INVARIANCE PRINCIPLES IN METRIC DIOPHANTINE APPROXIMATION

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Abstract. In [7], LeVeque proved a central limit theorem for the number of solutions p, q of

$$\left| x - \frac{p}{q} \right| \leq \frac{f(\log q)}{q^2}$$

subjected to the following conditions

$$0 < q \leq n, \quad (p, q) \leq d,$$

where $x \in [0, 1]$ and f satisfies certain assumptions. The case $d = 1$ was considerably improved a few years later by Philipp [8]. We give a common extension of both results by proving almost sure and distribution type invariance principles. Our results entail several corollaries e.g. a functional central limit theorem and a Strassen's type version of the iterated logarithm law.

1. Introduction

Suppose f is a real-valued, positive function defined on the non-negative real numbers satisfying the following conditions

$$f \downarrow 0, \quad \sum_{k=1}^{\infty} f(k) = \infty, \tag{1}$$

$$\sum_{k=1}^{n} f(k) k^{-\delta_1} \ll \left(\sum_{k=1}^{n} f(k)\right)^{1/2-\delta_2}. \tag{2}$$

We are interested in the diophantine approximation problem

$$\left| x - \frac{p}{q} \right| \leq \frac{f(\log q)}{q^2} \tag{3}$$

which, according to a famous result of Khintchine [5], has infinitely many solutions p, q with $q > 0$ for almost all $x \in [0, 1]$ (with respect to Lebesgue measure which we are going to denote by λ).

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In [7], LeVeque investigated the statistical behaviour of the following sequence of random variables

$$T_n^{(d)}(x) := \#\{\langle p,q\rangle | 1 \leq q \leq n, (p,q) \leq d, p/q \text{ is a solution of (3)}\},$$

where d is a fixed positive integer. He proved the following result.

Theorem 1. *Suppose f is a real-valued, positive, non-increasing function defined on the non-negative real numbers satisfying the following conditions*

$$f(x) = \mathcal{O}(x^{-1}), f'(x) = \mathcal{O}(x^{-2}), \text{ as } x \longrightarrow \infty; \quad \sum_{k=1}^{\infty} f(k) = \infty \qquad (4)$$

and set

$$F(n) = \sum_{k=1}^{n} \frac{f(\log k)}{k}.$$

Then, we have

$$\lim_{n\to\infty} \left[T_n^{(d)} < 2\left(1 - \frac{6}{\pi^2} \sum_{i=d+1}^{\infty} i^{-2}\right) F(n) + \omega \left(\left(\frac{12}{\pi^2} \sum_{i=1}^{d} \frac{2i-1}{i^2} \right) F(n) \right)^{1/2} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2} du.$$

Remark 1. It's easy to see that the class of functions considered in the above theorem forms a subset of the class of functions introduced in the beginning of this section.

In an earlier paper [6], LeVeque had already treated the case $d = 1$. A few years later this case was improved by Philipp [8] who relaxed the conditions on f, added an iterated logarithm law, and used Szüsz's generalization of Khintchine's theorem (see [13]) in order to take only solutions of (3) into account that have denominators contained in an arithmetic progression.

In details, consider the following sequence of random variables

$$T_n(x) = \#\{\langle p,q\rangle | 1 \leq q \leq n, q \equiv s(r), (p,q) = 1, \; p/q \text{ is a solution of (3)}\},$$

where $r \geq 1, s$ are arbitrary integers. Then, the result of Philipp reads as follows.

Theorem 2. *Suppose f satisfies (1) and (2) and set*

$$F(n) = \sum_{k=1}^{n} \frac{f(\log k)}{k}.$$

Then, we have

$$\lim_{n\to\infty} \lambda \left[T_n < \sigma F(n) + \omega(\sigma F(n))^{1/2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2} du,$$

and

$$\limsup_{n\to\infty} \frac{|T_n - \sigma F(n)|}{\sqrt{2\sigma F(n) \log\log F(n)}} = 1,$$

where

$$\sigma = \frac{12}{\pi^2} \frac{r\varphi((s,r))}{C(r)(s,r)}, \quad C(r) = r^2 \prod_{p|r} \left(1 - \frac{1}{p^2}\right).$$

Remark 2. Notice that the class of functions considered in [8] seems to be slightly larger than the one considered here; especially in [8] condition (2) is just assumed with $\delta_2 = 0$. However, that's a minor mistake in [8] and in fact the class considered there has to be replaced by the class considered here.

The aim of this paper is to provide a common generalization of these two results by proving almost sure and distribution type invariance principles. The paper is organized as follows: in the next section, we state the main results and several consequences. In Section 3 the proofs of the main results are prepared by approximating the involved sequence of random variables. The corresponding invariance principles for these approximating sequences of random variables are obtained in Section 4 and the proofs of the main results are completed in Section 5 by showing that the approximation is strong enough. In the last section, we outline the proofs of the corollaries and lemmas stated in the next section. Finally, in an appendix, we sketch a proof of an extension of a theorem due to Philipp and Stout needed in Section 4.

2. Results

Let f be a real-valued, positive function defined on the non-negative real numbers satisfying conditions (1) and (2). We define a sequence of random variables as

$$X_n(x) := \#\{\langle p,q\rangle | 1 \le q \le n, q \equiv s(r), (p,q) \le d, \\ p/q \text{ is a solution of (3)}\},$$

where $n \ge 0$ and $d \ge 1, r \ge 1, s$ are fixed integers.

Furthermore, put

$$F(n) = \sum_{k=1}^{n} \frac{f(\log k)}{k}$$

and

$$\sigma = \sigma_d = \frac{12r}{C(r)\pi^2} \sum_{0 \le q < r} \frac{\varphi((q,r))}{(q,r)} \sum_{\substack{1 \le c \le d \\ cq \equiv s(r)}} \frac{1}{c^2},$$

$$\tau^2 = \tau_d^2 = \frac{12r}{C(r)\pi^2} \sum_{0 \le q < r} \frac{\varphi((q,r))}{(q,r)} \sum_{\substack{1 \le c \le d \\ cq \equiv s(r)}} \frac{k_{c,q}}{c^2},$$

where $k_{c,q}$ is the number of pairs $\langle k_1, k_2 \rangle$ of solutions of $xq \equiv s(r)$ with $k_1 \le c, k_2 \le c$ and either $k_1 = c$ or $k_2 = c$.

Finally, we set

$$n_t = \begin{cases} \max\{n | \tau^2 F(n) \le t\} & \text{if } t \ge \tau^2 F(1) \\ 0 & \text{otherwise,} \end{cases}$$

where $t \ge 0$.

Now, let $([0,1]^2, \mathcal{B}_2, \lambda_2)$ denote the unit square with Lebesgue measure. Using the above notation, we define on this probability space a stochastic process X by setting for $\langle x_1, x_2 \rangle \in [0,1]^2$

$$X(t) = X(t; x_1, x_2) = X_{n_t}(x_1) - \sigma F(n_t).$$

Adjoining a uniformly distributed random variable independent of the entire sequence $(X_n(x_1))_{n \ge 1}$ guarantees that the probability space is rich enough.

Theorem 3. *There exists a sequence $(Y_i(x_1, x_2))_{i \ge 1}$ of independent, standard normal $N(0,1)$ random variables defined on the above probability space such that, as $n \longrightarrow \infty$,*

$$X(n) - \sum_{i \le n} Y_i = o((n \log\log n)^{1/2}), \quad a.s. \tag{5}$$

and

$$\lambda_2 \left[\frac{1}{\sqrt{n}} \max_{k \le n} \left| X(k) - \sum_{i \le k} Y_i \right| \ge \epsilon \right] \longrightarrow 0 \tag{6}$$

for all $\epsilon > 0$.

If f satisfies slightly stronger assumptions than we can prove even more.

Theorem 4. *Let f satisfy (1), (2) and the following additional condition*

$$\sum_{k=1}^{n} f^2(k) \ll \left(\sum_{k=1}^{n} f(k) \right)^{1-\delta_3} \tag{7}$$

where $0 < \delta_3 < 1$*. Then, there exists a sequence* $(Y_i(x_1, x_2))_{i\geq 1}$ *of independent, standard normal* $N(0,1)$ *random variables defined on the above probability space such that, as* $n \longrightarrow \infty$*,*

$$X(n) - \sum_{i\leq n} Y_i \ll n^{1/2-\lambda}, \quad a.s. \tag{8}$$

where $0 < \lambda < 1/2$ *is a real constant.*

Remark 3. Notice that an equivalent formulation of Theorem 3 and Theorem 4 can be given by using standard Brownian motion. For instance (8) can also be stated as

$$X(t) - W(t) \ll t^{1/2-\lambda} \tag{9}$$

where $W(t)$ denotes standard Brownian motion on the unit square.

