Dependence between Path Lengths and Size in Random Trees

(joint with H.-H. Chern, H.-K. Hwang and R. Neininger)

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Random m-ary Search Trees

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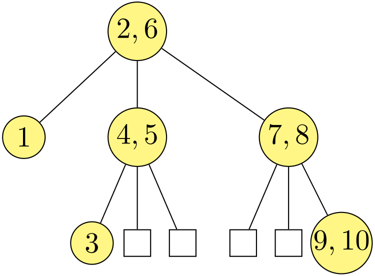

$m = 3$

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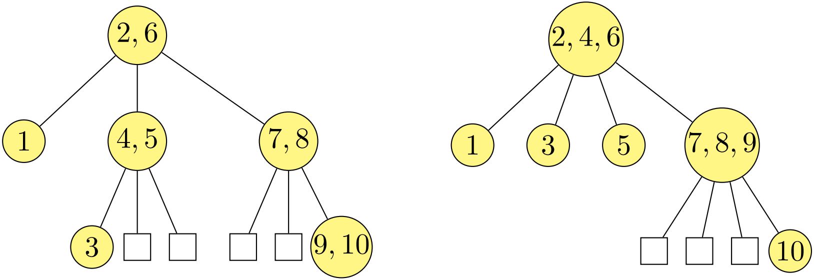

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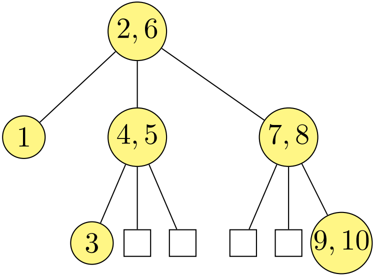

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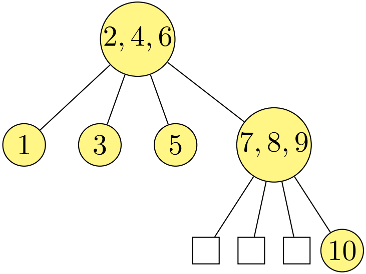

$m = 4$

If permutations are equally likely $\longrightarrow$ **random m-ary search trees**

Size, KPL, and NPL

- **Size** (or Storage Requirement)

  Number of nodes holding keys. Only random if $m \geq 3$.

  $S_n =$ size of a random m-ary search tree built from n keys.

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Size: Mean

Knuth (1973):

$$\mathbb{E}(S_n) \sim \phi n,$$

where

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Mahmoud and Pittel (1989):

$$\mathbb{E}(S_n) = \phi(n+1) - \frac{1}{m-1} + \mathcal{O}(n^{\alpha-1}),$$

where α is the real part of the second largest zero of

$$\Lambda(z) = z(z+1)\cdots(z+m-2) - m!.$$

Size: Phase Change for Variance

Mahmoud and Pittel (1989):

$$\mathrm{Var}(S_n) \sim \begin{cases} C_S n, & \text{if } m \le 26; \\ F_1(\beta \log n) n^{2\alpha-2}, & \text{if } m \ge 27, \end{cases}$$

where $\lambda = \alpha + i\beta$ is the second largest zero of $\Lambda(z)$.

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where $\lambda = \alpha + i\beta$ is the second largest zero of $\Lambda(z)$.

Here, $F_1(z)$ is the periodic function

$$\begin{aligned} F_1(z) = 2\frac{|A|^2}{|\Gamma(\lambda)|^2}&\left(-1 + \frac{m!(m-1)|\Gamma(\lambda)|^2}{\Gamma(2\alpha+m-2) - m!\Gamma(2\alpha-1)}\right) \\ &+ 2\Re\left(\frac{A^2 e^{2iz}}{\Gamma(\lambda)^2}\left(-1 + \frac{m!(m-1)\Gamma(\lambda)^2}{\Gamma(2\lambda+m-2) - m!\Gamma(2\lambda-1)}\right)\right) \end{aligned}$$

with $A = 1/(\lambda(\lambda-1)\sum_{0\le j\le m-2}\frac{1}{j+\lambda})$.

Size: Phase Change for Limit Law

Theorem (Mahmoud & Pittel (1989); Lew & Mahmoud (1994))

For $3 \leq m \leq 26$,

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \xrightarrow{d} N(0,1),$$

where $N(0,1)$ *is the standard normal distribution.*

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Theorem (Chern & Hwang (2001))

For $m \geq 27$*,*

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}}$$

does not converge to a fixed limit law.

