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p-version projection-based interpolation

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Kurzfassung

In der vorliegenden Arbeit definieren wir auf dem Referenztetraeder $\widehat{K}$ Interpolationsoperatoren für die p-Methode der FEM. Diese werden auf den Räumen $H^2(\widehat{K})$, $\mathbf{H}^1(\widehat{K}, \mathbf{curl})$ und $\mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$ definiert, und wir zeigen, dass diese Operatoren Projektionen auf Polynomräume sind, mit Ableitungsoperatoren vertauschen und bei Erhöhung des Polynomgrades gewissen Approximationseigenschaften genügen. Zusätzlich ist die Spur des Interpolanten am Rand vollständig durch die Spur der Funktion bestimmt, weshalb man Interpolationsoperatoren auf einem Gitter elementweise durch Transformation der Operatoren am Referenzelement konstruieren kann.
Projektionsbasierte Interpolationsoperatoren wurden von L. Demkowicz und Koautoren in [15, 28, 29, 33] eingeführt. Eine Zusammenfassung der Resultate findet sich in [26]. Diese Operatoren haben optimale Konvergenzeigenschaften (für $p \to \infty$), abgesehen von logarithmischen Faktoren. In der vorliegenden Arbeit werden die logarithmischen Faktoren entfernt. Allerdings benötigen wir höhere Regularitätsvoraussetzungen als in [26].
Wir untersuchen auch den Interpolationsfehler in negativen Sobolevnormen. In 2D wird die schwächstmögliche negative Norm durch die maximale Regularität von Lösungen des Poissonproblems bestimmt, was von der Verwendung von Dualitätsargumenten stammt. In 3D bekommen wir ebenfalls Abschätzungen in negativen Normen, wobei wir hier die Tatsache verwenden, dass Tetraeder konvex sind, um mehr Regularität von Lösungen zu erhalten.
In dieser Arbeit untersuchen wir auch die Regularität von Lösungen der Poisson-Gleichung $-\Delta u = f$ auf Polygonen Ω in zwei Raumdimensionen, sowohl für Dirichlet-, als auch Neumann-Randbedingungen ("Shift theorem"). Bezeichnen wir mit ω den Innenwinkel von Ω bei einer Ecke, so können wir das Shift Theorem für Funktionen in Sobolevräumen zeigen (für eine gegebene rechte Seite f, die nahe einer Ecke im Raum H^{-1+s}, $0 \leq s < \frac{\pi}{\omega}$, liegt, gilt für die Lösung lokal $u \in H^{1+s}$). Allerdings liegt der Fokus unserer Untersuchungen auf dem Grenzfall $s = \frac{\pi}{\omega}$, den wir in Besovräumen betrachten können: Angenommen $\frac{\pi}{\omega} \notin \mathbb{N}$, dann folgt aus $f \in B^{-1+\pi/\omega}_{2,1}$, dass die Lösung u in $B^{1+\pi/\omega}_{2,\infty}$ liegt. Dieses Resultat ist ähnlich zu denen aus [8, 9], wo Multilevel-Theorie zur Beweisführung des Grenzfalls verwendet wurde. Allerdings wurden diese Ergebnisse nur für Funktionen in Besovräumen, die keine gewöhnlichen Besovräume darstellen, gezeigt, während wir den Mellinkalkül nutzen, um einen Beweis in Standard-Besovräumen zu erhalten.

Abstract

In this thesis, we define p-version projection-based interpolation operators on the reference tetrahedron $\widehat{K}$. These are defined on the spaces $H^2(\widehat{K})$, $\mathbf{H}^1(\widehat{K}, \mathbf{curl})$ and $\mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$, and we show that they are projections onto polynomial spaces, that they satisfy a commuting diagram property and that they have suitable approximation properties, when increasing the polynomial degree. Additionally, the trace of the interpolant on the boundary is fully determined by the trace of the function, which allows the construction of interpolation operators on a grid in an element-wise fashion by transformation of the operators on the reference element.

