# ON METRIC DIOPHANTINE APPROXIMATION IN THE FIELD OF FORMAL LAURENT SERIES*

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Abstract. In [4] deMathan proved that Khintchine's Theorem has an analogue in the field of formal Laurent series. First, we show that in case of only one inequality this result can be also obtained by the continued fraction theory. Then, we are interested in the number of solutions and show under special assumptions that one gets a central limit theorem, a law of iterated logarithm and an asymptotic formula. This is an analogue of a result due to LeVeque [10]. The proof is based on probabilistic results for formal Laurent series due to Niederreiter [11].

## 1. Introduction

In [6] Hurwitz proved the following classical theorem:

**Theorem 1. (Hurwitz Theorem)** *For any irrational number $x$ the inequality*

$$\left|x - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2} \tag{1}$$

*has infinitely many integer solutions $p$ and $q > 0$. The factor $\sqrt{5}$ is best possible, which means, that a similar theorem does not hold, if the factor is replaced by any bigger one.*

It is well known that the bound on the right hand side of (1) can be considerably improved if someone concentrates not only on the set of irrational numbers but also on other sets with measure one (thereby the measure is the Lebesgue measure on $(0, 1)$ which we are going to denote by $\lambda$). This is a famous result due to Khintchine.

**Theorem 2. (Khintchine's Theorem)** *Let $g(k)$ be a positive function on the positive integers, such that $kg(k)$ decreases. Then the inequality*

$$\left|x - \frac{p}{q}\right| < \frac{g(q)}{q} \tag{2}$$

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*has finitely or infinitely many integer solutions $p$ and $q > 0$ for almost all $x$, according to the series*

$$\sum_{k=1}^{\infty} g(k)$$

*converges or diverges.*

In case of divergence of the above series, it is interesting to consider the following sets

$$X_n(x) = \#\{(p,q) | 1 \leq q \leq n, (p,q) = 1 \text{ and } p/q \text{ is a solution of (2)}\} \tag{3}$$

and

$$Y_n(x) = \#\{(p,q) | 1 \leq q \leq n \text{ and } p/q \text{ is a solution of (2)}\} \tag{4}$$

for integers $n \geq 1$ and $x \in (0,1)$. $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ can be viewed as sequences of random variables and a lot of work was done on the asymptotic distribution of these sequences by several authors. In this paper we are interested in a result of LeVeque.

Let $f$ be a function with the following properties:

$$0 \leq f(x) \leq \frac{1}{2} \text{ and decreasing for } x \geq 0; \tag{5}$$

$$f(x) = O(x^{-1}) \text{ and } f'(x) = O(x^{-2}) \text{ , as } x \longrightarrow \infty; \tag{6}$$

$$\sum_{k=1}^{\infty} f(k) = \infty \tag{7}$$

LeVeque proved in [10] the following theorem:

**Theorem 3.** *Suppose $f$ satisfies the conditions (5)-(7) and put*

$$g(x) = \frac{f(\log x)}{x} \quad \textit{and} \quad G(n) = \sum_{k=1}^{n} g(k).$$

*With $X_n$ defined as in (3) we have:*

*I. For fixed $\omega$ we have*

$$\lim_{n\longrightarrow\infty} \lambda \left[ X_n < \frac{12}{\pi^2} G(n) + \omega \left( \frac{12}{\pi^2} G(n) \right)^{1/2} \right] = \Phi(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-\frac{u^2}{2}} du.$$

*II. For almost all $x$ we have*

$$X_n(x) \sim \frac{12}{\pi^2} G(n).$$

By deMathan [4], we know that Khintchine's Theorem has an analogue in the field of formal Laurent series. The main result of this paper is an analogue of the above theorem of LeVeque (see Theorem 9 in section 4). The proof (section 6) is based on ideas of the classical proof and uses generalizations of probabilistic results contained in [11], a sharper version of the Lemma of Borel-Cantelli for formal Laurent series, and an analogue of a sharper version of the Khintchine-Levy Theorem (these auxiliary results are collected in section 5). Furthermore, we show that the number of solutions in the divergence case of Khintchine's Theorem obeys a law of iterated logarithm (Theorem 9.II).

In fact, deMathan proved Khintchine's Theorem already for systems of inequalities by following the classical idea of Khintchine. In this paper we are not interested in systems but only in one equation. It is well known that for this situation an easier proof of Khintchine's Theorem can be given by continued fraction theory. We show in section 3 that the classical arguments also work in the Laurent series case (see Theorem 7). Furthermore, we give an application of Khintchine's Theorem in section 3 (see Theorem 8) and show that also Hurwitz Theorem has an analogue in the field of formal Laurent series (see Theorem 6).

We start with a brief introduction in the continued fraction theory in the field of Laurent series (see also [4] and [15]).

## 2. Continued fractions and probabilistic results in the field of formal Laurent series

Let $K$ be an arbitrary field. We consider the field of rational functions $K(T)$ with the following exponential evaluation

$$v\left(\frac{P}{Q}\right) = \deg P - \deg Q \quad P, Q \in K[T], Q \neq 0,$$

where we put as usual $\deg 0 = -\infty$.

With $|\alpha| = b^{v(\alpha)}$, where $b \geq 2$ is an integer and $\alpha \in K(T)$, we get an evaluation of $K(T)$ and the complementation of this field with respect of this evaluation is the field of formal Laurent series which we are going to denote by $K((T^{-1}))$.

In the following, we write $a, b, \ldots$ for elements of $K$, $A, B, \ldots$ for elements of $K[T]$, and $\alpha, \beta, \ldots$ for elements of $K((T^{-1}))$.

There are lots of analogues between $K[T], K(T), K((T^{-1}))$ and $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Especially, one can consider finite continued fractions, which we are going to denote by $[A_0; A_1, \ldots, A_n]$, where $A_0$ is an arbitrary polynomial and $A_1, \ldots, A_n$ are polynomials of degree $> 0$. It is easy to see that every element of $K(T)$ has a unique representation as a finite continued fraction.

Furthermore, if one considers a sequence of polynomials $(A_k)_{k\geq 0}$, where $A_0$ is an arbitrary element of $K[T]$ and $A_k, k \geq 1$ are polynomials with degree $\geq 1$, then $[A_0; A_1, \ldots, A_n]$ converges to an irrational element of $K((T^{-1}))$ and one gets each irrational element exactly once.