In order to see that (8) and (9) are equivalent, we only have to show that (8) implies (9) since the other direction is obvious. Therefore, observe

$$X([t]) \leq X(t) + \mathcal{O}(1) \leq X([t]+1) + \mathcal{O}(1)$$

for $t \geq \min\{\tau^2 F(1), 1\}$. Hence

$$\Big|X(t) - \sum_{i\leq t} Y_i\Big| \leq \max\Big\{\Big|X([t]) - \sum_{i\leq [t]} Y_i\Big|, \\ \Big|X([t]+1) - \sum_{i\leq [t]+1} Y_i\Big| + |Y_{[t]+1}|\Big\} + \mathcal{O}(1).$$

It is plain by the Borel Cantelli Lemma that

$$Y_i \leq i^{1/2-\epsilon}, \quad \text{a.s.}$$

for all $0 < \epsilon < 1/2$ and consequently, by using (8)

$$X(t) - \sum_{i\leq t} Y_i \ll t^{1/2-\lambda}, \quad \text{a.s.} \tag{10}$$

for $t \geq \min\{\tau^2 F(1), 1\}$ and indeed for $0 < t < \min\{\tau^2 F(1), 1\}$ as well. Now, it is classical that $\sum_{i\leq t} Y_i$ can be approximated sufficiently close by standard Brownian motion and combining this with (10) immediately gives (9).

The reason why we have avoided standard Brownian motion in the formulation of Theorem 3 and Theorm 4 was to make the assertions of the results more lucid. But in fact, we will prove the theorems in their equivalent formulations with standard Brownian motion.

Remark 4. Theorem 3 and Theorem 4 entail several consequences (see [1],[10], [11],[14]). Theorem 3, for instance, implies the following functional central limit theorem that contains both central limit theorems mentioned in the introduction.

Corollary 1. *We have, as $n \longrightarrow \infty$,*

$$\left\{\frac{1}{\sqrt{\tau^2 F(n)}} X(\tau^2 F(n)t),\ 0 \leq t \leq 1\right\} \longrightarrow \{W(t),\ 0 \leq t \leq 1\}.$$

Furthermore, we can deduce from Theorem 3 a Strassen's type version of the iterated logarithm law. This generalizes the iterated logarithm law proved by Philipp. In order to state the result, let K denote Strassen's set that is the set of all real-valued, absolutely continuous functions $g(t)$ defined on the interval $[0,1]$ satisfying the following conditions

$$g(0) = 0, \qquad \int_0^1 (g'(t))^2 dt \leq 1.$$

Then, we have the following corollary.

Corollary 2. *The sequence of functions*

$$\frac{1}{(2\tau^2 F(n) \log\log F(n))^{1/2}} X(\tau^2 F(n)t)$$

defined on $[0,1]$ is a.s. relatively compact in the topology of uniform convergence and has K as its set of limit points.

From Theorem 4, we can deduce even more. As an example, we state a functional iterated logarithm law for the maximum (for other consequences see [10]). Therefore, let J be the set of extended (∞ included in the range), non-negative, non-decreasing functions $h(t), 0 \leq t < \infty$ which are right continuous except possibly at zero and satisfy

$$h(0) = 0, \qquad \int_0^\infty h^{-2}(t) dt \leq 1.$$

With this notation, we have

Corollary 3. *The sequence of functions*

$$\left(\frac{8 \log\log F(n)}{\pi^2 \tau^2 F(n)}\right)^{1/2} \sup_{s \leq t} |X(\tau^2 F(n)s)|$$

defined on $[0,\infty)$ is a.s. relatively compact in the topology of weak convergence (i.e., pointwise convergence at all continuity points of the limit function) and has J as its set of limit points.

Remark 5. The central limit theorem contained in Corollary 1 can be viewed as an approximation to the central limit theorem proved by the author in a recent paper [3] for the following sequence of random variables

$$W_n(x) := \#\{\langle p,q\rangle | 1 \le q \le n, q \equiv s(r), p/q \text{ is a solution of (3)}\}. \quad (11)$$

This is getting even more lucid if we point out the following asymptotic expansions for σ and τ^2.

Lemma 1. *We have*

$$\sigma = \frac{2}{r} + \mathcal{O}\left(\frac{1}{d}\right),$$

and

$$\tau^2 = \frac{24(s,r)\varphi(r)\log d}{\pi^2 r C(s,r)} + \mathcal{O}(1),$$

where

$$C(s,r) = r^2 \prod_{p|r}\left(1-\frac{1}{p}\right) \prod_{p|\frac{r}{(s,r)}}\left(1+\frac{1}{p}\right).$$

Especially notice, that the main term in the asymptotic expansion of σ is, according to [3], the constant belonging to the mean value of (11) and that τ^2 is increasing logarithmically with d which could be seen as explanation for $F(n)\log F(n)$ to be the order of magnitude of the variance of (11) (again compare with [3]).

3. Preliminaries

In order to fix notation let $x = [a_0, a_1, \ldots]$ be the continued fraction expansion of $x \in [0,1]$ and denote by

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k]$$

the k-th convergent. Furthermore put

$$\varphi_k = [a_{k+1}, a_{k+2}, a_{k+3}, \ldots] + [0, a_k, a_{k-1}, \ldots, a_0],$$

and

$$\xi_k = [0, a_{k+1}, a_{k+2}, \ldots].$$

First consider the sequence of random variables

$$Y_k(x) := \#\{1 \le c \le d | cq_k \equiv s(r), c^2 \le \varphi_k f(\log cq_k)\},$$

where $x \in [0,1]$. We need the following simple fact.

Lemma 2. *We have*

$$\sum_{q_{k+1}\le n} Y_k(x) + \mathcal{O}(1) \le X_n(x) \le \sum_{q_k \le n} Y_k(x) + \mathcal{O}(1). \quad (12)$$

Proof. It is plain, by elementary properties of the continued fraction expansion, that we have

$$\left|x-\frac{p_k}{q_k}\right|\leq\frac{f(\log cq_k)}{cq_k^2}\qquad\Longleftrightarrow\qquad c^2\leq\varphi_k f(\log cq_k).$$

Furthermore, if p/q is a solution of (3) with $(p,q)\leq d$ then it follows

$$\left|x-\frac{p}{q}\right|\leq\frac{f(\log q)}{q^2}\leq\frac{1}{2q^2}$$

for q large enough. Hence, by using another elementary property of the continued fraction expansion, there are integers $k\geq 0$ and $1\leq c\leq d$ such that $p=cp_k$ and $q=cq_k$.

Therefore, it is enough to count multiples cp_k/cq_k of the convergents of x with the restrictions

$$1\leq c\leq d,\ cq_k\equiv s(r),\ \text{and } c^2\leq\varphi_k f(\log cq_k),\tag{13}$$

in order to obtain X_n. This explains the right hand side of (12).

For the left hand side of (12) suppose that cp_k/cq_k is satisfying (13) and $q_{k+1}\leq n$. By the elementary inequality

$$\sqrt{1/4(x+2)}\leq x,\quad\text{for } x\geq 1,$$

we have

$$c\leq\sqrt{\varphi_k f(\log cq_k)}\leq\sqrt{1/4(a_{k+1}+2)}\leq a_{k+1}$$

for k large enough and therefore

$$cq_k\leq a_{k+1}q_k\leq q_{k+1}\leq n.$$

This proves the left hand side of (12). □

Next, we need the following theorem due to Gordin and Reznik [4].

Lemma 3. *For almost all $x\in[0,1]$, we have*

$$\limsup_{k\longrightarrow\infty}\frac{|\log q_k-k\log\gamma|}{\sqrt{2\sigma^2 k\log\log k}}=1,$$

where $\sigma>0$ and $\gamma=\exp\left(\pi^2/(12\log 2)\right)$ is the Khintchine-Levy constant.

By this lemma, we have for each $\epsilon>0$ that there exist κ large enough such that

$$k\log\gamma-\kappa k^{1-\delta_1}\leq\log q_k\leq k\log\gamma+\kappa k^{1-\delta_1},\quad k\geq 1\tag{14}$$

for a subset F of $[0,1]$ with $\lambda(F)\geq 1-\epsilon$.

Using this, we get

$$f((k+1)\log\gamma+\kappa(k+1)^{1-\delta_1}) \le f(\log q_{k+1}) \le f(\log cq_k) \\ \le f(\log q_k) \le f(k\log\gamma-\kappa k^{1-\delta_1})$$

for $x \in F$ and $1 \le c \le a_{k+1}$.