KPL: Moments

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Mahmoud (1992):

$$\mathrm{Var}(K_n) \sim C_K n^2,$$

where

$$C_K = 4\phi^2 \left(\frac{(m+1)H_m^{(2)} - 2}{m-1} - \frac{\pi^2}{6} \right)$$

with $H_m^{(2)} = \sum_{1 \le j \le m} 1/j^2$.

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So, no phase change here for the variance!

KPL: Limit Law

Theorem (Neininger & Rüschendorf (1999))

We have,

$$\frac{K_n - \mathbb{E}(K_n)}{n} \xrightarrow{d} K,$$

where K is the unique solution of

$$X \stackrel{d}{=} \sum_{1\le r\le m} V_r X^{(r)} + 2\phi \sum_{1\le r\le m} V_r \log V_r$$

with $X^{(r)}$ an independent copy of X and

$$V_r = U_{(r)} - U_{(r-1)},$$

where $U_{(r)}$ is the r-th order statistic of m i.i.d. uniform RVs.

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We have,

$$\begin{cases} S_n \stackrel{d}{=} S_{I_1}^{(1)} + \cdots + S_{I_m}^{(m)} + 1, \\ N_n \stackrel{d}{=} N_{I_1}^{(1)} + \cdots + N_{I_m}^{(m)} + S_{I_1}^{(1)} + \cdots + S_{I_m}^{(m)}. \end{cases}$$

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So, one expects a strong positive dependence between S_n and N_n!

Size and NPL: Correlation (i)

Theorem (Chern, F., Hwang, Neininger)

We have,

$$\mathrm{Cov}(S_n, N_n) \sim \begin{cases} C_R n \log n, & \text{if } 3 \le m \le 13; \\ \phi F_2(\beta \log n) n^{\alpha}, & \text{if } m \ge 14, \end{cases},$$

where C_R is a constant and $F_2(z)$ is a periodic function. Moreover,

$$\mathrm{Var}(N_n) \sim \phi^2 C_K n^2.$$

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where C_R is a constant and $F_2(z)$ is a periodic function. Moreover,

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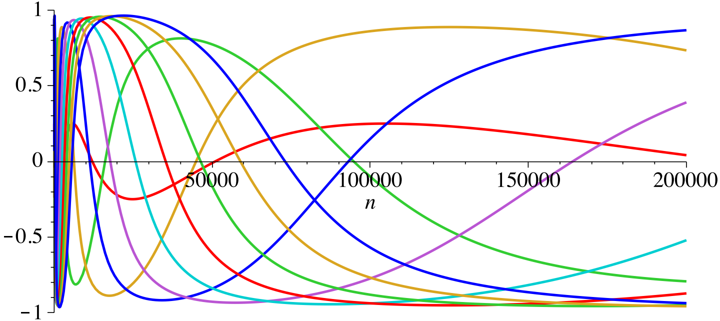
Thus (!),

$$\rho(S_n, N_n) \begin{cases} \longrightarrow 0, & \text{if } 3 \le m \le 26; \\ \sim \frac{F_2(\beta \log n)}{\sqrt{C_K F_1(\beta \log n)}}, & \text{if } m \ge 27. \end{cases}$$

Size and NPL: Correlation (ii)



Periodic function of $\rho(S_n, N_n)$ for $m = 27, 54, \ldots, 270$.

Pearson's Correlation Coefficient

Pearson: for RVs X and Y

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

Measures linear dependence between X and Y!

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Question: Can our counterintuitive result for $\rho(S_n, N_n)$ be ascribed to the weakness of Pearson's correlation coefficient? NO!

Size and NPL: Limit Law for $3 \leq m \leq 26$

Theorem (Chern, F., Hwang, Neininger)

Consider

$$Q_n = (S_n, N_n).$$

Then,

$$\mathrm{Cov}(Q_n)^{-1/2}\Big(Q_n - \mathbb{E}(Q_n)\Big) \xrightarrow{d} \Big(N, C_K^{-1/2}K\Big),$$

where N has a standard normal distribution.

Moreover, N and K are independent!

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Moreover, N and K are independent!

Thus, asymptotic independence for $3 \leq m \leq 26$ is also observed in the bivariate limit law!