Projection-based interpolation operators were introduced by L. Demkowicz and several coworkers in [15, 28, 29, 33]. The results were then summarized in [26]. These operators have optimal approximation properties (as $p \to \infty$) up to logarithmic factors. In this thesis, the logarithmic factor is removed. The regularity requirement is, however, stronger than in the work [26].

We also get interpolation error estimates in negative Sobolev norms. In 2D, the weakest possible negative norm is here determined by the maximal regularity for solutions of the Poisson problem, since we use duality arguments. In 3D, we also obtain estimates in negative norms. Here we use the fact that the convexity of tetrahedra allows more regularity for the Poisson problem.

In this work, we also analyze regularity of solutions for the Poisson equation $-\Delta u = f$ on polygons Ω in 2D, both for Dirichlet and Neumann boundary conditions ("Shift theorem"). Denoting ω the interior angle of Ω at a corner, we show the shift theorem for functions in Sobolev spaces (for a right-hand side f that is in H^{-1+s} with $0 \leq s < \frac{\pi}{\omega}$ near a corner, the solution satisfies $u \in H^{1+s}$ locally), however, the focus lies on the limit case $s = \frac{\pi}{\omega}$ which holds in terms of Besov spaces: Assume $\frac{\pi}{\omega} \notin \mathbb{N}$, then $f \in B^{-1+\pi/\omega}_{2,1}$ admits regularity $B^{1+\pi/\omega}_{2,\infty}$. This result is similar to those shown in [8, 9] where multilevel theory was used to prove the limit case. However, these results are formulated for functions in non-standard Besov spaces, whereas we use the Mellin calculus which leads to a proof in standard Besov spaces.

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 27. Januar 2020

Claudio Rojik

Contents

1 Introduction

One possible way to approximate the solutions of partial differential equations (PDEs) is the finite element method (FEM), where one subdivides the domain into finitely many elements with simple geometrical structure and solves the problem in spaces of piecewise polynomials. Besides the common h-method, where one refines the mesh to obtain convergence, also the p-method, where the polynomial degrees p are increased, and the hp-method, which is a combination of both concepts, are used. The p-method and hp-method usually lead to better convergence rates than the h-method, however, the computational cost to create the elements is much higher.

Interpolation operators which approximate a given function by a polynomial function that is easier to handle, play an important role in the theory of solving numerical problems. In the case of scalar functions, many different interpolation operators have been developed for the FEM-h-version (e.g. nodal interpolation, Clément interpolation [17], Scott-Zhang interpolation [58]), but also for the p-version, cf. [4, 6, 57]. However, for vector-valued functions, particularly in the spaces $\mathbf{H}(\mathbf{curl})$ and $\mathbf{H}(\mathrm{div})$, the existing theory is not as well developed. What complicates the situation there is the fact that interpolation operators should not only have suitable approximation properties, but also be projections and satisfy a commuting diagram property. It is convenient to construct these operators elementwise by determining them on the reference element and defining them in physical space by a push-forward from the reference element.

The operators introduced here are projection-based operators. The expression *projection-based interpolation operator* was first used by Hiptmair [38, Sec. 3.5], where such a type of interpolation operators is introduced and a first p-version error estimate is given, cf. [38, Thm. 3.18]. Projection-based interpolation operators for the p-method with the properties mentioned above then have been developed in the first few years of the 21st century by Demkowicz and several co-authors in [15, 28, 29, 33]. The results were then summarized in [26]. These interpolation operators are projections, have the commuting diagram property and admit element-by-element construction, under the regularity assumptions H^{1+s} with $s > 1/2$ for scalar functions and $\mathbf{H}^s(\mathbf{curl})$ with $s > 1/2$ and $\mathbf{H}^s(\mathrm{div})$ with $s > 0$ for vectorial functions (in 3D). However, the operators do not have the optimal convergence properties as $p \to \infty$, but only up to logarithmic factors $\log p$, cf. [26, Thm. 5.3]. The logarithmic factor is due to the approach taken in [26]: In [26] non-local norms on the boundary, which arise from integration by parts, are localized by writing them as a sum of contributions over the different boundary parts, which is possible at the price of logarithmic factors.