We set

$$f_1(k) := f((k+1)\log\gamma+\kappa(k+1)^{1-\delta_1}), \quad f_2(k) := f((k\log\gamma-\kappa k^{1-\delta_1}),$$

and

$$F_i(n) = \sum_{k=1}^{n} f_i(k), \quad i = 1, 2.$$

Then, we define

$$\varphi_k^{(i)} = a_{k+1} + [0, a_{k+2}, \cdots, a_{k+[c_k^{(i)}\log F_i(k)]}] \\ + [0, a_k, a_{k-1}, \cdots, a_{k-[c_k^{(i)}\log F_i(k)]}], \quad i = 1, 2,$$

where $c_k^{(i)}$ is chosen uniformly bounded and in such a fashion that $c_k^{(i)} \ge 8/\ln 2$, $[c_k^{(1)}\log F_1(k)]$ is odd, and $[c_k^{(2)}\log F_2(k)]$ is even.

With this notation, we consider the following random variables (compare with p47 in [8])

$$Z_k^{(i)}(x) := \#\{1 \le c \le d | cq_k \equiv s(r), c^2 \le \varphi_k^{(i)} f_i(k)\}, \quad i = 1, 2.$$

By the definition of $\varphi_k^{(i)}$, we have

$$\varphi_k^{(1)} \le \varphi_k \le \varphi_k^{(2)}$$

and together with the definition of $f_i(k)$, we get

$$Z_k^{(1)}(x) \le Y_k(x) \le Z_k^{(2)}(x) \tag{15}$$

for $x \in F$ and k large enough.

Next, we need a theorem proved by Szüsz [13].

Lemma 4. *For $t \ge 2$ and $a, b \in \mathbb{N}_0$ define*

$$m_k(a,b,t) := \lambda\{x \in [0,1] | q_{k-1} \equiv a \ mod\ r, q_k \equiv b \ mod\ r, \varphi_k \ge t\}.$$

Then, we have

$$m_k(a,b,t) = \begin{cases} \frac{1}{C(r)\log 2} t^{-1}(1+\mathcal{O}(q^k)) & (a,b,r) = 1 \\ 0 & (a,b,r) \ne 1, \end{cases}$$

where $C(r) = r^2 \prod_{p|r}\left(1-\frac{1}{p^2}\right)$, $q < 1$ is a constant, and the constant implied in the error term only depends on r.

Furthermore, we need the following lemma which is straightforward to prove.

Lemma 5. *If f satisfies (1), (2), and (7) then, we have*

$$\sum_{k=1}^{n} f_i^2(k) = \mathcal{O}(F_i(n)^{1-\delta_3}). \tag{16}$$

We shall use the last two lemmas together with ideas of [3] and [8] in order to obtain the following result.

Lemma 6. *We have*

$$\mathbf{E}\sum_{k=1}^{n} Z_k^{(i)} = (\sigma \log\gamma)F_i(n) + \mathcal{O}(1), \tag{17}$$

and either

$$\mathbf{V}\sum_{k=1}^{n} Z_k^{(i)} \sim (\tau^2 \log\gamma)F_i(n), \tag{18}$$

if f satisfies (1) and (2), or

$$\mathbf{V}\sum_{k=1}^{n} Z_k^{(i)} = (\tau^2 \log\gamma)F_i(n) + \mathcal{O}(F_i(n)^{1-\delta_3}), \tag{19}$$

if f satisfies (1), (2), and (7).

Proof. We will only consider the case $i = 1$ because $i = 2$ is treated in the same fashion.

Define a sequence of random variables as

$$U_k^{(1)}(x) := \#\{1 \le c \le d \mid cq_k \equiv s(r), c^2 \le \varphi_k f_1(k)\}.$$

Using Lemma 4, we get

$$\begin{aligned} \mathbf{E}U_k^{(1)} &= \sum_{0\le q<r} \sum_{\substack{1 \le c \le d \\ cq \equiv s(r)}} \frac{1}{C(r)\ln 2}\frac{r\varphi((q,r))}{(q,r)}\frac{f_1(k)}{c^2}(1+\mathcal{O}(q^k)) \\ &= (\sigma\log\gamma)f_1(k)(1+\mathcal{O}(q^k)) \end{aligned} \tag{20}$$

and therefore

$$\mathbf{E}\sum_{k=1}^{n} U_k^{(1)} = (\sigma \log\gamma)F_1(n) + \mathcal{O}(1). \tag{21}$$

Furthermore, observe

$$\varphi_k - \varphi_k^{(1)} \ll 2^{-1/2c_k^{(1)}\log F_1(k)} \ll F_1(k)^{-4}$$

which yields

$$\begin{aligned}\lambda[U_k^{(1)} \neq Z_k^{(1)}] &\leq \lambda[\exists c : 1 \leq c \leq d | \varphi_k^{(1)} f_1(k) < c^2 \leq \varphi_k f_1(k)] \\ &\ll \sum_{c=1}^{d} \frac{f_1(k)}{c^2} - \left(\frac{c^2}{f_1(k)} + \mathcal{O}(F_1(k)^{-4}) \right)^{-1} \\ &\ll \frac{f_1(k)^2}{F_1(k)^4}\end{aligned} \tag{22}$$

and hence

$$\mathbf{E}(U_k^{(1)} - Z_k^{(1)}) \ll \frac{f_1(k)^2}{F_1(k)^4}. \tag{23}$$

Thus, by taking the Dini-Abel Theorem into account

$$\sum_{k=1}^{n} \mathbf{E}(U_k^{(1)} - Z_k^{(1)}) \ll 1 \tag{24}$$

and by combining (21) and (24), we get (17) as a consequence.

For the proof of (18) and (19) observe (compare with [3])

$$\begin{aligned}\mathbf{E}(U_k^{(1)})^2 &= \sum_{0 \leq q < r} \sum_{\substack{1 \leq c \leq d \\ cq \equiv s(r)}} \frac{k_{c,q}}{C(r) \ln 2} \frac{r\varphi((q,r))}{(q,r)} \frac{f_1(k)}{c^2} (1 + \mathcal{O}(q^k) \\ &= (\tau^2 \log \gamma) f_1(k)(1 + \mathcal{O}(q^k)).\end{aligned} \tag{25}$$

As in [8] (proof of Lemma 3.1.1), it is quite easy to see that

$$\mathrm{Cov}(U_{k_1}^{(1)} U_{k_2}^{(1)}) \ll f_1^2(k_1) \rho^{k_2 - k_1}, \tag{26}$$

for $k_1 < k_2$ where $\rho < 1$ is a real constant. By combining (20), (25), (26), and using Lemma 5

$$\mathbf{V} \sum_{k=1}^{n} U_k^{(1)} = \sum_{k=1}^{n} \mathbf{E}(U_k^{(1)})^2 - \sum_{k=1}^{n} (\mathbf{E} U_k^{(1)})^2 + 2 \sum_{1 \leq k_1 < k_2 \leq n} \mathrm{Cov}(U_{k_1}^{(1)} U_{k_2}^{(1)})$$

which is either asymptotic

$$(\tau^2 \log \gamma) F_1(n)$$

or equal to

$$(\tau^2 \log \gamma) F_1(n) + \mathcal{O}(F_1(n)^{1-\delta_3})$$

depending on the assumptions on f.

Finally, notice

$$\mathbf{E}(U_k^{(1)} - Z_k^{(1)})^2 \ll \frac{f_1(k)^2}{F_1(k)^4} \tag{27}$$

which is plain by (22) and therefore, we can proceed as in [8] in order to obtain (18) and (19). □

The next section is devoted to the proofs of invariance principles corresponding to (5), (6), and (8) but for $Z_k^{(i)}$.

4. Invariance principles for $Z_k^{(i)}$

Throughout this section, we are going to suppress the dependence on i. Furthermore, we denote by

$$\xi_k = Z_k - \mathbf{E}Z_k \tag{28}$$

the centered version of Z_k.

We shall use the following theorem due to Philipp and Stout (Theorem 7.1 in [10]) in order to obtain invariance principles for Z_k.

Theorem 5. *Let ξ_k be a sequence of centered random variables on the probability space $(\Omega, \mathcal{A}, P)$ and suppose that there exist constants $0 < \delta \le 2$ and $C > 0$ such that*

$$\mathbf{E}|\xi_k|^{2+\delta} \le C \log k \tag{29}$$

and

$$\mathbf{E}\left(\sum_{k=1}^{n} \xi_k\right)^2 \sim n. \tag{30}$$

Furthermore, denote by $\mathcal{F}_a^b$ the σ-algebra generated by $\xi_k, a \le k \le b$ and assume that ξ_k satisfies a so called retarded strong mixing condition of the form

$$|P(AB) - P(A)P(B)| \le \beta(kt^{-\kappa}) \tag{31}$$

for all $A \in \mathcal{F}_1^t$ and all $B \in \mathcal{F}_{k+t}^{\infty}$.