In a nutshell we have as in the classical theory

*Each element in $K((T^{-1}))$ has a unique continued fraction expansion and the expansion for an element is finite if and only if the element is in*

$K(T)$.

As in the classical theory one defines the k-th rational convergent of the continued fraction expansion of $\alpha$ (denoted by $\frac{P_k}{Q_k}$). Most of the classical results for the convergents have an analogue in the field of formal Laurent series. We collect few results which we are going to use frequently.

**Lemma 1.** *Let $\frac{P_k}{Q_k}$ denote the k-th convergent of $\alpha$. Then we have*

(1) $(P_k, Q_k) = 1$
(2) $1 = |Q_0| < |Q_1| < |Q_2| \dots$
(3) $|Q_k| = \prod_{i=1}^{k} |A_i|$
(4) $\left|\alpha - \frac{P_k}{Q_k}\right| = \frac{1}{|Q_k||Q_{k+1}|} < \frac{1}{|Q_k|^2}$
(5) *If $P, Q \in K[T], Q \neq 0, (P, Q) = 1$ and*
$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{|Q|^2}$$
*then there exists an integer $k \geq 0$ such that $\frac{P}{Q} = \frac{P_k}{Q_k}$.*
(6) *If $P, Q \in K[T], Q \neq 0$ and $|Q_k| \leq |Q| < |Q_{k+1}|$ then*
$$\left|\alpha - \frac{P}{Q}\right| \geq \left|\alpha - \frac{P_k}{Q_k}\right|$$

For each $\alpha \in K((T^{-1}))$ we write $[\alpha]$ for the polynomial part of $\alpha$ and $\{\alpha\} = \alpha - [\alpha]$ for the fractional part of $\alpha$.

In this paper we only consider the case of $K = \mathbb{F}_q$ with $q = p^t, p \in \mathbb{P}$ and $t \geq 1$ an integer. In this case we use $q$ for the basis of the evaluation.

The following subset of $\mathbb{F}_q((T^{-1}))$ can be viewed as the analogue of the interval $(0, 1)$ in the field of formal Laurent series

$$H = \{\alpha \in \mathbb{F}_q((T^{-1})) | |\alpha| < 1\}. \tag{8}$$

By restriction of the valuation of $\mathbb{F}_q((T^{-1}))$ on $H$ one gets a compact topological space. We denote by $\mathfrak{B}$ the $\sigma$-Algebra of Borel sets on $H$.

$H$ is also an abelian subgroup of $\mathbb{F}_q((T^{-1}))$ and therefore, we have a compact abelian group. On such a group there exists a unique, translation invariant probability measure which we are going to denote by $h$.

With $P$, we denote the set of polynomials over $\mathbb{F}_q$ of positive degree and with $A_k(\alpha)$ resp. $Q_k(\alpha)$ and $P_k(\alpha)$ the k-th partial quotient resp. k-th convergent of the continued fraction expansion of $\alpha$. The following result is contained in [11].

**Lemma 2.** *Let $A_1, \dots, A_n$ be given polynomials in $P$ and put*
$$R(A_1, \dots, A_n) = \{\alpha \in H | A_k(\alpha) = A_k, 1 \leq k \leq n\}.$$
*Then we have*
$$h(R(A_1, \dots, A_n)) = q^{-2(\deg A_1 + \dots + \deg A_n)}.$$

*Proof.* Lemma 2 in [11]. □

Most of the classical metrical results of the continued fraction theory have an analogue in the field of formal Laurent series and because of the ultrametric structure the proof is usually more simple. We mention only two results, which we are going to use. They are both contained in the work of Niederreiter [11].

We write in the following h-a.e. for a property which is true, except for a set of measure zero.

**Theorem 4. (Lemma of Borel-Cantelli for formal Laurent series)** *Let $f(k)$ be a positive function on the positive integers. Then we have*

(1) $\sum_{k=1}^{\infty} \frac{1}{f(k)} < \infty \Longrightarrow |A_k(\alpha)| \leq f(k)$ *for $k$ large enough h-a.e.*
(2) $\sum_{k=1}^{\infty} \frac{1}{f(k)} = \infty \Longrightarrow |A_k(\alpha)| > f(k)$ *for infinitely many $k$ h-a.e.*

*Proof.* Theorem 6 of [11]. □

The next theorem can be viewed as an analogue of the classical Theorem of Khintchine-Levy.

**Theorem 5. (Theorem of Khintchine-Levy for formal Laurent series)** *We have h-a.e.*

$$\lim_{k \longrightarrow \infty} \sqrt[k]{|Q_k(\alpha)|} = q^{\frac{q}{q-1}}.$$

*Proof.* Corollary 1 of [11]. □

The classical analogues of these two theorems are the main ingredients in the classical proof of Khintchine's Theorem. Therefore, one can expect that the classical arguments carry over in the field of formal Laurent series. We show in the next section that, in fact, this is true.

## 3. The Theorems of Khintchine and Hurwitz for formal Laurent series

We start with the following Lemma (compare with Theorem 23 in [8]):

**Lemma 3.** *If $\alpha$ is an irrational element of $\mathbb{F}_q((T^{-1}))$ with bounded continued fraction expansion then there exists a real constant $c > 0$ such that*

$$\left|\alpha - \frac{P}{Q}\right| \geq \frac{1}{c|Q|^2}$$

*for all $P, Q \in K[T], Q \neq 0$.*

*If $\alpha$ is an irrational element with unbounded continued fraction expansion then for any real constant $c > 0$ the inequality*

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{c|Q|^2}$$

*has infinitely many solutions $P, Q \in K[T], Q \neq 0$.*

*Proof.* In the first case, let $c$ be a bound for the absolute values of the partial quotients of the continued fraction expansion of $\alpha$. Then, we have

$$\left|\alpha - \frac{P_k}{Q_k}\right| = \frac{1}{|Q_k||Q_{k+1}|} = \frac{1}{|A_{k+1}||Q_k|^2} \geq \frac{1}{c|Q_k|^2}$$

for $k \geq 0$.

