1\\ B_1,\dots,B_l \text{ acceptable}}} f(B_1,\dots,B_l,A)h(T_{k_i}(B_1,\dots,B_l,A))\\ &= (1+\mathcal{O}(\rho^{k_i}))C(f).\end{aligned}$$

Therefore, we have

$$\mathbf{E}\left(\sum_{i\leq n} X_i\right) = C(f)n + \mathcal{O}(1). \tag{11}$$

Next, we estimate

$$\mathbf{E}|X_i| = \sum_{\substack{A\in\mathbb{F}_q[T],\ \deg A\geq 1\\ B_1,\dots,B_l \text{ acceptable}}} |f(B_1,\dots,B_l,A)|h(T_{k_i}(B_1,\dots,B_l,A)) \ll 1$$

and it is clear that we have $\mathbf{E}(X_i)^2 \ll 1$ as well. Because of Lemma 5, we can apply Lemma 1.2.1 in [9] in order to estimate the covariance

$$\mathrm{Cov}(X_{i_1}, X_{i_2}) \ll \rho^{i_2-i_1}\mathbf{E}|X_{i_1}|\mathbf{E}|X_{i_2}| \ll \rho^{i_2-i_1} \tag{12}$$

for $i_2 \geq i_1 + l - 1$. Furthermore, notice

$$\begin{aligned}|\mathrm{Cov}(X_{i_1}, X_{i_2})| &\leq \mathbf{E}|X_{i_1}X_{i_2}| + \mathbf{E}|X_{i_1}|\mathbf{E}|X_{i_2}|\\ &\ll \mathbf{E}^{1/2}(X_{i_1})^2\mathbf{E}^{1/2}(X_{i_2})^2 + 1 \ll 1\end{aligned} \tag{13}$$

for $i_2 > i_1$. Next, consider

$$\mathbf{V}\left(\sum_{i=m+1}^{m+n} X_i\right) = \sum_{i=m+1}^{m+n}\mathbf{V}X_i + 2\sum_{m+1\leq i_1<i_2\leq m+n}\mathrm{Cov}(X_{i_1}, X_{i_2}) \tag{14}$$

and break the last sum into two parts $\sum = \sum' + \sum''$ according to whether $i_2 \geq i_1 + l - 1$ or not. Because of (12) and (13) both parts are bounded by n. Trivially, the first sum on the right hand side of (14) is also bounded by n and so, we finally get

$$\mathbf{V}\left(\sum_{i=m+1}^{m+n} X_i\right) \ll n.$$

Using Gaal-Koksma's method (see for instance Theorem 1.155 in [1]) yields

$$\sum_{i\leq n}(X_i - \mathbf{E}X_i) = \mathcal{O}(n^{1/2}(\log n)^{3/2+\epsilon})$$

for all $\epsilon > 0$ and together with (11), we get the result. □

Proof of Corollary 1. First, we can assume w.l.o.g. that $\deg S < \deg R$.

Furthermore, choose a polynomial T with $\deg T < \deg R$ and $(S,T,R)=1$. Define a function as

$$f(C_1,C_2,A) = \begin{cases} 1 & C_1 = S \text{ and } C_2 = T \\ 0 & \text{otherwise} \end{cases}$$

for polynomials C_1, C_2 with $\deg C_1, \deg C_2 < \deg R$ and extend it mod R in the first two coordinates. By applying Theorem 4 the corollary follows. □

5. Proof of Theorem 5

The proof is very similar to the one in [2], therefore we will mainly emphasis on differences that occur when this proof is transferred to the Laurent series case: more specifically, we will give the fundamental lemmas used in this proof.

As in [2], we start by approximating the sequence $(X_n)_{n\geq 0}$ several times. Therefore, define a sequence of random variables as

$$U_k(\alpha) := \#\{0 \leq \deg C \leq \deg A_{k+1} : CQ_k \equiv S\ (R), \\ |C|^2 < |A_{k+1}| f(\deg CQ_k)\}.$$

According to the following lemma, we can use this sequence to approximate $(X_n)_{n\geq 0}$

Lemma 6. *We have*

$$\sum_{\deg Q_{k+1}\leq n} U_k(\alpha) + \mathcal{O}(1) \leq X_n(\alpha) \leq \sum_{\deg Q_k \leq n} U_k(\alpha) + \mathcal{O}(1) \qquad (15)$$

Proof. The proof essentially runs along the same lines than the proof of Lemma 2 in [3].