Here, β is a real-valued function defined on the non-negative real numbers with

$$\beta(s) \ll s^{-168(1+2/\delta)} \tag{32}$$

and

$$\kappa = \delta/(11 + 4\delta). \tag{33}$$

Define a process $\{Z(t), t \ge 0\}$ on the product probability space $(\Omega, \mathcal{A}, P) \times ([0,1], \mathcal{B}, \lambda)$ (here $\mathcal{B}$ denotes the sigma-algebra of Borel sets on $[0,1]$) by setting

$$Z(t) = \sum_{k \le t} \xi_k. \tag{34}$$

Then, we have, as $t \longrightarrow \infty$,

$$Z(t) - W(t) = o((t \log\log t)^{1/2}), \quad a.s. \tag{35}$$

and

$$(P \times \lambda)\left[\frac{1}{\sqrt{t}} \sup_{s \le t} |Z(s) - W(s)| \ge \epsilon\right] \longrightarrow 0 \tag{36}$$

for all $\epsilon > 0$, whereas $\{W(t), t \geq 0\}$ denotes Brownian motion on the above probability space.

Furthermore, if we replace (30) by the following stronger assumption

$$\mathbf{E}\left(\sum_{k=1}^{n} \xi_k\right)^2 = n + \mathcal{O}(n^{1-\delta/30}) \tag{37}$$

then, we obtain the stronger result

$$Z(t) - W(t) \ll t^{1/2-\lambda}, \quad a.s., \text{ as } t \longrightarrow \infty, \tag{38}$$

for each $\lambda < \delta/588$.

Remark 6. Actually, this result is an extension of the cited result due to Philipp and Stout. First of all, the result is much more explicit because the formulation "...we can redefine the process on a richer probability space..." used in [10] was avoided by providing an explicit probability space that is rich enough. This can easily be achieved by combining the proof method in [10] with the approach introduced by Dudley and Philipp in [2] (compare also with [9]). Secondly in [10], the $2 + \delta$-moments of the sequence ξ_k were assumed to be uniformly bounded. An inspection of the proof shows that this can be relaxed to (29). Thirdly in the paper of Philipp and Stout, the theorem was only stated with the stronger assumption (37). In order to proof the result with assumption (30) one can essentially follow the proof in [10]. In an appendix, we are going to outline the differences that occur in doing that.

Due to (30) resp. (37) it's impossible to apply the theorem directly on our sequence ξ_k (compare with (18) resp. (19)). Therefore, we will introduce a suitable blocking in order to be able to make use of the theorem.

Define for a positive integer k the integer h_k by

$$(\tau^2 \log \gamma) F(h_k) \leq k < (\tau^2 \log \gamma) F(h_k + 1).$$

Furthermore, put $h_0 = 0$ and consider a sequence of random variables defined as

$$\eta_k = \sum_{l=h_{k-1}+1}^{h_k} \xi_l$$

for $k \geq 1$. The sequence $(\eta_k)_{k \geq 1}$ satisfies the assumptions of Theorem 5 according to the following lemma.

Lemma 7. *We have*

(1) *The sequence* $(\eta_k)_{k\geq 1}$ *satisfies a retarded strong mixing condition of the form (31) with arbitrary* $\kappa > 0$ *and* $\beta(s) = q^s, q < 1$.
(2) *Either*

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k\right)^2 \sim n,$$

if f *satisfies (1) and (2), or*

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k\right)^2 = n + \mathcal{O}(n^{1-\delta_3}),$$

if f *satisfies (1), (2), and (7).*
(3)

$$\mathbf{E}|\eta_k|^3 \ll \log k.$$

Proof. We start by pointing out that the sequence $(\xi_k)_{k\geq 1}$ satisfies a mixing condition of the form

$$|\lambda(AB) - \lambda(A)\lambda(B)| \ll q^{k-\log F(t)} \tag{39}$$

for all $A \in \mathcal{F}_1^t$ and all $B \in \mathcal{F}_{t+k}^{\infty}$ (here, $q < 1$ is the constant in Lemma 4). This follows immediately from a result of Szüsz [12] (compare also with 3.3.4 in [8]).

Let $\mathcal{M}_a^b$ be the σ-algebra generated by $\eta_k, a \leq k \leq b$. Then, we have for $A \in \mathcal{M}_1^t$ and $B \in \mathcal{M}_{t+k}^{\infty}$

$$|\lambda(AB) - \lambda(A)\lambda(B)| \ll q^{h_{t+k} - h_t - \log F(h_t)}. \tag{40}$$

Furthermore, observe

$$0 \leq t - (\tau^2 \log \gamma) F(h_t) < (\tau^2 \log \gamma) f(h_t + 1) \ll 1 \tag{41}$$

and together with (40)

$$|\lambda(AB) - \lambda(A)\lambda(B)| \ll q^{k - \log t}.$$

It is plain that this implies

$$|\lambda(AB) - \lambda(A)\lambda(B)| \ll q^{kt^{-\kappa}}$$

for arbitrary $\kappa > 0$ and therefore, (1) is proved.

(2) easily follows by using (18) resp. (19) and (41)

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k\right)^2 = \mathbf{E}\left(\sum_{k=1}^{h_n} \xi_k\right)^2$$

which is either asymptotic

$$(\tau^2 \log \gamma) F(h_n) \sim n,$$

or equal to

$$(\tau^2 \log \gamma) F(h_n) + \mathcal{O}(F(h_n)^{1-\delta_3}) = n + \mathcal{O}(n^{1-\delta_3})$$

depending on the assumptions on f.

In order to prove (3), we observe that for $l \geq 1$

$$\mathbf{E}|\xi_k|^l \ll \mathbf{E} Z_k \ll f(k) \tag{42}$$

where (20) and (23) were used. Then, we expand the left hand side of (3) by the multinomial theorem

$$\begin{aligned}\mathbf{E}|\eta_k|^3 &= \mathbf{E}\left|\sum_{l=h_{k-1}+1}^{h_k} \xi_l\right|^3 \\ &= \sum_{e_{h_{k-1}+1}+\ldots+e_{h_k}=3} \binom{3}{e_{h_{k-1}+1},\ldots,e_{h_k}} \mathbf{E}|\xi_{h_{k-1}+1}|^{e_{h_{k-1}+1}} \cdots |\xi_{h_k}|^{e_{h_k}}\end{aligned}$$

and break the last sum into several parts according to the powers in the product $|\xi_{h_{k-1}+1}|^{e_{h_{k-1}+1}} \cdots |\xi_{h_k}|^{e_{h_k}}$.

For the third powers, we get, by using (41) and (42)

$$\sum_{l=h_{k-1}+1}^{h_k} \mathbf{E}|\xi_l|^3 \ll \sum_{l=h_{k-1}+1}^{h_k} f(l) = F(h_k) - F(h_{k-1}) \ll 1.$$

Next, consider

$$\sum_{h_{k-1}<l_1<l_2\leq h_k} \mathbf{E}|\xi_{l_1}|^2|\xi_{l_2}| \ll \sum_{h_{k-1}<l_1<l_2\leq h_k} \mathbf{E}|\xi_{l_1}||\xi_{l_2}|. \tag{43}$$

Define random variables as

$$\zeta_k = U_k - \mathbf{E} U_k$$

where U_k is defined in the proof of Lemma 6 and

$$\zeta_{k,\lambda} = U_{k,\lambda} - \mathbf{E} U_{k,\lambda}$$

where $U_{k,\lambda}$ is defined as follows

$$U_{k,\lambda}(x) := \#\{1 \leq c \leq d | c q_k \equiv s(r), c^2 \leq \varphi_k^{(\lambda)} f(k)\}$$

with

$$\varphi_k^{(\lambda)} = a_{k+1} + [a_{k+2}, \ldots, a_{k+\lambda}] + [a_k, \ldots a_{k-\lambda}].$$

Similarly to (22), it is plain that

$$\lambda[U_k \neq U_{k,\lambda}] \ll f(k)^2 2^{-\lambda/2}$$

and hence

$$\mathbf{E}|\zeta_k - \zeta_{k,\lambda}| \ll f(k)^2 2^{-\lambda/2}. \tag{44}$$

Furthermore, if we choose for integers $k_1 < k_2$

$$\lambda = \left[\frac{k_2 - k_1}{3}\right]$$

then

$$\mathbf{E}|\zeta_{k_1,\lambda}||\zeta_{k_2,\lambda}| \ll f(k_1)f(k_2). \tag{45}$$

This follows by Lemma 1.2.1 in [8] together with a result of Szüsz [12] about the mixing behaviour of the sequence $\zeta_{k,\lambda}$. By combining (44) and (45), we get

$$\begin{aligned}\mathbf{E}|\zeta_{k_1}||\zeta_{k_2}| &\ll \mathbf{E}|\zeta_{k_1} - \zeta_{k_1,\lambda}| + \mathbf{E}|\zeta_{k_2} - \zeta_{k_2,\lambda}| + \mathbf{E}|\zeta_{k_1,\lambda}||\zeta_{k_2,\lambda}| \\ &\ll f(k_1)\rho^{k_2-k_1} + f(k_2)\rho^{k_2-k_1} + f(k_1)f(k_2),\end{aligned}$$

where $\rho < 1$ is a constant and λ was defined above.