Next we consider arbitrary $P, Q \in K[T], Q \neq 0$. Because of Lemma 1 (2) there is an integer $k \geq 0$ such that $|Q_k| \leq |Q| < |Q_{k+1}|$. With (6) of Lemma 1 we have

$$\left|\alpha - \frac{P}{Q}\right| \geq \left|\alpha - \frac{P_k}{Q_k}\right| \geq \frac{1}{c|Q_k|} \geq \frac{1}{c|Q|}$$

and the first case is proved.

In the second case there exists, because of the unbounded continued fraction expansion of $\alpha$, an integer $k_0 \geq 1$ such that $|A_k| > c$ for all $k \geq k_0$. Therefore, we have for such $k$

$$\left|\alpha - \frac{P_k}{Q_k}\right| = \frac{1}{|Q_k||Q_{k+1}|} = \frac{1}{|A_{k+1}||Q_k|^2} < \frac{1}{c|Q_k|^2}$$

and also the second case is proved. □

As a corollary we get an analogue of the Theorem of Hurwitz in the field of formal Laurent series.

**Theorem 6. (Hurwitz Theorem for Formal Laurent Series)** *Let* $0 < q' < q$. *Then for all irrational* $\alpha \in \mathbb{F}_q((T^{-1}))$ *the inequality*

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{q'|Q|^2}$$

*has infinitely many solutions* $P, Q \in \mathbb{F}_q[T], Q \neq 0$. *If* $q' \geq q$ *then this is not true in general. Furthermore, there exist irrational* $\alpha$ *for which the above inequality has no solutions.*

*Proof.* Part one follows from Lemma 1 (4).

For the proof of the second part, we consider the following element

$$\alpha = [0; T, T, T, \ldots]$$

which is irrational and has a bounded continued fraction expansion with $q$ as a bound for the absolute values of the partial quotients. With the Lemma we get the claimed result. □

The Lemma also shows, that if one is interested in irrational elements with unbounded continued fraction expansion the factor $q'$ in Hurwitz Theorem for formal Laurent series can be replaced by any positive real constant.

It is an easy consequence of the Lemma of Borel-Cantelli for formal Laurent series that the elements in $H$ with a bounded continued fraction expansion form a set of measure zero (in fact much more is known about this set, see [12]).

Because of that, one can expect, as in the classical case, an improvement of the upper bound in Hurwitz Theorem for formal Laurent series, if one concentrates not only on all irrational elements but also on sets of elements with measure one.

Our next aim is the prove of an analogue of Khintchine's Theorem in the field of formal Laurent series. We start with an easy consequence of the Lemma of Borel-Cantelli for formal Laurent series:

**Lemma 4.** *Let $(c_k)_{k\geq 0}$ be a sequence of positive real numbers with $\sum_{k=0}^{\infty} c_k = \infty$. Then we have h-a.e. for infinitely many k*

$$\left|\alpha - \frac{P_k}{Q_k}\right| < \frac{c_k}{|Q_k|^2}.$$

*Proof.* Because of

$$|Q_k||Q_k\alpha - P_k| = \frac{|Q_k|}{|Q_{k+1}|} = \frac{1}{|A_{k+1}|}$$

this follows from the Lemma of Borel-Cantelli for formal Laurent series. □

Now we can prove an analogue of Khintchine's Theorem:

**Theorem 7. (Khintchine's Theorem for Formal Laurent Series)** *Let g be a positive function defined on the sequence $q^k, k \geq 0$, such that $q^k g(q^k)$ decreases. Then the inequality*

$$\left|\alpha - \frac{P}{Q}\right| < \frac{g(|Q|)}{|Q|} \tag{9}$$

*has finitely or infinitely many solutions $P, Q \in K[T], Q \neq 0$ for h-a.e. $\alpha$, according to the series*

$$\sum_{k=0}^{\infty} q^k g(q^k) \tag{10}$$

*converges or diverges.*

*Proof.* We assume first that the series (10) converges. In this case, we consider the following sets

$$B_k = \{\alpha \in H | (9) \text{ has a solution } P, Q \text{ with } \deg Q = k, \deg P < \deg Q\}$$

where $k$ is a nonnegative integer. Furthermore, we consider for fixed polynomials $P, Q \in K[T], Q \neq 0, \deg P < \deg Q$ the set

$$B_{P,Q} = \{\alpha \in H | (9) \text{ has } P, Q \text{ as an solution}\}.$$

It is clear that

$$B_k = \bigcup_{\deg Q = k, \deg P < \deg Q} B_{P,Q}$$

and an easy calculation shows

$$h(B_{P,Q}) = O\left(\frac{g(|Q|)}{|Q|}\right),$$

where the implied constant does not depend on $P, Q$. Therefore, we have

$$\sum_{k=0}^{\infty} h(B_k) \leq (q-1) \sum_{k=0}^{\infty} q^{2k} O\left(\frac{g(q^k)}{q^k}\right) < \infty$$

and the first part follows from the classical Lemma of Borel-Cantelli.

In the second case, we assume that the series (10) diverges. First, we show that this implies that also the following series

$$\sum_{k=0}^{\infty} q^{tk} g(q^{tk}) \tag{11}$$

diverges, where $t \geq 0$ is an integer.

We can assume that $t \geq 1$ because the case $t = 0$ is obvious. Let $n \geq 0$ be an integer and we put $n = qt + r$ with integers $q, r$ and $r < q$. Then, we have

$$\frac{1}{t} \sum_{k=0}^{n} q^k g(q^k) \leq \frac{1}{t} \sum_{k=0}^{(q+1)t-1} q^k g(q^k) \leq \sum_{k=0}^{q} q^{tk} g(q^{tk})$$

and $n \longrightarrow \infty$ entails the claimed result.

Because of the Khintchine-Levy Theorem for formal Laurent series we can choose a positive integer $t$ such that we have h-a.e.

$$|Q_k| < q^{tk}$$

for $k$ large enough. Lemma 4 and (11) show, that we have h-a.e.

$$\left|\alpha - \frac{P_k}{Q_k}\right| < \frac{q^{tk} g(q^{tk})}{|Q_k|^2}$$

for infinitely many $k$. If we combine these two results and use the assumption that $q^k g(q^k)$ decreases we obtain the claimed result. $\square$

We give a classical example:

*Example 1.* We use the following notation $\mathrm{Log}_q k = \max\{1, \log_q k\}$ and consider the function

$$g(q^k) = \frac{1}{q^k \mathrm{Log}_q(q^k)},$$

which fulfills the assumptions of Khinchine's Theorem for formal Laurent series. Furthermore, we have

$$\sum_{k=0}^{\infty} q^k g(q^k) = \infty$$

and therefore it follows that the inequality

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{|Q|^2 \mathrm{Log}_q |Q|}$$

has infinitely many solutions h-a.e.