In this work, we define projection-based interpolation operators of the type described in [26], but use a different approach for dealing with the non-local norms on the boundary to get rid of the logarithmic factors. We do not try to localize the norms on the boundary, but instead use interpolation arguments: We interpolate between integer order Sobolev norms because these norms can be localized. In turn, we have to analyze the interpolation

error in two different norms. The estimates in the stronger norm follow with arguments similar to [26], and similar tools are used, particularly the polynomial lifting operators for tetrahedra [30, 31, 32] and for triangles [2], and the Poincaré maps as right inverses for the differential operators. However, we use stronger versions of the right inverses, developed by Costabel and McIntosh [19] in 2010. The estimates in the weaker norm are obtained by duality arguments. Here, the shift theorems for elliptic partial differential equations, both for Dirichlet and Neumann boundary conditions, play a central role, since the duality arguments rely on regularity. The shift theorems limit how far we can go in terms of negative norm estimates for the interpolation error. Particularly in 2D, maximal regularity for the Poisson problem in triangles can be exactly specified and determines the weakest possible norm estimate in negative norms. For the tetrahedron in 3D, we restrict the negative norm estimates to a range that, essentially, merely exploits the convexity of the tetrahedron. Extending the range would require more precise information about the shift theorem in 3D which is much more complicated than in 2D.
However, there is also a price to be paid for our approach: We need the more stringent regularity assumptions H^{1+s} and $\mathbf{H}^s(\mathbf{curl})$ both with $s \geq 1$ and $\mathbf{H}^s(\mathrm{div})$ with $s \geq 1/2$. To our best knowledge, interpolation operators without logarithmic factors, but with less needed regularity haven't been developed yet in every case, however, in the recent paper [34] the authors present a projection operator on a polyhedral domain Ω with minimal regularity requirement $\mathbf{H}(\Omega, \mathrm{div})$, which is locally defined on patches of tetrahedra, satisfies a commuting property with the divergence derivative and features *hp*-convergence rates.
Another part of this theses is devoted to the solutions of the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u|_\Gamma &= 0 \quad \text{on } \Gamma := \partial\Omega \end{aligned} \tag{1.1}$$

and the Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \partial_n u|_\Gamma &= 0 \quad \text{on } \Gamma \end{aligned} \tag{1.2}$$

for given right-hand side $f \in H^s(\Omega)$ in a polygonal domain $\Omega \subseteq \mathbb{R}^2$.
The shift theorem for right hand sides $f \in L^2(\Omega)$ for domains Ω with smooth boundary can be found in every book about PDEs, e.g. [35], [37, Chapter 2]. If the boundary of Ω is only Lipschitz, but the domain is convex, the situation is equally well understood, [37, Chapter 3]. For polygonal domains Ω and right hand sides $f \in H^k(\Omega)$ with $k \in \mathbb{N}$, it is easily seen that the shift theorem holds far from the corners, cf. [37, Lemma 5.1.1.1]. Near the corners, however, we need to consider an additional singularity function that inhibits full regularity for the solution.
For right-hand sides $f \in H^s(\Omega)$ with $s < \frac{\pi}{\omega_{max}} - 1$, where ω_{max} denotes the largest interior angle of Ω, we obtain as maximal regularity for solutions u of the problems (1.1) and (1.2) $u \in H^{s+2}(\Omega)$. However, in the limit case $s = \frac{\pi}{\omega_{max}} - 1$ (if $\frac{\pi}{\omega_{max}} \notin \mathbb{N}$) the situation is different. Here we cannot expect $u \in H^{s+2}(\Omega)$, but show that u is still an element of the Besov space $B_{2,\infty}^{\pi/\omega_{max}+1}(\Omega)$ if $f \in B_{2,1}^{\pi/\omega_{max}-1}(\Omega)$.
The shift theorem results in terms of Sobolev spaces are well-known, see e.g. [8, 9, 13] or [23, 37] for more general elliptic problems (and also the references therein). The papers