Thus, from (41) and the last estimate,

$$\sum_{h_{k-1}<l_1<l_2\le h_k} \mathbf{E}|\zeta_{l_1}||\zeta_{l_2}| \ll \sum_{l=h_{k-1}+1}^{h_k} f(l) + \left(\sum_{l=h_{k-1}+1}^{h_k} f(l)\right)^2 \ll 1$$

and therefore

$$\begin{aligned}\mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k} |\zeta_l|\right)^2 &= \sum_{l=h_{k-1}+1}^{h_k} \mathbf{E}|\zeta_l|^2 + 2\sum_{h_{k-1}<l_1<l_2\le h_k} \mathbf{E}|\zeta_{l_1}||\zeta_{l_2}| \\ &\ll \sum_{l=h_{k-1}+1}^{h_k} f(l) + 1 \ll 1,\end{aligned} \tag{46}$$

where (20) was used.

Next, observe

$$\begin{aligned}&\mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k} |\xi_l|\right)^2 - \mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k} |\zeta_l|\right)^2 \\ &\ll \mathbf{E}^{1/2}\Big(\sum_{l=h_{k-1}+1}^{h_k} (|\xi_l| - |\zeta_l|)\Big)^2 \\ &\qquad \left(\mathbf{E}^{1/2}\Big(\sum_{l=h_{k-1}+1}^{h_k} (|\xi_l| - |\zeta_l|)\Big)^2 + \mathbf{E}^{1/2}\Big(\sum_{l=h_{k-1}+1}^{h_k} |\zeta_l|\Big)^2\right)\end{aligned} \tag{47}$$

and by using (27)

$$\mathbf{E}\Big(\sum_{l=h_{k-1}+1}^{h_k}(|\xi_l|-|\zeta_l|)\Big)^2 \leq \sum_{h_{k-1}<l_1,l_2\leq h_k}\mathbf{E}|\xi_{l_1}-\zeta_{l_1}||\xi_{l_2}-\zeta_{l_2}|$$
$$\leq \sum_{h_{k-1}<l_1,l_2\leq h_k}\mathbf{E}^{1/2}(\xi_{l_1}-\zeta_{l_1})^2\mathbf{E}^{1/2}(\xi_{l_2}-\zeta_{l_2})^2$$
$$\ll \sum_{h_{k-1}<l_1,l_2\leq h_k}\frac{f(l_1)}{F(l_1)^2}\frac{f(l_2)}{F(l_2)^2}\ll 1, \tag{48}$$

where the last estimate follows by the Dini-Abel Theorem.

Finally, by combining (46),(47), and (48), we get for the right hand side of (43)

$$\sum_{h_{k-1}<l_1<l_2\leq h_k}\mathbf{E}|\xi_{l_1}||\xi_{l_2}| \leq \mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k}|\xi_l|\right)^2 \ll 1 \tag{49}$$

The next sum

$$\sum_{h_{k-1}<l_1<l_2\leq h_k}\mathbf{E}|\xi_{l_1}||\xi_{l_2}|^2.$$

is treated in the same manner.

In order to prove (3), we are left with the following sum

$$\sum_{h_{k-1}<l_1<l_2<l_3\leq h_k}\mathbf{E}|\xi_{l_1}||\xi_{l_2}||\xi_{l_3}|.$$

We break the sum into two parts $\sum=\sum^*+\sum^{**}$ according to whether $l_3-l_2>[c_{l_2}\log F(l_2)]$ or not. In the first case the random variables $|\xi_{l_1}||\xi_{l_2}|$ and $|\xi_{l_3}|$ are satisfying a mixing condition which is stronger than (39), namely

$$|\lambda(AB)-\lambda(A)\lambda(B)|\ll q^{l_3-l_2-\log F(l_2)}\lambda(A)\lambda(B)$$

for all A in the σ-algebra generated by $|\xi_{l_1}||\xi_{l_2}|$ and all B in the σ-algebra generated by $|\xi_{l_3}|$. This is a consequence of a theorem due to Szüsz [12].

Therefore, we can use Lemma 1.2.1 in [8] and obtain

$$\begin{aligned}\sum_{h_{k-1}<l_1<l_2<l_3\le h_k}^{*} \mathbf{E}|\xi_{l_1}||\xi_{l_2}||\xi_{l_3}| &\ll \sum_{h_{k-1}<l_1<l_2<l_3\le h_k}^{*} \mathbf{E}|\xi_{l_1}||\xi_{l_2}|\mathbf{E}|\xi_{l_3}|\\ &\ll \left(\sum_{h_{k-1}<l_1<l_2\le h_k} \mathbf{E}|\xi_{l_1}||\xi_{l_2}|\right)\left(\sum_{l=h_{k-1}+1}^{h_k} \mathbf{E}|\xi_l|\right)\\ &\ll \sum_{l=h_{k-1}+1}^{h_k} f(l) \ll 1,\end{aligned}$$

where (41), (42), and (49) were used.

For the last sum, another application of (41) and (49) gives

$$\begin{aligned}\sum_{h_{k-1}<l_1<l_2<l_3\le h_k}^{**} \mathbf{E}|\xi_{l_1}||\xi_{l_2}||\xi_{l_3}| &\ll \sum_{h_{k-1}<l_1<l_2\le h_k} \log F(l_2)\mathbf{E}|\xi_{l_1}||\xi_{l_2}|\\ &\ll \log F(h_k) \ll \log k,\end{aligned}$$

which concludes the proof of (3). □

Because of Lemma 7, we can apply Theorem 5 to the sequence $(\eta_k)_{k\ge1}$ to obtain, as $t \longrightarrow \infty$, either

$$Y(t) - W(t) = \mathrm{o}((t \log\log t)^{1/2}), \quad \text{a.s.} \tag{50}$$

and

$$\lambda_2\left[\frac{1}{\sqrt{t}} \sup_{s\le t} |Y(s) - W(s)| \ge \epsilon\right] \longrightarrow 0 \tag{51}$$

for all $\epsilon > 0$, or

$$Y(t) - W(t) \ll t^{1/2-\delta}, \quad \text{a.s.} \tag{52}$$

for a suitable constant δ depending on the assumptions on f. Here, the stochastic process $\{Y(t), t \ge 0\}$ is defined on the probability space $([0,1]^2, \mathcal{B}_2, \lambda_2)$ as

$$Y(t) = \sum_{k\le t} \eta_k.$$

Our next aim is to show that (50), (51), and (52) for η_k entail the corresponding results for ξ_k. Therefore, consider

$$n_t = \begin{cases}\max\{n|(\tau^2\log\gamma)F(n) \le t\} & \text{if } t \ge (\tau^2\log\gamma)F(1)\\ 0 & \text{otherwise,}\end{cases},$$

for a real number $t \ge 0$, and define a stochastic process $\{Z(t), t \ge 0\}$ on the probability space $([0,1]^2, \mathcal{B}_2, \lambda_2)$ as

$$Z(t) = \sum_{k\le n_t} \xi_k. \tag{53}$$

Lemma 8. *We have, as $t \longrightarrow \infty$,*

$$Z(t) - Y(t) \ll t^{1/2-\epsilon}, \quad a.s., \tag{54}$$

for all $\epsilon < 1/6$.