The only difference is that, because of the following elementary property of the continued fraction expansion in the Laurent series case

$$\left|\alpha - \frac{P_k}{Q_k}\right| = \frac{1}{|Q_k||Q_{k+1}|} = \frac{1}{|Q_k|^2|A_{k+1}|},$$

we have

$$\left|\alpha - \frac{P_k}{Q_k}\right| < \frac{f(\deg CQ_k)}{|CQ_k|^2} \Longleftrightarrow |C|^2 < |A_{k+1}| f(\deg CQ_k).$$

□

In order to approximate once more, the law of the iterated logarithm for the denominators of the convergents in the continued fraction expansion due to Gordin and Reznik was used in [2]. As it was already pointed out in [4], there is a similar result in the Laurent series case (proved by Niederreiter in [8]).

Lemma 7. *For almost all $\alpha \in H$, we have*

$$\limsup_{k\longrightarrow\infty} \frac{|\deg Q_k - k\log_q \gamma|}{\sqrt{2\sigma^2 k \log\log k}} = 1$$

where $\sigma = q/(q-1)^2$ and $\gamma = q^{q/(q-1)}$ is the Khintchine-Levy constant.

By this lemma, we have for each $\epsilon > 0$ that there exists κ large enough such that

$$k \log_q \gamma - \kappa k^{1-\delta} \leq \deg Q_k \leq k \log_q \gamma + \kappa k^{1-\delta}, \quad k \geq 1$$

for a subset F of H with $h(F) \geq 1 - \epsilon$. Using this, we set

$$f_1(k) := f((k+1)\log \gamma + \kappa(k+1)^{1-\delta}), \quad f_2(k) := f(k \log \gamma + \kappa k^{1-\delta}),$$

and define

$$V_k^{(i)}(\alpha) := \#\{0 \leq \deg C : \ CQ_k \equiv S\ (R), |C|^2 < |A_{k+1}| f_i(k)\}.$$

Then, we have

$$V_k^{(1)}(\alpha) \leq U_k(\alpha) \leq V_k^{(2)}(\alpha)$$

for $\alpha \in F$ and k large enough.

Next, we put

$$F_i(n) = \sum_{k=0}^{n} q^{\lceil \log_q f_i(k) \rceil}$$

and define

$$V_{k,n}^{(i)}(\alpha) := \#\{0 \leq \deg C \leq \phi_n : \ CQ_k \equiv S\ (R), |C|^2 < |A_{k+1}| f_i(k)\} \quad (16)$$

where $\phi_n = [\log_q((F_i(n))^{1/2}(\log F_i(n))^{1/2-\rho})]$ and $\rho > 0$ is a real constant.

In [2], moments of the above sequence of random variables were computed with help of several lemmas. We need analogues of these lemmas. The first lemma in [2] was the Theorem of Szüsz and we will use Theorem 2 instead of it. The second lemma was an identity observed by Philipp. We have the following analogue in the Laurent series case

Theorem 8. *Set*

$$K(D) := \frac{|R|\varphi((D,R))}{C(R)|(D,R)|}.$$

Then, we have

$$\sum_{\deg D < \deg R} \ \sum_{\substack{0 \leq \deg C \\ CD \equiv S\ (R)}} \frac{K(D)}{|C|^2} = \frac{q}{|R|}. \quad (17)$$

Remark 4. In the classical case, Philipp obtained this identity by comparing two deep results on metric diophantine approximation. Here, we give a direct and elementary proof.

For the proof, we need a few lemmas.

Lemma 8. *Assume that* $\deg S < \deg R$ *and set*

$$L = \{C \in \mathbb{F}_q[T] : \ C \equiv S\ (R)\}.$$

Then, we have

$$\sum_{C\in L} \frac{1}{|C|^2} = \frac{1}{|S|^2} + \frac{q}{|R|^2} \tag{18}$$

if $\deg S \geq 0$ *and*

$$\sum_{C\in L, C\neq 0} \frac{1}{|C|^2} = \frac{q}{|R|^2} \tag{19}$$

if $S = 0$*, respectively.*

Proof. This is an easy calculation. □

Lemma 9. *We have*

$$\sum_{(C,R),\ 0\leq \deg C<\deg R (C,R)=1} \frac{1}{|C|^2} = \frac{q}{|R|^2}(C(R) - \varphi(R)). \tag{20}$$