We show an easy consequence of Khintchine's Theorem for formal Laurent series. Therefore, we define:

**Definition 1.** *Let $\alpha$ be an irrational element of $\mathbb{F}_q((T^{-1}))$. The number*

$$\nu(\alpha) = \limsup_{|Q| \longrightarrow \infty} \left( -\frac{\log|\alpha - P/Q|}{\log|Q|} \right),$$

*where $P$ and $Q$ vary over all polynomials in $\mathbb{F}_q[T]$ with $Q \neq 0$, is called the approximation exponent of $\alpha$.*

This notation is slightly different from that introduced by deMathan [5] and is contained in the survey of Lasjaunias [9].

The result is now as follows:

**Theorem 8.** *We have h-a.e.*

$$\nu(\alpha) = 2.$$

*Proof.* We consider the following function

$$g_\epsilon(q^k) = \frac{1}{q^{k(1+\epsilon)}}$$

where $\epsilon > 0$ is a real constant. Because of

$$\sum_{k=0}^{\infty} q^k g_\epsilon(q^k) < \infty$$

it follows from Khintchine's Theorem for formal Laurent series that the inequality

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{|Q|^{2+\epsilon}}$$

has finitely many solutions h-a.e.

The result is now a direct consequence of the definition of the approximation exponent. □

In the next section we consider the case of divergence in Khintchin's Theorem for formal Laurent series and state the main result of the paper.

## 4. The Theorem of LeVeque for formal Laurent series

Let $f$ be a function defined on the non-negative real numbers with the following properties

$$0 < f(x) \leq 1, \quad f \text{ is decreasing }, \quad \frac{1}{x^{1+\epsilon}} \ll f(x) \ll \frac{1}{x}, \tag{12}$$

$$f'(x) \ll \frac{1}{x^2}, \quad \sum_{k=0}^{\infty} f(k) = \infty, \tag{13}$$

where $\epsilon < 1$ is a positive real constant.

If we define

$$g(q^k) = q^{-k} f(k)$$

for $k \geq 0$, then it follows from Khintchine's Theorem for formal Laurent series that the inequality

$$\left|\alpha - \frac{P}{Q}\right| < \frac{g(|Q|)}{|Q|} = \frac{f(\deg Q)}{|Q|^2} \tag{14}$$

has infinitely many solutions $P, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$ h-a.e.

We consider the following set

$$W_n(\alpha) = \#\{(P,Q) \in \mathbb{F}_q[T] \times \mathbb{F}_q[T] \mid 0 \leq \deg Q \leq n,\ (P,Q) = 1,\ P/Q \text{ is a solution of } (14)\}$$

for $n \geq 0$ and $\alpha \in H$.

With this notation, we have the following analogue of LeVeque's Theorem in the field of formal Laurent series.

**Theorem 9.** *Let $(W_n)_{n\geq 0}$ be the sequence of random variables introduced above and denote by*

$$A(n) = \frac{(q-1)^2}{q} \sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil}$$

*for $n \geq 0$, where $\lceil x \rceil$, for real $x$, is the smallest integer $\geq x$.*

*I. For fixed real number $\omega$ we have*

$$\lim_{n \longrightarrow \infty} h\left[W_n - A(n) < \omega((q-1)A(n))^{1/2}\right] = \Phi(\omega)$$

*II. We have h-a.e.*

$$\limsup_{n\longrightarrow\infty} \frac{1}{(2(q-1)A(n)\log\log A(n))^{1/2}}(W_n(\alpha) - A(n)) = 1,$$

$$\liminf_{n\longrightarrow\infty} \frac{1}{(2(q-1)A(n)\log\log A(n))^{1/2}}(W_n(\alpha) - A(n)) = -1.$$

*In particular we have h-a.e.*

$$W_n(\alpha) \sim A(n). \tag{15}$$

Before we prove this result, we need a few auxiliary results which we collect in the next section.

## 5. Auxilary Results

The first Lemma is a simply generalization of Lemma 4 in [11]:

**Lemma 5.** *Let $(g_k)_{k\geq 1}$ be sequence of real-valued functions on $P$. Define $X_k(\alpha) := g_k(A_k(\alpha))$ for $k \geq 1$ and $\alpha \in H$. Then $(X_k)_{k\geq 1}$ is an independent sequence of random variables on the probability space $(H, \mathfrak{B}, h)$.*

*Proof.* To be exact, the $X_k$ are only defined on the irrational elements of $H$. But the other elements form a set of measure zero and we can define $X_k$ on this set arbitrarily.

In order to show that the sequence is independent, it suffices to show that the events $A_1(\alpha) = A_1, \ldots, A_n(\alpha) = A_n$ are independent for all polynomials $A_1, \ldots, A_n$ in $P$ and $n \geq 1$. But this is a simple consequence of Lemma 2. □

As in [11] we apply now the classical results of probability theory on this sequence of random variables. Compare the next two results with Theorem 4 and Theorem 5 in [11].

**Theorem 10. (Law of Iterated Logarithm for formal Laurent series)** *Let $(g_k)_{k\geq 1}$ be a sequence of real-valued functions on $P$ with*

$$\sum_{p\in P} g_k^2(p) q^{-2\deg p} < \infty$$

*for all $k \geq 1$. We denote by*

$$\xi_k := \sum_{p\in P} g_k(p) q^{-2\deg p}, \quad \sigma_k^2 := \sum_{p\in P} g_k^2(p) q^{-2\deg p} - \xi_k^2, \quad s_n^2 := \sum_{k=1}^n \sigma_k^2$$

*for $k, n \geq 1$ and assume that*

$$s_n^2 \longrightarrow \infty \quad , \text{as } n \longrightarrow \infty \tag{16}$$

$$|g_k| \leq m_k \tag{17}$$

$$m_k = o\left(\sqrt{\frac{s_k^2}{\log\log s_k^2}}\right) \quad , \text{as } k \longrightarrow \infty. \tag{18}$$