[8, 9] also deal with regularity in Besov spaces in the limit case $s = \frac{\pi}{\omega_{max}} - 1$ by the use of multilevel theory (see also [7] on this subject), however, the employed spaces are not always standard Besov spaces. For regularity in Besov spaces avoiding the limit case, see [22].
In this thesis, we consider the limiting case $s = \frac{\pi}{\omega_{max}} - 1$ for the shift theorem in terms of Besov spaces, but use techniques different from those in [8, 9]. In fact, we reduce the original problem to a similar problem on a simpler sector domain by localization and use Mellin techniques which enable us to find the explicit form of the singularity function. The use of the Mellin transform goes back at least to the seminal papers by Kondrat'iev [41]. He showed in 1967 that the Mellin calculus is useful when dealing with domains with corners, and it has then become an established method to consider regularity in Sobolev spaces, cf. [20, 21, 23, 37, 42, 44, 53].

1.1 Structure of the work

In **Chapter 2** we introduce fundamental notation, definitions and results. We start with a short presentation of Sobolev spaces and present some results. We then introduce the theory of interpolation spaces and show how Sobolev and Besov spaces are covered by this framework. This is followed by a short compilation about the vector-valued functions spaces $\mathbf{H}(\mathbf{curl})$ and $\mathbf{H}(\mathrm{div})$ and the connections between them. This introductory chapter is then completed with a section about discrete spaces for the finite element method, a section about regular Helmholtz decompositions and a part about discrete Friedrichs inequalities.

Chapter 3 is then devoted to the shift theorem in polygons Ω in 2D. Here we use the Mellin calculus as a tool to obtain the explicit form of the singularity function near the corners. Mellin techniques used for this purpose are already well-established [20, 21, 23, 37, 41], and in Sections 3.1-3.3 (which are based on [24] in addition to the references above), we review the theory since for $f \in L^2(\Omega)$, it allows us to obtain the solution u_1 of the problem (1.1) in the form

$$u_1 = u_0 + s^+ S(f), \tag{1.3}$$

where $u_0 \in H^2(\Omega)$ is the regular part, $s^+ = r^\alpha \sin(\alpha\phi) \in B^{\alpha+1}_{2,\infty}(\Omega)$ (in polar coordinates) with $\alpha > 0$ (dependent on the angles) and $S(f)$ denotes the "stress intensity function"

$$S(f) = \int_\Omega r^{-\alpha} \sin(\alpha\varphi) f(x)\, dx \in \left(B^{\alpha-1}_{2,1}(\Omega)\right)^\star .$$

From this explicit decomposition, we obtain the desired regularity results near the corners by a localization procedure and interpolation arguments. Note that the maximal regularity depends on the largest interior angle of Ω. We also mention that the localization leads to regularity considerations in weighted Sobolev spaces on sectors since the sector with origin 0 and opening angle ω is a convenient model for the polygon near a corner. The arguments for Neumann boundary conditions (1.2) are similar.
In Section 3.4, the applied techniques for right-hand sides $f \in L^2(\Omega)$ are then reviewed and generalized in order to provide regularity results for more regular right-hand sides

$f \in H^s(\Omega)$ for $0 < s < \frac{\pi}{\omega_{max}} - 1$. Basing the argumentation on the ideas of Sections 3.1-3.3, we obtain decompositions similar to (1.3) whose mapping properties are studied and which enables us to prove the desired shift theorems (Section 3.5) for Dirichlet (Theorem 3.44) and Neumann (Theorem 3.47) boundary conditions.