Size and NPL: Limit Law for $m \geq 27$

Theorem (Chern, F., Hwang, Neininger)

Consider

$$Y_n = \left(\frac{S_n - \phi n}{n^{\alpha-1}}, \frac{N_n - \mathbb{E}(N_n)}{n} \right).$$

Then,

$$\ell_2(Y_n, (\Re(n^{i\beta}\Lambda), \phi K)) \longrightarrow 0,$$

where ℓ_2 is the minimal L_2-metric and Λ is the unique solution of

$$W \stackrel{d}{=} \sum_{1 \leq r \leq m} V_r^{\lambda-1} W^{(r)}$$

with $W^{(r)}$ independent copies of W.

Size and KPL

Same results hold for size and KPL, e.g.,

$$\rho(S_n, K_n) \begin{cases} \longrightarrow 0, & \text{if } 3 \le m \le 26; \\ \sim \rho(S_n, N_n), & \text{if } m \ge 27. \end{cases}$$

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Theorem (Chern, F., Hwang, Neininger)

We have $\rho(K_n, N_n) \sim 1$ *and*

$$\|N_n - \phi K_n - (\mathbb{E}(N_n - \phi K_n))\|_2 = o(n).$$

In particular,

$$\left(\frac{K_n - \mathbb{E}(K_n)}{n}, \frac{N_n - \mathbb{E}(N_n)}{n}\right) \xrightarrow{d} (K, \phi K).$$

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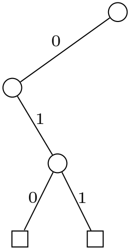

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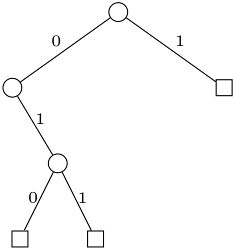

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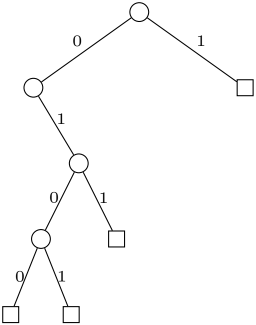

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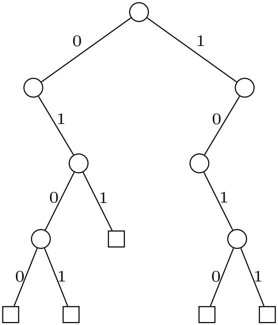

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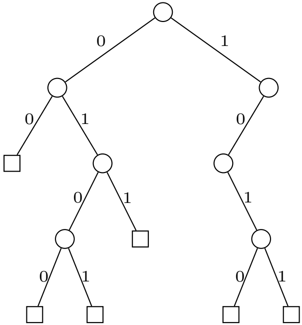

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Question: correlation between size and path-length in random tries?

Size, EPL, and IPL

- **Size**

  Number of internal nodes.

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We again use S_n, K_n, N_n and

$$\begin{array}{ccc} \text{EPL} & \leftrightsquigarrow & \text{KPL;} \\ \text{IPL} & \leftrightsquigarrow & \text{NPL.} \end{array}$$

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$$\chi_k = \frac{2rk\pi i}{\log p}, \qquad (k \in \mathbb{Z}).$$

- For a sequence g_k:

$$\mathscr{F}[g](z) = \begin{cases} \sum_{k\in\mathbb{Z}} g_k z^{-\chi_k}, & \text{if } \log p/\log q \in \mathbb{Q}; \\ g_0, & \text{if } \log p/\log q \notin \mathbb{Q}. \end{cases}$$

Means and Variances

Shape parameters	$\frac{1}{n}$(mean) $\sim$	$\frac{1}{n}$(variance) $\sim$
Size S_n	$\mathscr{F}[\cdot](n)$	$\mathscr{F}[g^{(1)}](n)$
NPL N_n	$\frac{\mathbb{E}(S_n)}{n}\frac{\log n}{h}$	$\frac{\mathrm{Var}(S_n)}{n}\frac{(\log n)^2}{h^2}$
KPL K_n	$\frac{\log n}{h}+\mathscr{F}[\cdot](n)$	$\frac{pq\log^2\frac{p}{q}}{h^2}\cdot\frac{\log n}{h}+\mathscr{F}[g^{(3)}](n)$
Depth D_n	$\mathbb{E}(D_n)=\frac{\mathbb{E}(K_n)}{n}$	$\mathrm{Var}(D_n)=\frac{\mathrm{Var}(K_n)}{n}+O(1)$