Proof. We use the principle of inclusion and exclusion in order to get

$$\sum_{\substack{0 \leq \deg C < \deg R \\ (C,R)=1}} \frac{1}{|C|^2} = \sum_{P_1\cdots P_k|R} (-1)^k \sum_{0\leq C<\deg R - \deg P_1\cdots P_k} \frac{1}{|P_1\cdots P_k C|^2}$$

where the first sum on the right hand side is extended over all monic, irreducible, pairwise disjoint k-tuples of polynomials $P_1, \ldots, P_k$ satisfying $P_1 \cdots P_k | R$. A straightforward calculation gives

$$\sum_{0\leq C<\deg R - \deg P_1\cdots P_k} \frac{1}{|P_1\cdots P_k C|^2} = q\left(\frac{1}{|P_1\cdots P_k|^2} - \frac{1}{|R||P_1\cdots P_k|}\right)$$

and consequently

$$\begin{aligned}\sum_{\substack{0 \leq \deg C < \deg R \\ (C,R)=1}} \frac{1}{|C|^2} &= q\sum_{P_1\cdots P_k|R} \frac{(-1)^k}{|P_1\cdots P_k|^2} - \frac{q}{|R|}\sum_{P_1\ldots P_k|R} \frac{(-1)^k}{|P_1\ldots P_k|} \\ &= q\prod_{P|R}\left(1 - \frac{1}{|P|^2}\right) - \frac{q}{|R|}\prod_{P|R}\left(1-\frac{1}{|P|}\right) \\ &= \frac{q}{|R|^2}(C(R) - \varphi(R))\end{aligned}$$

which proves the result. □

Lemma 10. *We have*

$$\sum_{\deg C<\deg R,\ (C,R)|S} |(C,R)|\varphi((C,R)) = |(S,R)|\varphi(R) \prod_{P|R,\ P\nmid \frac{R}{(S,R)}} \left(1 + \frac{1}{|P|}\right) \tag{21}$$

where the product is extended over all monic, irreducible polynomials P satisfying the desired property.

Proof. The proof of Lemma 5 in [2] carries over without difficulties. □

After these preliminaries, we can start with the proof of Theorem 8.

Proof of Theorem 8. Since the congruence relation in the second sum on the left hand side of (17) has no solution in case of D with $(D,R)\nmid(S,R)$, we can add the condition $(D,R)|(S,R)$ to the first sum on the left hand side without changing the result of the left hand side. So, we have

$$\sum_{\deg D<\deg R}\ \sum_{\substack{0\leq \deg C\\ CD\equiv S\ (R)}}\frac{K(D)}{|C|^2}=\sum_{K|(S,R)}\ \sum_{\deg D<\deg R,\ (D,R)=K}\ \sum_{\substack{0\leq \deg C\\ C\bar{D}\equiv \bar{S}\ (\bar{R})}}\frac{K(D)}{|C|^2} \tag{22}$$

where $\bar{D}=D/K,\bar{S}=S/K,\bar{R}=R/K$ and the first sum on the right hand side runs over all monic divisors of (S,R). Next, we have to distinguish between $S=0$ and $S\neq 0$.

In case $S=0$, we compute

$$\begin{aligned}\sum_{K|R}\left(\sum_{\deg D<\deg R,\ (D,R)=K}\left(\sum_{C\equiv 0\ (\bar{R}),\ 0\leq \deg C}\frac{K(D)}{|C|^2}\right)\right)\\ &=\frac{q}{|R|C(R)}\sum_{K|R}|K|\varphi(K)\varphi\left(\frac{R}{K}\right)\\ &=\frac{q}{|R|C(R)}\sum_{\deg C<\deg R}|(C,R)|\varphi((C,R))=\frac{q}{|R|}\end{aligned}$$

where Lemma 8 and Lemma 10 were used.

In case $S\neq 0$ notice that the last sum on the right hand side of (22) runs over a residue class $\bar{E}$ mod $\bar{R}$ with $(\bar{E},\bar{R})=(\bar{S},\bar{R})$. Conversely, it is easy to see that each residue class $\bar{E}$ mod $\bar{R}$ with $(\bar{E},\bar{R})=(\bar{S},\bar{R})$ appears on the right hand side of (22) and moreover, each residue class with the above property appears equally often and therefore

$$T(K):=\varphi\left(\frac{R}{K}\right)/\varphi\left(\frac{R}{(S,R)}\right)$$

many times. Hence

$$\begin{aligned}&\sum_{K|(S,R)}\ \sum_{\substack{\deg D<\deg R\\ (D,R)=K}}\ \sum_{\substack{0\leq \deg C\\ C\bar{D}\equiv \bar{S}\ (\bar{R})}}\frac{K(D)}{|C|^2}\\ &\quad=\frac{|R|}{C(R)}\sum_{K|(S,R)}\frac{\varphi(K)T(K)}{|K|}\sum_{\substack{\deg\bar{E}<\deg\bar{R}\\ (\bar{E},\bar{R})=(\bar{S},\bar{R})}}\ \sum_{C\equiv\bar{E}\ (\bar{R})}\frac{1}{|C|^2}.\end{aligned}$$