In **Chapter 4** we define the interpolation operators and prove that they indeed provide optimal convergence properties for the p-method. Here, we restrict ourselves to the case of a reference tetrahedron $\widehat{K}$ (or reference triangle $\widehat{f}$ in 2D), since the operators admit an element-by-element construction and thus yield a globally conforming discretization, when we apply them elementwise to a globally defined function. This is due to the fact that by definition of the operators, the trace of the interpolant on a subsimplex of $\widehat{K}$ is completely determined by the trace of the function, which ensures interelement continuity.
Section 4.1 serves as an introductory section. Here we explain the main ideas of the proofs, using the rather simple example of the interpolation operator for the gradient in 2D, $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$. However, we do not go into details, and a rigorous proof of the interpolation error estimates can then be found in Section 4.6.
In Sections 4.2 - 4.4 the definitions of the interpolation operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, $\widehat{\Pi}_{p}^{\mathrm{curl},3d}$ and $\widehat{\Pi}_{p}^{\mathrm{div},3d}$ ($\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$ and $\widehat{\Pi}_{p}^{\mathrm{curl},2d}$ in 2D) are given, and we show that the definitions are indeed meaningful and that the operators satisfy the commuting diagram property. In the short Section 4.5 we deal with the only one-dimensional interpolation problem. We define an interpolation operator on the reference interval $\widehat{e} = (-1, 1)$ mapping in the space of polynomials $\mathcal{P}_p(\widehat{e})$ and show error bounds, cf. Lemma 4.15.
Sections 4.6 and 4.7 are then devoted to error estimates of the interpolation operators in 2D and 3D. They are subdivided into several subsections about the different interpolation operators. At the end of each section the main results are collected in a separate subsection for a quick overview. The main results in 2D are collected in Theorem 4.24, the 3D results are found in Theorem 4.42.
This chapter is then complete with another short Section 4.8. Whereas the whole work up to now has only dealt with first kind Nédélec elements and Raviart-Thomas elements, we use this section to provide some remarks about elements of the second kind (see e.g. [26, Sec. 2], [55, Sec. 3] or [62, Sec. 4] for a short overview about these concepts). It will turn out that all important results also hold in this setting.

Parts of this work, especially Chapter 4, are contained in the paper [49] which was created during the PhD-studies and has been recently published by AMS.

2 Background

In this chapter we introduce the various continuous and discrete function spaces that will appear in the construction of the interpolation operators. In Section 2.1 we define Sobolev spaces of integer order and then also of real order, and we also discuss trace operators and trace inequalities as well as Sobolev embeddings. Section 2.2 is a short introduction to the theory of interpolation spaces. We show how Sobolev spaces fit into this concept. Section 2.3 is devoted to the vector-valued function spaces $\mathbf{H}(\Omega, \mathbf{curl})$ and $\mathbf{H}(\Omega, \mathrm{div})$ and how they are connected with standard Sobolev spaces via differential operators. Section 2.4 is a short introduction to the finite element method, especially to the p-method, since it is our goal to define p-version projection-based interpolation operators which have optimal polynomial approximation properties. Section 2.5 deals with the regularized right inverses of [19], which are then used to construct several Helmholtz-like decompositions that are needed later on. The final Section 2.6 in this chapter then deals with discrete Friedrichs inequalities for the **curl** and div operators.

2.1 Sobolev spaces

In this section Sobolev spaces are defined, of integer order and real order, of both positive and negative orders. For a reference see e.g. [1], [37], [45] or [51].
In this chapter $\Omega \subseteq \mathbb{R}^n$, $n \in \{2, 3\}$ always denotes a bounded domain (i.e. open and connected) with Lipschitz boundary $\Gamma := \partial\Omega$. The following definition is from [51, Def. 3.1].

Definition 2.1. *The domain $\Omega \subseteq \mathbb{R}^n$ has a Lipschitz boundary $\partial\Omega$ if for every $\mathbf{x} \in \partial\Omega$ there is an open set $\mathcal{O} \subseteq \mathbb{R}^n$ with $\mathbf{x} \in \mathcal{O}$ and an orthogonal coordinate system with coordinate $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ having the following properties: There is a vector $\mathbf{a} \in \mathbb{R}^n$ with*

$$\mathcal{O} = \{\boldsymbol{\zeta} : -a_j < \zeta_j < a_j, 1 \leq j \leq n\}$$

and a Lipschitz continuous function ϕ defined on

$$\mathcal{O}' = \{\boldsymbol{\zeta}' \in \mathbb{R}^{n-1} : -a_j < \zeta_j < a_j, 1 \leq j \leq n-1\}$$

with $|\phi(\boldsymbol{\zeta}')| \leq a_n/2$ for all $\boldsymbol{\zeta}' \in \mathcal{O}'$ such that

$$\Omega \cap \mathcal{O} = \{\boldsymbol{\zeta} : \zeta_n < \phi(\boldsymbol{\zeta}'), \boldsymbol{\zeta}' \in \mathcal{O}'\} \text{ and } \partial\Omega \cap \mathcal{O} = \{\boldsymbol{\zeta} : \zeta_n = \phi(\boldsymbol{\zeta}'), \boldsymbol{\zeta}' \in \mathcal{O}'\}.$$