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NPL N_n	$\frac{\mathbb{E}(S_n)}{n}\frac{\log n}{h}$	$\frac{\mathrm{Var}(S_n)}{n}\frac{(\log n)^2}{h^2}$
KPL K_n	$\frac{\log n}{h}+\mathscr{F}[\cdot](n)$	$\frac{pq\log^2\frac{p}{q}}{h^2}\cdot\frac{\log n}{h}+\mathscr{F}[g^{(3)}](n)$
Depth D_n	$\mathbb{E}(D_n)=\frac{\mathbb{E}(K_n)}{n}$	$\mathrm{Var}(D_n)=\frac{\mathrm{Var}(K_n)}{n}+O(1)$

Note that

$$N_n \approx S_n\frac{\log n}{h}$$

and for $p \neq q$

$$\mathrm{Var}(D_n) \sim \frac{\mathrm{Var}(K_n)}{n}.$$

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- **Poisson Variance and Covariance**

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- **Other Approaches**

  Elementary (Schachinger), probabilistic (Devroye, Janson, etc.)

Size: Variance

Theorem (Régnier & Jacquet (1989); Kirschenhofer & Prodinger (1991); F., Hwang, Zacharovas (2014))

We have,

$$\mathrm{Var}(S_n) \sim \mathscr{F}[g^{(1)}](n)n,$$

where

$$\begin{aligned} g_k^{(1)} =& \frac{\chi_k \Gamma(-1+\chi_k)}{h}\left(1 - \frac{\chi_k+3}{2^{1+\chi_k}}\right) \\ & - \frac{1}{h^2}\sum_{j\in\mathbb{Z}} \Gamma(\chi_j+1)\Gamma(\chi_{k-j}+1) \\ & - \frac{2}{h}\sum_{j\geq 1} \frac{(-1)^j (j+1+\chi_k)\Gamma(j+\chi_k)\left(p^{j+1}+q^{j+1}\right)}{(j-1)!(j+1)(1-p^{j+1}-q^{j+1})}. \end{aligned}$$

Size and NPL: Variances and Covariance

Theorem (F., Hwang, Zacharovas (2014))

We have,

$$\mathrm{Var}(S_n) \sim \mathscr{F}[g^{(1)}](n)n,$$

and

$$\mathrm{Cov}(S_n, N_n) \sim \mathscr{F}[g^{(1)}](n)\frac{n\log n}{h}$$

and

$$\mathrm{Var}(N_n) \sim \mathscr{F}[g^{(1)}](n)\frac{n\log^2 n}{h^2}.$$

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Theorem (F., Hwang, Zacharovas (2014))

We have,

$$\mathrm{Var}(S_n) \sim \mathscr{F}[g^{(1)}](n)n,$$

and

$$\mathrm{Cov}(S_n, N_n) \sim \mathscr{F}[g^{(1)}](n)\frac{n\log n}{h}$$

and

$$\mathrm{Var}(N_n) \sim \mathscr{F}[g^{(1)}](n)\frac{n\log^2 n}{h^2}.$$

Corollary

We have,

$$\rho(S_n, N_n) \longrightarrow 1.$$

Size and NPL: Limit Law

Theorem (F. & Lee (2015))

We have,

$$\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}}, \frac{N_n - \mathbb{E}(N_n)}{\sqrt{\mathrm{Var}(N_n)}}\right) \xrightarrow{d} \mathcal{N}(0, E_2),$$

where E_2 is the 2×2 unit matrix

$$E_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

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NPL was also investigated by Nguyen-The (2004) in his PhD-thesis, but his result is incorrect.

Question: same result for size and KPL?

Size and KPL: Covariance

Theorem (F. & Hwang)

We have,

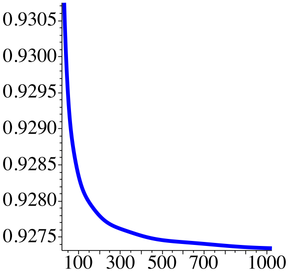
$$\mathrm{Cov}(S_n, K_n) \sim \mathscr{F}[g^{(2)}](n)n,$$

where

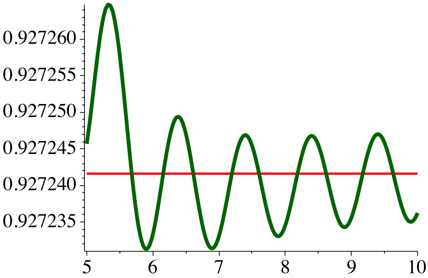
$$\begin{aligned} g_k^{(2)} &= \frac{\Gamma(\chi_k)}{h}\left(1 - \frac{\chi_k + 2}{2^{\chi_k+1}}\right) \\ &\quad - \frac{1}{h^2}\sum_{j\in\mathbb{Z}\setminus\{0\}} \Gamma(\chi_{k-j}+1)(\chi_j - 1)\Gamma(\chi_j) \\ &\quad - \frac{\Gamma(\chi_k+1)}{h^2}\left(\gamma + 1 + \psi(\chi_k+1) - \frac{p\log^2 p + q\log^2 q}{2h}\right) \\ &\quad + \frac{1}{h}\sum_{j\geq 2}\frac{(-1)^j(2j^2 - 2j + 1 + (2j-1)\chi_k)\Gamma(j-1\chi_k)(p^j + q^j)}{j!(1-p^j-q^j)}. \end{aligned}$$