*Then we have h-a.e.*

$$\limsup_{n\longrightarrow\infty} \frac{1}{(2s_n^2 \log\log s_n^2)^{1/2}} \sum_{k=1}^n (g_k(A_k(\alpha)) - \xi_k) = 1,$$

$$\liminf_{n\longrightarrow\infty} \frac{1}{(2s_n^2 \log\log s_n^2)^{1/2}} \sum_{k=1}^n (g_k(A_k(\alpha)) - \xi_k) = -1.$$

*Proof.* Consider the independent sequence of random variables, which is defined as in Lemma 5 and apply the classical law of iterated logarithm due to Kolmogorov (see for instance Theorem 1 of Chapter X in [13]). □

**Theorem 11. (Central Limit Theorem for formal Laurent series)** *Let $(g_k)_{k\geq 1}$ be a sequence of non-constant, real valued functions on $P$ with*

$$\sum_{p\in P} g_k^2(p) q^{-2\deg p} < \infty$$

*for all $k \geq 1$. Let $\xi_k, \sigma_k^2$ and $s_n^2$ be as in Theorem 8 and denote by*

$$L_n(\epsilon) := \frac{1}{s_n^2} \sum_{k=1}^n \sum_{p\in P, |g_k(p)-\xi_k|\geq \epsilon s_n} (g_k(p) - \xi_k)^2 q^{-2\deg p}$$

*for a positive real constant $\epsilon$. Assume that* $\lim_{n\longrightarrow\infty} L_n(\epsilon) = 0$ *for all $\epsilon > 0$.*

*Then, we have*

$$\lim_{n\longrightarrow\infty} h\left[\sum_{k=1}^{n}(g_k(A_k(\alpha))-\xi_k) < \omega s_n\right] = \Phi(\omega).$$

*Proof.* Consider again the sequence of random variables of Lemma 5. From the assumption that $g_k$ is non-constant, it follows that the standard deviations of these random variables are positive and $\lim_{n\longrightarrow} L_n(\epsilon) = 0$ for all $\epsilon > 0$ is the classical Lindeberg condition for this sequence. Therefore, the result follows from the classical central limit theorem. (see for instance Satz 51.3 in [1]) □

We have the following consequence:

**Corollary 1.** *Let $(g_k)_{k\geq 1}$ be a sequence satisfying the assumptions of Theorem 10 and further assume that*

$$s_n^2 = O\left(\sum_{k=1}^{n}\xi_k\right). \tag{19}$$

*We denote by $\lambda$ a real number in the interval $(\frac{1}{2}, 1)$. Then, for h-a.e. $\alpha$ there exist a real number $\kappa$ such that*

$$\left|\sum_{k=1}^{n} g_k(A_k(\alpha)) - \sum_{k=1}^{n}\xi_k\right| \leq \kappa\left|\sum_{k=1}^{n}\xi_k\right|^{1-\lambda}$$

*for all $n \geq 1$. Especially we have h-a.e.*

$$\sum_{k=1}^{n} g_k(A_k(\alpha)) \sim \sum_{k=1}^{n}\xi_k.$$

*Proof.* Because of the law of iterated logarithm for formal Laurent series, we have h.a.e

$$\left|\sum_{k=1}^{n} g_k(A_k(\alpha)) - \sum_{k=1}^{n}\xi_k\right| \leq \kappa(s_n^2\log\log s_n)^{1/2}$$

for all $n \geq 1$ with a suitable constant $\kappa$. The result now follows from the assumption (19).

Because of (19) we have

$$\left|\sum_{k=1}^{n}\xi_k\right| \longrightarrow \infty,\ \text{as } n \longrightarrow \infty$$

and the second part follows from the first one. □

The additional assumption (19) in Corollary 1 is, for instance, fulfilled by a non-negative sequence of functions $g_k$ with a uniformly bounded sequence $m_k$ (note that in this situation (18) is obvious). We are going to apply these probabilistic results on a sequence of functions $g_k$, which has this property.

Let $f$ be a function on the non-negative real numbers satisfying (12) and (13).

Because of Lemma 4, we know that we have h-a.e. for infinitely many $k$

$$\left|\alpha - \frac{P_k}{Q_k}\right| < \frac{f(k)}{|Q_k|^2}. \tag{20}$$

Therefore, it is natural to consider the following set

$$X_n(\alpha) := \{0 \leq k \leq n | P_k/Q_k \text{ is a solution of (20)}\}$$

for an integer $n \geq 0$ and $\alpha \in H$. The sequence $(X_n)_{n\geq 0}$ can be viewed as a sequence of random variables and we have the following asymptotic result:

**Theorem 12.** *Let $(X_n)_{n\geq 0}$ be as above and we put*

$$F(n) = \sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil}$$

*for $n \geq 0$.*

*I. For a fixed real number $\omega$ we have*

$$\lim_{n\longrightarrow\infty} h\left[X_n - F(n) < \omega(F(n))^{1/2}\right] = \Phi(\omega)$$

*II. We have h-a.e.*

$$\limsup_{n\longrightarrow\infty} \frac{1}{(2F(n)\log\log F(n))^{1/2}}(X_n(\alpha) - F(n)) = 1.$$

$$\liminf_{n\longrightarrow\infty} \frac{1}{(2F(n)\log\log F(n))^{1/2}}(X_n(\alpha) - F(n)) = -1.$$

*In particular we have h-a.e.*

$$X_n(\alpha) \sim F(n). \tag{21}$$

*Proof.* We define the following sequence of functions on $P$

$$g_k(p) = \begin{cases} 1 & \text{if } |p| > \frac{1}{f(k-1)} \\ 0 & \text{otherwise} \end{cases}$$

for $k \geq 1$. Because of the the properties of $f$, these functions are non-constant and we have

$$\begin{aligned} \sum_{p\in P} g_k^2(p) q^{-2\deg p} &= \sum_{p\in P, \deg p > -\log_q f(k-1)} q^{-2\deg p} \\ &= q^{-[-\log_q f(k-1)]} = q^{\lceil \log_q f(k-1) \rceil} < \infty. \end{aligned}$$

We use the same notation as in Theorem 11 and get

$$s_n^2 = F(n-1) - \sum_{k=0}^{n-1} q^{2\lceil \log_q f(k) \rceil} = F(n-1) + O(1). \tag{22}$$

Again, because of the properties of $f$, it follows from (22) that

$$s_n^2 \longrightarrow \infty \quad , \text{ as } n \longrightarrow \infty. \tag{23}$$