Note that convex sets always have a Lipschitz boundary, so this is especially true for triangles and tetrahedra.
We next denote some standard function spaces. The space $L^p(\Omega)$ is the set of functions ϕ on Ω for which $|\phi|^p$ is integrable on Ω, equipped with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{1/p}.$$

The space $C^\infty(\Omega)$ is the class of infinitably differentiable functions on Ω, and $C_0^\infty(\Omega)$ - often reffered to as the space of test functions - denotes the subset of $C^\infty(\Omega)$ where the functions u are additionally compactly supported in Ω, i.e. $\operatorname{supp}\Omega := \overline{\{\mathbf{x} \in \Omega : u(\mathbf{x}) \neq 0\}}$ is a compact subset of Ω. For the notation of derivatives we use the well-known multi-index notation. For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we set $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i$, and if u is a sufficiently regular function, we define the derivative

$$D^{\boldsymbol{\alpha}} u := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

In the case that the derivative does not exist in the classical sense, we can still define the weak derivative by using a test function by

$$\int_\Omega D^{\boldsymbol{\alpha}} u \phi = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega u D^{\boldsymbol{\alpha}} \phi \quad \forall \phi \in C_0^\infty(\Omega). \tag{2.1}$$

Note that in this case, the derivative $D^{\boldsymbol{\alpha}} u$ is defined as linear functional in $(C_0^\infty(\Omega))'$, the dual space of $C_0^\infty(\Omega)$.
Since often only the degree of the multi-index matters, we also write for $k \in \mathbb{N}_0$

$$D^k u := \sum_{|\boldsymbol{\alpha}|=k} |D^{\boldsymbol{\alpha}} u|. \tag{2.2}$$

The Sobolev space $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, $p \in \mathbb{N}$ is now defined as the space of $L^p(\Omega)$-functions where also the derivatives up to order $k < \infty$ are in $L^p(\Omega)$, i.e.

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^{\boldsymbol{\alpha}} u \in L^p(\Omega), \boldsymbol{\alpha} \in \mathbb{N}_0^n, |\boldsymbol{\alpha}| \leq n\}.$$

Equipping this space with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}| \leq k} \|D^{\boldsymbol{\alpha}} u\|_{L^p(\Omega)} \right)^{1/p}$$

makes $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ a Banach space. The corresponding semi-norm

$$|u|_{W^{k,p}(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}| = k} \|D^{\boldsymbol{\alpha}} u\|_{L^p(\Omega)} \right)^{1/p}$$

is also often useful, e.g. when dealing with scaling arguments. In the above definitions of Sobolev spaces, $\Omega = \mathbb{R}^n$ is allowed, too, which gives us Sobolev spaces on the full space. Note that $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$, cf. [50].
The most important Sobolev spaces occur in the case $p = 2$. If we equip this space with the scalar product

$$(u, v)_{W^{k,2}(\Omega)} = \sum_{|\boldsymbol{\alpha}| \leq k} \int_\Omega D^{\boldsymbol{\alpha}} u D^{\boldsymbol{\alpha}} \overline{v},$$

we get a Hilbert space with induced norm $\|\cdot\|_{W^{k,2}(\Omega)}$.
We also need fractional order Sobolev spaces on Ω. For $\sigma \in (0,1)$ define the Slobodeckij semi-norm $|\cdot|_{W^{\sigma,p}(\Omega)}$ by

$$|u|_{W^{\sigma,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(\mathbf{x}) - u(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{n+\sigma p}}\, d\mathbf{x}\, d\mathbf{y}\right)^{1/p}.$$