Correlation Coefficient

$p = q = 1/2$:



$\rho(S_n, K_n)$



$$\frac{\mathrm{Cov}(S_n, K_n)}{\sqrt{\mathrm{Var}(S_n)(\mathrm{Var}(K_n)+1.046)}}$$

Size and KPL: Correlation Coefficient

Theorem (F. & Hwang)

We have,

$$\rho(S_n, K_n) \sim \begin{cases} 0, & \textit{if } p \neq q; \\ F(n), & \textit{if } p = q, \end{cases}$$

where

$$F(n) = \frac{\mathscr{F}[g^{(2)}](n)}{\sqrt{\mathscr{F}[g^{(1)}](n)\mathscr{F}[g^{(3)}](n)}}$$

is a periodic function with

$$\textit{average value} = 0.927\cdots \ \textit{and amplitude} \leq 1.5 \times 10^{-5}.$$

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Question: is now this behavior due to the weakness of Pearson's correlation coefficient? Again NO!

Size and KPL: Limit Laws

Theorem (F. & Hwang)

- $p \neq q$:

$$\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}}, \frac{K_n - \mathbb{E}(K_n)}{\sqrt{\mathrm{Var}(K_n)}}\right) \xrightarrow{d} \mathcal{N}(0, I_2),$$

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where I_2 is the 2×2 identity matrix.

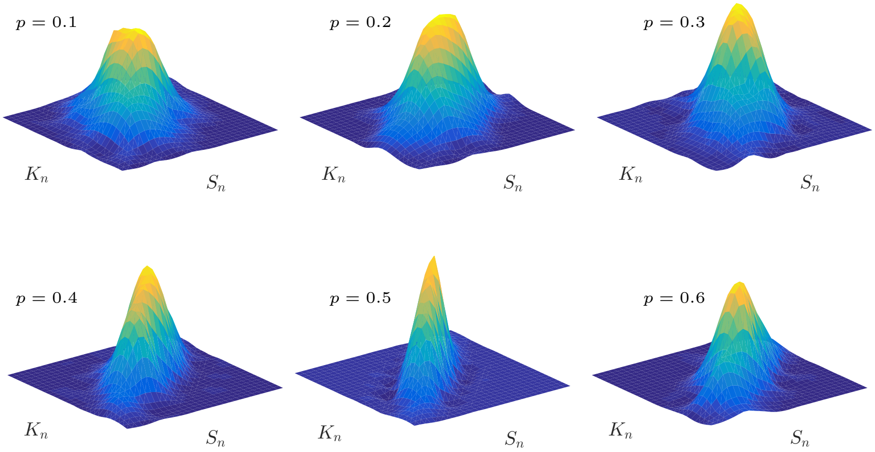
- $p = q$:

$$\Sigma_n^{-1/2}\Big(S_n - \mathbb{E}(S_n), K_n - \mathbb{E}(K_n)\Big) \xrightarrow{d} \mathcal{N}_2(0, I_2),$$

where Σ_n is the (asymptotic) covariance matrix:

$$\Sigma_n := n \begin{pmatrix} \mathscr{F}[g^{(1)}](n) & \mathscr{F}[g^{(2)}](n) \\ \mathscr{F}[g^{(2)}](n) & \mathscr{F}[g^{(3)}](n) \end{pmatrix}.$$

Joint Distribution of S_n and K_n



Summary

trees	$\rho(S_n, K_n)$	$\rho(S_n, N_n)$
tries	$\begin{cases} p \neq q :\to 0 \\ p = q : \text{periodic} \end{cases}$	~ 1
m-ary search trees	$\begin{cases} 3 \leq m \leq 26 :\to 0 \\ m \geq 27 : \text{periodic} \end{cases}$	

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  H.-H. Chern, M. Fuchs, H.-K. Hwang, R. Neininger. Dependence and phase changes in random m-ary search trees, arxiv:1501.05135.

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- Better explanation of our results?