Therefore, and because of the trivial fact that the sequence of functions $g_k$ is uniformly bounded by 1, it follows for an arbitrary positive real constant $\epsilon$ and $n$ large enough that

$$|g_k(p) - \xi_k| < \epsilon s_n$$

for all integers $k \geq 1$ and all $p \in P$. Hence, the Lindeberg condition in the central limit theorem for formal Laurent series is true and this theorem implies

$$\lim_{n \longrightarrow \infty} h \left[ \sum_{k=1}^{n+1} g_k(A_k) - F(n) < \omega (F(n))^{1/2} \right] = \Phi(\omega).$$

Because of

$$\frac{1}{|A_{k+1}|} > f(k) \Longleftrightarrow \left| \alpha - \frac{P_k}{Q_k} \right| < \frac{f(k)}{|Q_k|^2} \tag{24}$$

and the definition of $g_k$, we have

$$\sum_{k=1}^{n+1} g_k(A_k) = X_n$$

and the first claimed result is proved.

The second result follows by an application of the law of iterated logarithm for formal Laurent series on the sequence $g_k$. Thereby, the boundary condition on $g_k$ is trivially satisfied and the other assumptions follow from (23). (notice that because of (22) one can replace $s_n^2$ in the law of iterated logarithm for formal Laurent series by $F(n-1)$)

Moreover, it follows from (22) that (19) of Corollary 1 is satisfied and hence (21) follows. □

*Remark 1* Notice that Theorem 12 remains true, also when $f$ fulfills the following weaker assumptions

$$0 < f(x) \leq 1, \quad f(x) = O\left(\frac{1}{x}\right), \quad \sum_{k=0}^{\infty} f(k) = \infty. \tag{25}$$

*Remark 2.* Because of (24) $X_n(\alpha)$ is also the number of $1 \leq k \leq n+1$ with

$$|A_k(\alpha)| > \frac{1}{f(k-1)}.$$

Therefore, Theorem 12 can be seen as a stronger version of the Lemma of Borel-Cantelli for formal Laurent series for functions $f$ with (25).

*Remark 3.* The constant function $f(x) = c$, where $c \in (0, 1]$ is a real number, doesn't satisfy the assumptions but Theorem 12 is still true for this function. The reason is that this situation is much more easier, because the involved sequence of random variables is not only an independent sequence, but also an equidistributed sequence of random variables.

Especially, in this case, we have a stronger version of (21), which follows from the strong law of large numbers applied on the sequence of random variables. In detail, we have h-a.e.

$$\lim_{n\longrightarrow\infty}\frac{1}{n+1}X_n(\alpha)=q^{\lceil\log_q c\rceil}.$$

This can be also seen as follows:

*The function $k\longmapsto |Q_k||Q_k\alpha-P_k|$ has a limiting distribution h-a.e.*

This was already pointed out in [2] and was in the classical case a conjecture of H. W. Lenstra (with the exact limit distribution), which was proved in [3].

The next two Lemmas are technical details for the proof of Theorem 9. The first one is contained in [10]:

**Lemma 6.** *For positive real constants $c$ and $\lambda$, we have*

$$f(k+O(k^{1-\lambda})) = f(k)+O(k^{-1-\lambda}), \tag{26}$$

$$\sum_{k=n+1}^{cn} f(k) = O(1), \tag{27}$$

$$\sum_{k=0}^{n} cf(ck) = \sum_{k=0}^{n} f(k)+O(1). \tag{28}$$

*Proof.* Lemma 1 in [10]. □

**Lemma 7.** *I. There exists a real number $\lambda\in(1/2,1)$, such that for all real constant $\kappa$, we have*

$$\sum_{k=0}^{n} q^{\lceil\log_q f(k+\kappa k^{1-\lambda})\rceil}=\sum_{k=0}^{n} q^{\lceil\log_q f(k)\rceil}+O(1), \tag{29}$$

$$\sum_{k=0}^{n} q^{\lceil\log_q f(k-\kappa k^{1-\lambda})\rceil}=\sum_{k=0}^{n} q^{\lceil\log_q f(k)\rceil}+O(1). \tag{30}$$

*II. For all positive real constants $c$, we have*

$$\sum_{k=0}^{cn} q^{\lceil\log_q f(k)\rceil} = \sum_{k=0}^{n} q^{\lceil\log_q f(k)\rceil}+O(1), \tag{31}$$

$$\int_0^n q^{\lceil\log_q f(x)\rceil}dx = \sum_{k=0}^{n} q^{\lceil\log_q f(k)\rceil}+O(1), \tag{32}$$

$$\sum_{k=0}^{n} cq^{\lceil\log_q f(ck)\rceil} = \sum_{k=0}^{n} q^{\lceil\log_q f(k)\rceil}+O(1). \tag{33}$$

*Proof.* First, because of (26), we observe

$$
\begin{aligned}
f(k+\kappa k^{1-\lambda}) &= f(k)+O(k^{-1-\lambda}) \\
&= f(k)\left(1+O\left(\frac{k^{-1-\lambda}}{f(k)}\right)\right) \qquad (34)\\
&= f(k)(1+O(k^{\epsilon-\lambda}))
\end{aligned}
$$

and therefore, we chose $\lambda > \epsilon$. Next we put

$$h(x)=\frac{1}{f(x)}$$

and because of the assumptions on $f$, the function $h$ is increasing and $\geq 1$. The definition of $\lambda$ and (34) implies for large enough $k$

$$h(k) < q^i \Longrightarrow h(k+\kappa k^{1-\lambda}) < q^{i+1}$$

where $i \geq 0$ is an integer.

We have to find now an upper bound for the number of $k$ with

$$h(k) < q^i \leq h(k+\kappa k^{1-\lambda}) \qquad (35)$$

for all integers $i \geq 1$, because the elements in the sums on the right-hand side and on the left-hand side of (29) differ exactly for this $k$.

Therefore, let $k_i$ for $i \geq 1$ denote the smallest integer with the property (35). Then, we see from the right-hand side of (35) that the number of $k$ with (35) is bounded by $[\kappa k_i^{1-\lambda}]+2$.