Proof. By (3) of Lemma 7, we have

$$\begin{aligned} \lambda_2[\sum_{l=h_{k-1}+1}^{h_k} |\xi_l| \geq k^{1/2-\epsilon}] &= \lambda[\sum_{l=h_{k-1}+1}^{h_k} |\xi_l| \geq k^{1/2-\epsilon}] \\ &\leq \mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k} |\xi_l|\right)^3 / k^{3/2-3\epsilon} \ll \frac{\log k}{k^{3/2-3\epsilon}} \end{aligned} \tag{55}$$

and an application of the Borel-Cantelli Lemma yields

$$\sum_{l=h_{k-1}+1}^{h_k} |\xi_l| \ll k^{1/2-\epsilon}, \quad \text{a.s.} \tag{56}$$

Next observe

$$|Z(t) - Y(t)| = \left| \sum_{k \leq n_t} \xi_k - \sum_{k \leq h_{[t]}} \xi_k \right| \leq \sum_{l=h_{[t]}+1}^{h_{[t]+1}} |\xi_l|$$

which together with (56) proves the lemma. □

By combining (50), (51), (52), and (54), we finally get

Lemma 9. *We have, as $t \longrightarrow \infty$, either*

$$Z(t) - W(t) = o((t \log\log t)^{1/2}), \quad a.s. \tag{57}$$

and

$$\lambda_2\left[\frac{1}{\sqrt{t}} \sup_{s \leq t} |Z(s) - W(s)| \geq \epsilon\right] \longrightarrow 0 \tag{58}$$

for all $\epsilon > 0$, or

$$Z(t) - W(t) \ll t^{1/2-\lambda}, \quad a.s. \tag{59}$$

for all $\lambda \leq \min\{\delta, 1/6\}$, depending on the assumptions on f.

Proof. The proof of (57) and (59) is obvious. In order to proof (58) observe that because of (54) there exist for each $\rho < 1$ a subset E of $[0,1]^2$ with measure at least $1-\rho$ and κ large enough such that for all elements in E and for all $t \geq 0$, we have

$$Z(t) - Y(t) \leq \kappa t^{1/2-\bar{\epsilon}}$$

where $\bar{\epsilon} < 1/6$. Hence

$$\frac{1}{\sqrt{t}}\sup_{s\le t}|Z(s)-W(s)| \le \frac{1}{\sqrt{t}}\sup_{s\le t}(|Z(s)-Y(s)|+|Y(s)-W(s)|)$$
$$\le \frac{1}{\sqrt{t}}\sup_{s\le t}|Y(s)-W(s)|+\frac{\kappa}{t^{\bar{\epsilon}}}$$

for all elements in E. Therefore, we have

$$\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Z(s)-W(s)|\ge\epsilon\right]\cap E\subseteq\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Y(s)-W(s)|\ge\epsilon/2\right]$$

for t large enough. By taking (51) into account, we get, as $t \longrightarrow \infty$,

$$\lambda_2\left(\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Z(s)-W(s)|\ge\epsilon\right]\cap E\right)\longrightarrow 0. \qquad (60)$$

Thus, by the trivial inequality $\lambda_2(A\cap B)\ge\lambda_2(A)+\lambda_2(B)-1$,

$$\lambda_2\left(\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Z(s)-W(s)|\ge\epsilon\right]\cap E\right)$$
$$\ge\lambda_2\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Z(s)-W(s)|\ge\epsilon\right]-\rho$$

which together with (60) gives for large enough t

$$\lambda_2\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Z(s)-W(s)|\ge\epsilon\right]\le 2\rho$$

and hence (58) is proved. □

In the next section, we finish the proofs of our main results by showing that Z_k approximates X_n good enough that the invariance principles for X_n can be deduced from the corresponding ones for Z_k proved in the last lemma.

5. Proof of Theorem 3 and Theorem 4

Before we start with the proof of the theorems, we need a few additional observations. First the following lemma about standard brownian motion.

Lemma 10. *Let $\{W(t), t\ge 0\}$ be standard brownian motion on some probability space $(\Omega,\mathcal{A},P)$ and $0<\delta<1$. Then, we have, as $t\longrightarrow\infty$,*

$$W(t+\mathcal{O}(t^{1-\delta}))-W(t)\ll t^{(1-\delta)/2+\lambda},\quad a.s.$$

for all $\lambda>0$.

Proof. Consider

$$\sup_{n\leq t<n+1}|W(t+\mathcal{O}(t^{1-\delta}))-W(t)|\leq \sup_{a\leq t,s\leq b}|W(t)-W(s)|:=R(a,b),$$

where $a=n-Cn^{1-\delta}, b=(n+1)+C(n+1)^{1-\delta}$ and C is a suitable constant.

The following estimate is elementary

$$P[R(a,b)\geq n^{(1-\delta)/2+\lambda}]=P[R(0,1)\geq C_1n^{\lambda}]\ll e^{-C_2n^{2\lambda}}$$

where C_1, C_2 are positive constants. Hence, applying the Borel Cantelli Lemma immediately gives the result. □

Furthermore, we need a technical lemma due to Philipp [8].

Lemma 11. *Let g_1 (resp. g_2) be the inverse function of $\gamma^{(k+1)}\exp(\kappa(k+1)^{1-\delta_1})$ (resp. $\gamma^k exp(-\kappa k^{1-\delta_1})$). Then, we have*

$$F_i(g_i(n))=\frac{1}{\log\gamma}F(n)+\mathcal{O}(F(n)^{1/2-\delta_2}).$$

Now, we are ready for the proof of Theorem 3 and Theorem 4.

Proof of Theorem 3 and Theorem 4 . First observe, by (12), (15), and Lemma 11,

$$\begin{aligned}X_n(x)\leq\sum_{q_k\leq n}Y_k(x)+\mathcal{O}(1)&\leq\sum_{k\leq g_2(n)}Y_k(x)+\mathcal{O}(1)\\&\leq\sum_{k\leq g_2(n)}Z_k^{(2)}(x)+\mathcal{O}(1)\end{aligned}$$

and

$$\begin{aligned}\sum_{k\leq g_1(n)}Z_k^{(1)}(x)+\mathcal{O}(1)\leq\sum_{k\leq g_1(n)}Y_k(x)+\mathcal{O}(1)&\\ \leq\sum_{q_{k+1}\leq n}Y_k(x)+\mathcal{O}(1)\leq X_n(x)&\end{aligned}$$

for all $x\in F$.

Thus, for all $x_1 \in F$ and $x_2 \in [0,1]$,

$$\begin{aligned} X(t) - W(t) &= X_{n_t} - \sigma F(n_t) - W(t) \\ &\leq \sum_{k \leq g_2(n_t)} Z_k^{(2)} - \sigma F(n_t) - W(t) + \mathcal{O}(1) \\ &= \sum_{k \leq g_2(n_t)} \xi_k^{(2)} - W(t) + \mathcal{O}(F(n_t)^{1/2-\delta_2}) + \mathcal{O}(1) \qquad (61) \\ &= Z^{(2)}(\tau^2 F(n_t) + \mathcal{O}(F(n_t)^{1/2-\delta_2})) \\ &\qquad - W(t) + \mathcal{O}(F(n_t)^{1/2-\delta_2}) + \mathcal{O}(1) \end{aligned}$$

where Lemma 6 and Lemma 11 were used. Here, $\xi_k^{(2)}$ resp. $Z^{(2)}(t)$ are defined by (28) resp. (53). Next consider

$$0 \leq t - \tau^2 F(n_t) \leq \tau^2 \frac{f(\log n_t + 1)}{n_t + 1} \ll 1$$

and therefore

$$\tau^2 F(n_t) = t + \mathcal{O}(1). \qquad (62)$$

Furthermore, we have by Lemma 10

$$W(t + \mathcal{O}(t^{1/2-\delta_2})) - W(t) \ll t^{1/2-\delta}, \quad \text{a.s.} \qquad (63)$$

for a suitable constant $0 < \delta < 1/2$.

By combining (61), (62), and (63), we obtain that for each $\epsilon > 0$ there exists a subset E of $[0,1]^2$ with measure at least $1-\epsilon$ so that for κ large enough (and only depending on E), we have

$$\begin{aligned} X(t) - W(t) \leq Z^{(2)}(t + \mathcal{O}(t^{1/2-\delta_2})) - W(t+\mathcal{O}(t^{1/2-\delta_2})) \\ + \mathcal{O}(t^{1/2-\lambda}) + \mathcal{O}(1) \end{aligned} \qquad (64)$$

for all elements in E and $t \geq 0$. Here, λ is a suitable constant only depending on δ_2 (thereby notice that the definition of $Z^{(2)}$ is depending on κ). Similarly, we get

$$\begin{aligned} X(t) - W(t) \geq Z^{(1)}(t + \mathcal{O}(t^{1/2-\delta_2})) - W(t+\mathcal{O}(t^{1/2-\delta_2})) \\ + \mathcal{O}(t^{1/2-\lambda}) + \mathcal{O}(1) \end{aligned} \qquad (65)$$

for all elements in E.