Because of Lemma 8 and Lemma 9, we have for the last two sums

$$\begin{aligned}
\sum_{\substack{\deg \bar{E} < \deg \bar{R} \\ (\bar{E},\bar{R})=(\bar{S},\bar{R})}} \sum_{C\equiv \bar{E}\ (\bar{R})} \frac{1}{|C|^2} &= \sum_{\substack{\deg \bar{E} < \deg \bar{R} \\ (\bar{E},\bar{R})=(\bar{S},\bar{R})}} \left(\frac{1}{|\bar{E}|^2} + \frac{q}{|\bar{R}|^2}\right) \\
&= \frac{\varphi(\tilde{R})q}{|\bar{R}|^2} + \sum_{\substack{\deg E < \deg \tilde{R} \\ (E,\tilde{R})=1}} \frac{1}{|(\bar{S},\bar{R})E|^2} \\
&= \frac{\varphi(\tilde{R})q}{|\bar{R}|^2} + \frac{q}{|(\bar{S},\bar{R})\tilde{R}|^2}(C(\tilde{R}) - \varphi(\tilde{R})) \\
&= \frac{q}{|\bar{R}|^2} C(\tilde{R})
\end{aligned}$$

where $\tilde{R} = R/(S,R)$. Consequently, by using Lemma 10

$$\begin{aligned}
&\sum_{K|(S,R)} \sum_{\substack{\deg D < \deg R \\ (D,R)=K}} \sum_{\substack{0\le \deg C \\ C\bar{D}\equiv \bar{S}\ (\bar{R})}} \frac{K(D)}{|C|^2} \\
&\qquad= \frac{qC(\tilde{R})}{|R|C(R)\varphi(\tilde{R})} \sum_{K|(S,R)} |K|\varphi(K)\varphi(\bar{R}) \\
&\qquad= \frac{qC(\tilde{R})}{|R|C(R)\varphi(\tilde{R})} |(S,R)|\varphi(R) \prod_{P|R,\ P\nmid\tilde{R}} \left(1 + \frac{1}{|P|}\right)
\end{aligned}$$

and after some easy calculations, we are done. □

Now, we are able to compute moments of (16).

Lemma 11. *For the sequence (16), we have*

$$\mu_{i,n} := \mathbf{E}\left(\sum_{k\le n} V_{k,n}^{(i)}\right) = \frac{q}{|R|} F_i(n) + \mathcal{O}\left(F_i(n)^{1/2}\right), \tag{23}$$

and

$$\tau_{i,n}^2 := \mathbf{V}\left(\sum_{k\le n} V_{k,n}^{(i)}\right) \sim \sigma F_i(n) \log F_i(n) \tag{24}$$

with

$$\sigma = \frac{(q^2-1)|(S,R)|\varphi(R)}{2\log q|R|C(S,R)}.$$

Proof. The proof is similar to the one of Lemma 6 in [2] and so, we are only going to point out differences. First, we write

$$V_{k,n}^{(i)} = \sum_{\substack{\deg D < \deg R \\ (D,R)|S}} \sum_{\substack{0\le \deg C \le \phi_n \\ CD\equiv S\ (R)}} \xi_k^{(C,D,i)}$$

where

$$\xi_k^{(C,D,i)}(\alpha) = \begin{cases} 1 & Q_k \equiv D\ (R) \text{ and } |C|^2 < |A_{k+1}| f_i(k) \\ 0 & \text{otherwise.} \end{cases}$$

By using Theorem 2, Theorem 8, and a little bit straightforward calculation (23) is easy to derive. Furthermore, it is easy to see that

$$\mathbf{E}(V_{k,n}^{(i)})^2 = \sum_{\substack{\deg D < \deg R \\ (D,R)|S}} \sum_{l=\deg \bar{R}}^{\phi_n} k_{l,D} K(D) q^{\lceil \log_q f_i(k) \rceil - 2l} (1 + \mathcal{O}(\rho^k)) + \mathcal{O}\left(q^{\lceil \log_q f_i(k) \rceil} \right).$$