Furthermore, we have

$$q^i > h(k_i) \gg k_i$$

and hence

$$k_i \ll q^i.$$

Now, we can estimate the difference of the two sums in (29)

$$
\begin{aligned}
\sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil} - \sum_{k=0}^{n} q^{\lceil \log_q f(k+\kappa k^{1-\lambda}) \rceil} &\ll \sum_{i=1}^{\infty}([\kappa k_i^{1-\lambda}]+2)q^{-i} \\
&\ll \sum_{i=1}^{\infty}(q^{i(1-\lambda)}+1)q^{-i} \\
&= \sum_{i=1}^{\infty} q^{-i\lambda}+O(1)=O(1)
\end{aligned}
$$

and (29) is proved.

The proof of (30) is similar.

For the proof of (31), we have to make the following estimation

$$\sum_{k=n+1}^{cn} q^{\lceil \log_q f(k) \rceil} < q \sum_{k=n+1}^{cn} f(k) = O(1)$$

where the last estimation follows from (27).

The proof of (32) is trivial and finally, the proof of (33) follows from

$$c\sum_{k=0}^{n} q^{\lceil \log_q f(ck)\rceil} = c\int_0^n q^{\lceil \log_q f(cx)\rceil}dx + O(1)$$

with the substitution $y = cx$ and applying (32) and (33). □

The last ingredient in the proof of Theorem 9 is a stronger version of the Theorem of Khintchine-Levy for formal Laurent series (compare with Lemma 4 in [10]).

**Lemma 8.** *Let $\lambda \in (1/2, 1)$ be an arbitrary real number. Then, we have h-a.e., that there exist a positive real number $\kappa$, such that*

$$\left|\deg Q_k(\alpha) - \frac{q}{q-1}k\right| \le \kappa k^{1-\lambda} \tag{36}$$

*for all $k \ge 0$.*

*Proof.* Easy consequence of the law of iterated logarithm for continued fractions in [11] (Corollary 3). □

## 6. Proof of Theorem 9

We follow the ideas of the classical result in [10]. Therefore, we use the notation of Theorem 12 and have

$$\lim_{n\longrightarrow\infty} h[X_n < F(n) + \omega(F(n))^{1/2}] = \Phi(\omega).$$

Furthermore, we use $\lambda$ from Lemma 7 I and consider the following random variables

$$\begin{aligned} Y_{n,\kappa}(\alpha) &= \#\left\{0 \le k \le n \middle| \left|\alpha - \frac{P_k}{Q_k}\right| < \frac{f(k+\kappa k^{1-\lambda})}{|Q_k|^2}\right\}, \\ Y_n(\alpha) &= \#\left\{0 \le k \le n \middle| \left|\alpha - \frac{P_k}{Q_k}\right| < \frac{f(((q-1)/q)\deg Q_k)}{|Q_k|^2}\right\}, \end{aligned}$$

where $\kappa$ is an arbitrary positive constant, $n \ge 0$ is an integer and $\alpha \in H$. We introduce the following sets

$$\begin{aligned} B_{n,\kappa} &= \{\alpha \in H | Y_{n,\kappa}(\alpha) - F(n) < \omega(F(n))^{1/2}\}, \\ B_n &= \{\alpha \in H | Y_n(\alpha) - F(n) < \omega(F(n))^{1/2}\}, \\ C_\kappa &= \{\alpha \in H | \alpha \text{ satisfies (36) with } \kappa\}, \end{aligned}$$

where $\omega$ is a real constant and denote by

$$F_\kappa(n) = \sum_{k=0}^{n} q^{\lceil \log_q f(k+\kappa k^{1-\lambda})\rceil}.$$

Because of Lemma 6, we can apply Theorem 12 on the function $f(k + \kappa k^{1-\lambda})$ and we get

$$\lim_{n\longrightarrow\infty} h[Y_{n,\kappa} - F_\kappa(n) < \omega(F_\kappa(n))^{1/2}] = \Phi(\omega). \tag{37}$$

Lemma 7 implies that $F(n) = F_\kappa(n) + O(1)$ and therefore, we can replace $F_\kappa(n)$ in (37) by $F(n)$. Hence

$$\lim_{n\longrightarrow\infty} h(B_{n,\kappa}) = \Phi(\omega) \tag{38}$$

for all positive real numbers $\kappa$.

Let now $\epsilon > 0$ be a real constant. First, we can choose a real constant $\kappa_0$ such that $h(C_\kappa) \geq 1-\epsilon$ for all $\kappa \geq \kappa_0$ because of Lemma 8. Next, we fix $\kappa \geq \kappa_0$ and choose, because of (38), an integer $n_0$ such that $h(B_{n,\kappa}) \leq \Phi(\omega) + \epsilon$ for all $n \geq n_0$.

It is easy to see that we have

$$C_\kappa \cap B_n \subseteq B_{n,\kappa}$$

and therefore, we get

$$h(B_{n,\kappa}) \geq h(C_\kappa) + h(B_n) - 1.$$

An easy calculation shows

$$h(B_n) \leq \Phi(\omega) + 2\epsilon \quad \text{for} \quad n \geq n_0.$$

By using $-\kappa$ in the definition of $Y_{n,\kappa}$ and similar arguments one gets for $n$ large enough

$$h(B_n) \geq \Phi(\omega) - 2\epsilon$$

and hence

$$\lim_{n\longrightarrow\infty} h(B_n) = \Phi(\omega).$$

If we set now

$$Z_n(\alpha) = \#\left\{0 \leq k \leq n \middle| \left|\alpha - \frac{P_k}{Q_k}\right| < \frac{f(\deg Q_k)}{|Q_k|^2}\right\}$$

and

$$G(n) = \frac{q-1}{q}\sum_{k=0}^{n} q^{\lceil \log_q f(k)\rceil},$$

then, we have also proved that

$$\lim_{n\longrightarrow\infty} h[Z_n - G(n) < \omega(G(n))^{1/2}] = \Phi(\omega). \tag{39}$$

Of course if we start with the function $f$ then, the assumptions for $f$ are also true for the function $f(\frac{q}{q-1}x)$ and we can apply what we have already proved. Finally, Lemma 7 (33) implies the claimed result.