The proof of (5) is now an easy consequence of (57),(64), and (65).

In order to prove (6), we observe, by combining (64) and (65)

$$\begin{aligned} |X(t) - W(t)| \leq \max\{&|Z^{(2)}(t + \mathcal{O}(t^{1/2-\delta_2})) - W(t + \mathcal{O}(t^{1/2-\delta_2}))|, \\ &|Z^{(1)}(t + \mathcal{O}(t^{1/2-\delta_2})) - W(t + \mathcal{O}(t^{1/2-\delta_2}))|\} + \mathcal{O}(t^{1/2-\lambda}) + \mathcal{O}(1) \end{aligned}$$

for all elements in E. Hence, it follows that for each $\delta < \epsilon$ there exist a subset G of E with measure at least $1-\delta$ and a constant $\bar{\kappa}$ such that

$$|X(t) - W(t)| \leq \max\{|Z^{(2)}(t + \mathcal{O}(t^{1/2-\delta_2})) - W(t + \mathcal{O}(t^{1/2-\delta_2}))|, \\ |Z^{(1)}(t + \mathcal{O}(t^{1/2-\delta_2})) - W(t + \mathcal{O}(t^{1/2-\delta_2}))|\} + \bar{\kappa} t^{1/2-\lambda} + \bar{\kappa}$$

for all elements in G and $t \geq 0$. Thus,

$$\begin{aligned} &\frac{1}{\sqrt{t}} \sup_{s\leq t} |X(s) - W(s)| \\ \leq &\frac{1}{\sqrt{t}} \sup_{s\leq t} |Z^{(2)}(s + \mathcal{O}(s^{1/2-\delta_2})) - W(s + \mathcal{O}(s^{1/2-\delta_2}))| \\ &+ \frac{1}{\sqrt{t}} \sup_{s\leq t} |Z^{(1)}(s + \mathcal{O}(s^{1/2-\delta_2})) - W(s + \mathcal{O}(s^{1/2-\delta_2}))| + \frac{\bar{\kappa}}{t^{\lambda}} + \frac{\bar{\kappa}}{\sqrt{t}} \end{aligned}$$

for all elements in G. Therefore, we get

$$\begin{aligned} &\left[\frac{1}{\sqrt{t}} \sup_{s\leq t} |X(s) - W(s)| \geq \bar{\epsilon}\right] \cap G \\ \subseteq &\left[\frac{1}{\sqrt{t}} \sup_{s\leq t} |Z^{(2)}(s + \mathcal{O}(s^{1/2-\delta_2})) - W(s + \mathcal{O}(s^{1/2-\delta_2}))| \geq \bar{\epsilon}/8\right] \\ &\cup \left[\frac{1}{\sqrt{t}} \sup_{s\leq t} |Z^{(1)}(s + \mathcal{O}(s^{1/2-\delta_2})) - W(s + \mathcal{O}(s^{1/2-\delta_2}))| \geq \bar{\epsilon}/8\right] \end{aligned}$$

for t large enough. It is easy to see from (58)

$$\lambda_2 \left[\frac{1}{\sqrt{t}} \sup_{s\leq t} |Z^{(i)}(s + \mathcal{O}(s^{1/2-\delta_2})) - W(s + \mathcal{O}(s^{1/2-\delta_2}))| \geq \bar{\epsilon}/8\right] \longrightarrow 0,$$

as $t \longrightarrow \infty$ (i=1,2). Hence

$$\lambda_2 \left(\left[\frac{1}{\sqrt{t}} \sup_{s\leq t} |X(s) - W(s)| \geq \bar{\epsilon}\right] \cap G\right) \leq 2\delta$$

for t large enough and using the trivial inequality $\lambda(A \cap B) \geq \lambda(A) + \lambda(B) - 1$ gives

$$\lambda_2 \left[\frac{1}{\sqrt{t}} \sup_{s\leq t} |X(s) - W(s)| \geq \bar{\epsilon}\right] \leq 3\delta$$

for t large enough. This concludes the proof of (6).

For the proof of (8) observe (by (59)) that for $t \geq \tau^2 F(1)$ the right hand side of (64) is bounded by $\bar{C} t^{1/2-\bar{\lambda}}$ (for suitable $\bar{C}$ and $\bar{\lambda}$). The same is true for $t < \tau^2 F(1)$ because of the fact that the Brownian motion is locally Hölder continues. A similar lower bound is obtained from (65) and therefore, Theorem 4 is proved. □

6. Proof of the Corollaries

In order to proof the corollaries, we use the equivalent formulation of (5), (6), and (8) with standard Brownian motion $\{W(t), t \geq 0\}$.

Proof of Corollary 1. Observe that (6) implies

$$\frac{1}{\sqrt{\tau^2 F(n)}} \|X(\tau^2 F(n)t) - W(\tau^2 F(n)t)\| \xrightarrow{\lambda_2} 0$$

where $\|\cdot\|$ denotes the maximum norm. By pointing out

$$\{1/\sqrt{\tau^2 F(n)} W(\tau^2 F(n)t), 0 \leq t \leq 1\} \stackrel{\mathcal{L}}{=} \{W(t), 0 \leq t \leq 1\}$$

the result follows. □

Proof of Corollary 2. From (5), it is immediate that, as $n \longrightarrow \infty$,

$$\frac{1}{(2\tau^2 F(n) \log\log F(n))^{1/2}} \|X(\tau^2 F(n)t) - W(\tau^2 F(n)t)\| \longrightarrow 0, \quad \text{a.s.}$$

where $\|\cdot\|$ denotes the maximum norm.

Therefore, it is enough to prove the iterated logarithm law for the sequence of functions

$$\frac{1}{(2\tau^2 F(n) \log\log F(n))^{1/2}} W(\tau^2 F(n)t)$$

defined on $[0, 1]$. But because of

$$\tau^2 F(n+1) - \tau^2 F(n) \ll 1 \tag{66}$$

such a proof can easily performed by applying Strassen's method (see [11]). □

Proof of Corollary 3. (8) has the following simple consequence

$$\left(\frac{8 \log\log F(n)}{\pi^2 \tau^2 F(n)}\right)^{1/2} (\sup_{s \leq t} |X(\tau^2 F(n)s)| - \sup_{s \leq t} |W(\tau^2 F(n)s|) \longrightarrow 0,$$

a.s., as $n \longrightarrow \infty$, pointwise in t. Therefore, it is enough to prove Corollary 3 for the functions

$$\left(\frac{8 \log\log F(n)}{\pi^2 \tau^2 F(n)}\right)^{1/2} \sup_{s \leq t} |W(\tau^2 F(n)s|$$

defined on the interval $[0, \infty)$. By taking (66) into account, such a proof follows from (4.6) in [14]. □

Proof of Lemma 1. For the first part, we use an identity observed by Philipp (see Lemma 3 in [3])

$$\sigma = \frac{2}{r} - \frac{12r}{C(r)\pi^2} \sum_{0\leq q<r} \frac{\varphi((q,r))}{(q,r)} \sum_{\substack{d<c \\ cq\equiv s(r)}} \frac{1}{c^2} = \frac{2}{r} + \mathcal{O}\left(\frac{1}{d}\right).$$

For the second part observe (compare with [3])

$$k_{c,q} = \begin{cases} 2\frac{(q,r)}{r}c + c_q & \text{if } (q,r)|s \\ 0 & \text{otherwise} \end{cases},$$

with a suitable constant c_q. Hence

$$\begin{aligned} \tau^2 &= \frac{24}{C(r)\pi^2} \sum_{0\leq q<r,(q,r)|s} \varphi((q,r)) \sum_{\substack{1\leq c\leq d \\ cq\equiv s(r)}} \frac{1}{c} + \mathcal{O}(1) \\ &= \frac{24\log d}{C(r)r\pi^2} \sum_{0\leq q<r,(q,r)|s} \varphi((q,r))(q,r) + \mathcal{O}(1), \end{aligned}$$

which - together with Lemma 5 in [3] - proves the lemma. □

7. Appendix

In this section, we outline an extension of Theorem 7.1 in [10] (compare with Theorem 5 and Remark 2 in Section 4). We assume that the reader is familiar with [10] and use throughout the notation of [10] without introducing it.

We are interested in what is possible to obtain if condition (7.1.7) of Theorem 7.1 is relaxed to

$$\mathbf{E}\left(\sum_{n\leq N} \eta_n\right)^2 \sim N.$$

Our result is an almost sure invariance principle and a distribution type invariance principle and reads as follows.