Here, $\bar{R} = R/(D,R)$ and

$$k_{l,D} = (q^2-1)^2 q^{2(l - \deg \bar{R})} + \mathcal{O}(q^{l - \deg \bar{R}})$$

where the implied constant is absolute. Consequently, after some easy calculations,

$$\begin{aligned} \mathbf{E}(V_{k,n}^{(i)})^2 &= q^{\lceil \log_q f_i(k) \rceil} (1 + \mathcal{O}(\rho^k)) \frac{(q^2-1)\phi(n)}{|R| C(R)} \\ &\qquad \sum_{\substack{\deg D < \deg R \\ (D,R)|S}} |(D,R)| \varphi((D,R)) + \mathcal{O}\left(q^{\lceil \log_q f_i(k) \rceil} \right) \\ &= q^{\lceil \log_q f_i(k) \rceil} (1 + \mathcal{O}(\rho^k)) \frac{(q^2-1)\phi(n) |(S,R)| \varphi(R)}{|R| C(S,R)} \\ &\qquad + \mathcal{O}\left(q^{\lceil \log_q f_i(k) \rceil} \right) \end{aligned}$$

where Lemma 10 was used. Next, we sum over k and hence

$$\sum_{k \le n} \mathbf{E}(V_{k,n}^{(i)})^2 = \frac{(q^2-1)\phi(n) |(S,R)| \varphi(R)}{|R| C(S,R)} F_i(n) + \mathcal{O}(\phi_n) + \mathcal{O}(F_i(n)).$$

Therefore, we have

$$\sum_{k \le n} \mathbf{E}(V_{k,n}^{(i)})^2 \sim \sigma F_i(n) \log F_i(n)$$

and we can proceed as in the proof of Lemma 6 in [2] in order to get (24). □

In [2], the analogue of this result together with the mixing behaviour of the sequence analogues to (16) was used to obtain a central limit theorem for the approximating sequence of random variables. Since (16) has exactly the same mixing behaviour (follows immediately from Theorem 3) the proof method introduced in [2] can be used without difficulties to get asymptotic normality of (16), too.

Lemma 12. *We have*

$$\lim_{n\longrightarrow\infty} h\left[\left(\sum_{k\le n} V_{k,n}^{(i)}-\mu_{i,n}\right)/\tau_{i,n}\le\omega\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega}e^{-t^2/2}dt.$$

Furthermore, in [2] the analogue of this result entailed the corresponding central limit theorem for the sequence analogues to $(X_n)_{n\ge 0}$. The proof there can be used without any changes in the formal Laurent series case as well. We just have to point out the following lemma (compare with Lemma 9 in [2]) that is easy to obtain.

Lemma 13. *Let g_1 (resp. g_2) be the inverse function of $(k+1)\log\gamma+\kappa(k+1)^{1-\delta}$ (resp. $k\log\gamma-\kappa k^{1-\delta}$). Then, we have*

$$F_i(g_i(n))=\frac{1}{\log\gamma}F(n)+\mathcal{O}(F(n)^{1/2}).$$

6. Proof of Theorem 6 and Theorem 7

In order to prove the theorems, we closely follow the method introduced in [3], whereas we again mainly focus on differences. First define

$$V_k^{(i)}(\alpha):=\#\{0\le\deg C\le d:\ CQ_k\equiv S\ (R),|C|^2\le|A_{k+1}|f_i(k)\}\qquad(25)$$

where

$$f_1(k):=f((k+1)\log\gamma+\kappa(k+1)^{1-\delta}),\quad f_2(k):=f(k\log\gamma+\kappa k^{1-\delta})$$

with κ as in the last section. By Lemma 7 and the proof method of Lemma 2 in [3] it is immediate that Y_n can be approximated by a suitable sum of $V_k^{(i)}$.

In [3], we proceeded in two steps. In a first step, we proved almost sure and distribution type invariance principles for the approximating sequence and then, in a second step, we proved the corresponding results for the sequence analogues to Y_n by using the approximation. Thereby, the second step was only technical and the method can be used without difficulties in the Laurent series case, too (especially, notice that Lemma 13 can be used instead of Lemma 11 in [3]). Though the method of the first step could also be carried over in the Laurent series case, we give different proofs because here, the situation is much more easier. This is thanks to the fact that the approximating sequence (25) is exponential mixing in difference to the approximating sequence in [3] where this was not the case.

So, the following analogue of Lemma 6 in [3] can be obtained just by using Theorem 2 and some easy calculations (in difference to [3] where we had to approximate once more).