Next, we consider the inequality

$$\left|\alpha - \frac{P}{Q}\right| < \frac{f(\deg Q)}{|Q|^2} \tag{40}$$

and because of Lemma 1 (5) and the assumptions on $f$ the random variable $Z_n$ can be also viewed as follows

$$Z_n(\alpha) = \#\{(P,Q) | 1 \leq |Q| \leq |Q_n|, Q \text{ is monic}, (P,Q) = 1, \\ \text{and } P/Q \text{ is a solution of (40)}\}$$

We finish the proof by showing, that if someone replaces $|Q_n|$ in the above definition by $q^n$, (39) remains true. Therefore, we define the following random variable

$$V_{n,\beta} = \#\{(P,Q) | 1 \leq |Q| \leq q^{\beta n}, Q \text{ is monic, } (P,Q)=1, \\ \text{and P/Q is a solution of (40)}\},$$

where $\beta$ is a positive real constant and we use $V_n$ as a short form for $V_{n,1}$. Furthermore, we define the following sets

$$\begin{aligned} D_{n,\beta,\omega} &= \{\alpha \in H | V_{n,\beta}(\alpha) - G(n) < \omega (G(n))^{1/2}\} \\ E_{n,\omega} &= \{\alpha \in H | Z_n(\alpha) - G(n) < \omega (G(n))^{1/2}\} \\ F_N &= \{\alpha \in H | q^n < |Q_n| < q^{3n} \text{ for all } n \geq N\} \end{aligned}$$

where $N$ is a positive integer.

First, we observe that

$$F_N \cap E_{n,\omega} \subseteq D_{n,1,\omega}$$

for all $n \geq N$. Then, we consider for a positive real constant $\eta$

$$G(n/3) + \eta (G(n/3))^{1/2} = G(n) + (\eta + O((G(n))^{-1/2}))(G(n))^{1/2}$$

where Lemma 7 (31) was used. Because of the assumptions on $f$, we have that $\lim_{n \longrightarrow \infty} G(n) = \infty$ and therefore, it follows, that for all positive real constants $\delta$ there exists an index $n_0$, such that

$$G(n/3) + (\omega + \delta)(G(n/3))^{1/2} \geq G(n) + \omega (G(n))^{1/2} \tag{41}$$

for all $n \geq n_0$. By (41) we have

$$D_{n,1,\omega} \subseteq D_{n/3,3,\omega+\delta}$$

for all $n \geq n_0$. It is easy to see that we have

$$F_N \cap D_{n/3,3,\omega+\delta} \subseteq E_{[n/3],\omega+\delta}$$

for all $n \geq 3N$.

If we now put everything together, we have

$$F_N \cap E_{n,\omega} \subseteq F_N \cap D_{n,1,\omega} \subseteq F_N \cap D_{n/3,3,\omega+\delta} \subseteq F_N \cap E_{[n/3],\omega+\delta} \tag{42}$$

for all positive real constants $\delta$ if $n \geq \max\{n_0, 3N\}$.

Because of the fact

$$1 < \frac{q}{q-1} < 3$$

the Khintchine-Levy Theorem 5 implies that

$$\lim_{N \longrightarrow \infty} h(F_N) = 1.$$

Therefore, we can conclude from (42) and from (39) that

$$\Phi(\omega) \leq \lim_{n \longrightarrow \infty} h(D_{n,1,\omega}) \leq \Phi(\omega + \delta)$$

for all positive real constants $\delta$. By considering $\delta \longrightarrow 0$, we have

$$\lim_{n \longrightarrow \infty} h[V_n - G(n) < \omega (G(n))^{1/2}] = \Phi(\omega).$$

Finally, by multiplying both sides with $q-1$ we get the desired result.

The rest of the Theorem is proved similar with the corresponding results of Theorem 12. We give only a sketch of the proof.

First one apply Theorem 12 to the function $f(k+\kappa k^{1-\lambda})$ and use Lemma 7 (29) to get h-a.e.

$$\limsup_{n\longrightarrow\infty} \frac{1}{(2F(n)\log\log F(n))^{1/2}}(Y_{n,\kappa}(\alpha)-F(n))=1. \tag{43}$$

If we consider for $\kappa$ only integers and intersect all sets, where (43) is true, then we have h-a.e.

$$\limsup_{n\longrightarrow\infty} \frac{1}{(2F(n)\log\log F(n))^{1/2}}(Y_{n,k}(\alpha)-F(n))=1 \tag{44}$$

for all $k\in\mathbb{Z}$.

We now choose an $\alpha\in H$ with (44), which has the property (8). Then it is easy to see that

$$\limsup_{n\longrightarrow\infty} \frac{1}{(2F(n)\log\log F(n))^{1/2}}(Y_{n}(\alpha)-F(n))=1.$$

We can, as in the first part, replace the function $f$ by $f\left(\frac{q}{q-1}x\right)$ and it follows h-a.e.

$$\limsup_{n\longrightarrow\infty} \frac{1}{(2G(n)\log\log G(n))^{1/2}}(Z_{n}(\alpha)-G(n))=1. \tag{45}$$

Because of the Khintchine-Levy Theorem 5, we have h-a.e.

$$q^n<|Q_n|<q^{3n}$$

for $n$ large enough and hence

$$Z_{[n/3]}(\alpha)\le V_n(\alpha)\le Z_n(\alpha)$$

for $n$ large enough. By using this and Lemma 7 (32), we can replace $Z_n$ in (45) by $V_n$ and multiplying denominator and enumerator with $q-1$ give the claimed result.

The second part of II is proved in the same manner and (15) is a simple consequence of II. □

*Remark 4.* In [10], LeVeque only proved a central limit theorem and an asymptotic formula for the number of solutions. A few years later, an iterated logarithm law corresponding to (II) of Theorem 9 was added by Philipp [14].

We conclude the paper by giving an application of Theorem 9:

*Example 2.* We consider the function

$$f(x)=\left\{\begin{array}{ll} 1 & 0\le x\le 1\\ \frac{1}{x} & x>1\end{array}\right.$$

for which the assumptions are true. In this case, the inequality (14) has the following form

$$\left|\alpha-\frac{P}{Q}\right|<\frac{1}{|Q|^2\ \mathrm{Log}_q|Q|}. \tag{46}$$

It is easy to see that one has

$$\sum_{k=1}^{n} q^{-[\log_q k]} \sim (q-1)\log_q n$$

and therefore, Theorem 9 implies for the number of solutions of (46) h-a.e.

$$W_n(\alpha) \sim \frac{(q-1)^3}{q} \log_q n.$$

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