Theorem 6. *Let ξ_n be a sequence of random variables and let η_n be defined by* (7.1.1). *We shall assume of the function f and the sequence ξ_n that*

$$\mathbf{E}\eta_n = 0.$$

Suppose that there exist constants $0 < \delta \leq 2$ and $C > 0$ such that

$$\mathbf{E}|\eta_n|^{2+\delta} \leq C$$

and

$$\|\eta_n - \eta_{ln}\|_{2+\delta} \leq Cl^{(2+7/\delta)}$$

for all n,l=1,2,3,.... Moreover, suppose that

$$\mathbf{E}\left(\sum_{n\leq N}\eta_n\right)^2 \sim N,$$

as $N \longrightarrow \infty$. *Finally, assume that* ξ_n *satisfies a retarded strong mixing condition of the form* (7.1.2) *with*

$$\kappa = \delta/(11+4\delta)$$

and

$$\beta(s) \ll s^{-168/(1+2/\delta)}.$$

Define a process $\{S(t), t \geq 0\}$ *by setting*

$$S(t) = \sum_{n\leq t}\eta_n.$$

Then, without changing the distribution of $\{S(t), t \geq 0\}$*, we can redefine the process* $\{S(t), t \geq 0\}$ *on a richer probability space together with standard Brownian motion* $\{X(t), t \geq 0\}$ *such that, as* $t \longrightarrow \infty$,

$$S(t) - X(t) = o((t\log\log t)^{1/2}), \quad a.s. \tag{67}$$

and

$$P\left[\frac{1}{\sqrt{t}}\sup_{\tau\leq t}|S(\tau) - X(\tau)| \geq \epsilon\right] \longrightarrow 0 \tag{68}$$

for all $\epsilon > 0$.

In order to prove the theorem, we will follow the proof in [10]. We only outline the differences.

For the proof of (67), it is easy to see that Lemma 7.3.5, Lemma 7.4.3, and Lemma 7.5.1 can be replaced by the following three lemmas.

Lemma 12. *As* $N \longrightarrow \infty$,

$$\sum_{j=1}^{M_N} y_j^2 \sim N, \quad a.s.$$

Lemma 13. *As* $N \longrightarrow \infty$,

$$\sum_{j=1}^{M_N} Y_j^2 \sim N, \quad a.s.$$

Lemma 14. *As* $N \longrightarrow \infty$,

$$\sum_{j=1}^{M_N} T_j \sim N, \quad a.s.$$

Furthermore instead of Lemma 7.5.2, we have the following one.

Lemma 15. *As* $t \longrightarrow \infty$,

$$S^*(t) - X(t) = o((t \log\log t)^{1/2}), \quad a.s.$$

Proof. This is easily seen by Lemma 14 and the classical method of Strassen (see pp217 in [11]). □

All other lemmas remain unchanged and therefore, (67) is obvious. For the proof of (68), we point out that the proof of Lemma 7.3.5 gives (even under the weaker assumptions of Theorem 6)

Lemma 16. *As* $N \longrightarrow \infty$

$$\sum_{j=1}^{M_N} y_j^2 - \mathbf{E}y_j^2 \ll N^{1-2\alpha}, \quad a.s. \tag{69}$$

and

$$\sum_{j=1}^{M_N} \mathbf{E}y_j^2 \sim N. \tag{70}$$

This result implies

Lemma 17. *As* $N \longrightarrow \infty$

$$\max_{k \le N} \left| \frac{\sum_{j \le M_k} y_j^2}{N} - \frac{k}{N} \right| \xrightarrow{P} 0. \tag{71}$$

Proof. Let $\epsilon > 0$ be a given real number. Then, it's immediate from (70)

$$\max_{k \le N} \left| \frac{\sum_{j \le M_k} \mathbf{E}y_j^2}{N} - \frac{k}{N} \right| < \epsilon/2$$

for N large enough. Furthermore, observe by (69), that for each $\delta > 0$ there exist a set E with measure at least $1 - \delta$ and a constant κ such that

$$\left| \sum_{j=1}^{M_N} y_j^2 - \mathbf{E}y_j^2 \right| \le \kappa N^{1-2\alpha}$$

for all $N \geq 1$ and all elements in E. Hence

$$\max_{k\leq N}\left|\frac{\sum_{j\leq M_k} y_j^2 - \mathbf{E}y_j^2}{N}\right| \leq \frac{\kappa}{N^{2\alpha}} < \epsilon/2$$

for all elements in E and N large enough. Therefore,

$$P\left(\left[\max_{k\leq N}\left|\frac{\sum_{j\leq M_k} y_j^2}{N} - \frac{k}{N}\right| \geq \epsilon\right] \cap E\right) \longrightarrow 0,$$

as $N \longrightarrow \infty$. By using the trivial inequality $P(A\cap B) \geq P(A)+P(B)-1$ the result is easily obtained. □

Furthermore, we get a similar result for the sequences Y_j^2 and T_j

Lemma 18. *As* $N \longrightarrow \infty$,

$$\max_{k\leq N}\left|\frac{\sum_{j\leq M_k} Y_j^2}{N} - \frac{k}{N}\right| \xrightarrow{P} 0 \tag{72}$$

and

$$\max_{k\leq N}\left|\frac{\sum_{j\leq M_k} T_j}{N} - \frac{k}{N}\right| \xrightarrow{P} 0. \tag{73}$$

Proof. Since, as $N \longrightarrow \infty$,

$$\sum_{j=1}^{M_N} Y_j^2 - y_j^2 \ll N^{1-2\alpha}, \quad \text{a.s.}$$

(compare with (7.4.3) and (7.4.4) which are still true even under the weaker assumptions of Theorem 6), we obtain (72) by using ideas of the proof of the last lemma and (71). The same method can be applied for (73) due to

$$\sum_{j=1}^{M_N} T_j = \sum_{j=1}^{M_N}(T_j - \mathbf{E}(T_j|\mathcal{P}_{j-1}) + \sum_{j=1}^{M_N}(\mathbf{E}(Y_j^2|\mathcal{L}_{j-1}) - Y_j^2) + \sum_{j=1}^{M_N} Y_j^2$$

and

$$\sum_{j=1}^{M_N}(T_j - \mathbf{E}(T_j|\mathcal{P}_{j-1}) \ll N^{1-2\alpha}, \quad \text{a.s.}$$

$$\sum_{j=1}^{M_N}(\mathbf{E}(Y_j^2|\mathcal{L}_{j-1}) - Y_j^2) \ll N^{1-2\alpha}, \quad \text{a.s.}$$

(compare with Lemma 7.4.4, (7.5.1) and the proof of Lemma 7.5.1 which works in our situation as well). □

The last lemma is used to prove the following result

Lemma 19. *As* $t \longrightarrow \infty$,

$$P\left[\frac{1}{\sqrt{t}}\sup_{\tau\le t}|S^*(\tau)-X(\tau)|\ge\epsilon\right]\longrightarrow 0$$

for all $\epsilon > 0$.

Proof. First, we observe, as $t \longrightarrow \infty$,

$$P\left[\sup_{\tau\le t}\left|\frac{\sum_{j\le M_{[\tau]}}T_j}{t}-\frac{\tau}{t}\right|\ge\delta\right]\longrightarrow 0 \tag{74}$$

which is easily obtained from (73). Next consider

$$\begin{aligned}
P&\left[\frac{1}{\sqrt{t}}\sup_{\tau\le t}|S^*(\tau)-X(\tau)|\ge\epsilon\right]\\
&=P\left[\sup_{\tau\le t}\left|X\left(\frac{\sum_{j\le M_{[\tau]}}T_j}{t}\right)-X\left(\frac{\tau}{t}\right)\right|\ge\epsilon\right]\\
&\le P\left[\sup_{\tau\le t}\left|\frac{\sum_{j\le M_{[\tau]}}T_j}{t}-\frac{\tau}{t}\right|\ge\delta\right]\\
&\qquad+P\left[\sup_{\tau_2\le 1}\sup_{|\tau_1-\tau_2|<\delta}|X(\tau_1)-X(\tau_2)|\ge\epsilon\right].
\end{aligned}$$

The right hand side of the above inequality converges to zero because of (74) and the a.s. continuity of the Brownian motion pathes. This proves the result. □

Since, as $t \longrightarrow \infty$,

$$S(t)-S^*(t)\ll t^{1/2-2\alpha},\quad \text{a.s.}$$

(compare with the formula in the last line of page 93 which is still true even under the weaker assumptions of Theorem 6), it is straightforward to deduce (68) from the last lemma.

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