Lemma 14. *Set*

$$F_i(n)=\sum_{k=0}^{n}q^{\lceil\log_q f_i(k)\rceil}.$$

Then, we have

$$\mathbf{E}\left(\sum_{k=0}^{n} V_k^{(i)}\right) = (\sigma \log \gamma) F_i(n) + \mathcal{O}(1), \tag{26}$$

and either

$$\mathbf{V}\left(\sum_{k=0}^{n} V_k^{(i)}\right) \sim (\tau^2 \log \gamma) F_i(n), \tag{27}$$

if f satisfies (1) and (4), or

$$\mathbf{V}\left(\sum_{k=0}^{n} V_k^{(i)}\right) = (\tau^2 \log \gamma) F_i(n) + \mathcal{O}(F_i(n)^{1-\delta_3}), \tag{28}$$

if f satisfies (1), (4), and (8).

Proof. Similar to the proof of Lemma 11 in the last section. □

Next, we normalize

$$\xi_k^{(i)} = V_k^{(i)} - \mathbf{E} V_k^{(i)}$$

and define a suitable blocking (compare with the definition of η_k in [3]): for a fixed positive integer k denote by h_k the integer satisfying

$$(\tau^2 \log \gamma) F_i(h_k) \le k < (\tau^2 \log \gamma) F_i(h_k + 1)$$

and set $h_0 = -1$. Then, define a sequence of random variables as

$$\eta_k^{(i)} = \sum_{l=h_{k-1}+1}^{h_k} \xi_l^{(i)}.$$

Using this notation, we can prove the following analogue of Lemma 7 in [3] (again observe that the proof is much easier than in the real case)

Lemma 15. *We have*

(1) *the sequence $(\eta_k^{(i)})_{k\ge 1}$ is exponential mixing.*
(2) *Either*

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k^{(i)}\right)^2 \sim n,$$

if f satisfies (1) and (4), or

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k^{(i)}\right)^2 = n + \mathcal{O}(n^{1-\delta_3}),$$

if f satisfies (1), (4), and (8).
(3)

$$\mathbf{E}\left|\eta_k^{(i)}\right|^3 \ll 1.$$

Proof. (1) is obvious since $\xi_l^{(i)}$ is exponential mixing (follows immediately from Theorem 3) and (2) is a consequence of the last lemma together with the definition of $\eta_k^{(i)}$.

In order to prove (3) notice

$$0 \leq k - (\tau^2 \log \gamma) F_i(h_k) \leq (\tau^2 \log \gamma) f_i(h_k + 1) \ll 1$$

and therefore

$$(\tau^2 \log \gamma) F_i(h_k) = k + \mathcal{O}(1). \tag{29}$$

Next, observe by the multinomial theorem

$$\begin{aligned}
\mathbf{E}\left|\eta_k^{(i)}\right|^3 &= \mathbf{E}\left|\sum_{l=h_{k-1}+1}^{h_k} \xi_l^{(i)}\right|^3 \\
&= \sum_{e_{h_{k-1}+1}+\ldots+e_{h_k}=3} \binom{3}{e_{h_{k-1}+1}, \ldots, e_{h_k}} \mathbf{E}\left|\xi_{h_{k-1}+1}^{(i)}\right|^{e_{h_{k-1}+1}} \cdots \left|\xi_{h_k}^{(i)}\right|^{e_{h_k}} \\
&\ll \sum_{e_{h_{k-1}+1}+\ldots+e_{h_k}=3} \binom{3}{e_{h_{k-1}+1}, \ldots, e_{h_k}} \mathbf{E}\left|\xi_{h_{k-1}+1}^{(i)}\right|^{e_{h_{k-1}+1}} \cdots \mathbf{E}\left|\xi_{h_k}^{(i)}\right|^{e_{h_k}} \\
&\ll \left(\sum_{l=h_{k-1}+1}^{h_k} \mathbf{E}\left|\xi_l^{(i)}\right|\right)^3 \ll (F_i(h_k) - F_i(h_{k-1}))^3 \ll 1
\end{aligned}$$

which is the desired result. Here, the mixing property of $\xi_k^{(i)}$, the following estimation which is straightforward to prove

$$\mathbf{E}\left|\xi_k^{(i)}\right| \ll \mathbf{E} V_k^{(i)} \ll q^{\lceil \log_q f_i(k) \rceil},$$

and (29) were used. □

For the rest of the proof of almost sure and distribution type invariance principles for the sequence (25), we can proceed as in [3] and therefore, with the remarks at the beginning of this section, we are done.

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