For $s \in \mathbb{R}^+ \backslash \mathbb{N}$, write $s = \lfloor s \rfloor + \sigma$ with $\sigma \in (0,1)$, and define the corresponding Slobodeckij norm by[1]

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{\lfloor s \rfloor,p}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=\lfloor s \rfloor} \int_\Omega \int_\Omega \frac{|D^{\boldsymbol{\alpha}} u(\mathbf{x}) - D^{\boldsymbol{\alpha}} u(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{n+\sigma p}}\, d\mathbf{x}\, d\mathbf{y}\right)^{1/p}.$$

The fractional order Sobolev space $W^{s,p}(\Omega)$ is now defined in the natural way as

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : \|u\|_{W^{s,p}(\Omega)} < \infty\}.$$

It is possible to define the Sobolev spaces in an alternative way. Defining the Schwartz space by

$$\mathcal{S}(\mathbb{R}^n) := \{\phi \in C^\infty(\mathbb{R}^n) : \sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^{\boldsymbol{\alpha}} D^{\boldsymbol{\beta}} \phi(\mathbf{x})| < \infty \quad \text{for all multi-indices } \boldsymbol{\alpha}, \boldsymbol{\beta}\}$$

and the Bessel potential of order $s \in \mathbb{R}$ by

$$\mathcal{J}^s u(\mathbf{x}) := \int_{\mathbb{R}^n} (1 + |\boldsymbol{\xi}|^2)^{s/2} \widehat{u}(\boldsymbol{\xi}) \exp(2\pi i \boldsymbol{\xi} \cdot \mathbf{x})\, d\boldsymbol{\xi}$$

for $\mathbf{x} \in \mathbb{R}^n$, where $\widehat{u}$ denotes the Fourier transform of u, cf. Definition 3.1, we can define the Sobolev space of order s on $\mathbb{R}^n$ by

$$H^s(\mathbb{R}^n) := \{u \in (\mathcal{S}(\mathbb{R}^n))' : \mathcal{J}^s u \in L^2(\mathbb{R}^n)\}.$$

The Sobolev spaces on Ω are then defined by restriction,

$$H^s(\Omega) := \{u \in (C_0^\infty(\Omega))' : u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\}. \tag{2.3}$$

Theorem 3.16 and Theorem 3.30 from [45] show that for $s \geq 0$, both definitions of Sobolev spaces are equivalent on Lipschitz domains and the full space, for $p = 2$, i.e.

$$W^{s,2}(\Omega) = H^s(\Omega), \qquad W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n). \tag{2.4}$$

[1] One can also add a preceding normalizing factor to the seminorm expression, i.e.

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{\lfloor s \rfloor,p}(\Omega)} + a_\sigma^{-1} \sum_{|\boldsymbol{\alpha}|=\lfloor s \rfloor} \frac{\lfloor s \rfloor!}{\boldsymbol{\alpha}!} |D^{\boldsymbol{\alpha}} u|_{W^{\sigma,p}(\Omega)}\right)^{1/p},$$

where $a_\sigma := \int_0^\infty t^{-2\sigma-1} \int_{\boldsymbol{\omega}\in\mathbb{R}^n, |\boldsymbol{\omega}|=1} |e^{2\pi i \omega_1 t} - 1|^2\, d\boldsymbol{\omega}\, dt$. This prevents the blow-up of the equivalence constant with the minimum energy extension norm for $\sigma \to 0$, cf. (2.3), (2.4) and [27, p. 53]. However, this is not necessary here since Sobolev orders are always fixed.

Thus, we will use only the H^s-notation from now on.
We also need Sobolev spaces on the boundary Γ. For $0 < s < 1$ they are defined by

$$W^{s,p}(\Gamma) := \{u \in L^p(\Gamma) : \|u\|_{W^{s,p}(\Gamma)} < \infty\}, \tag{2.5}$$

where $\|\cdot\|_{W^{s,p}(\Gamma)}$ is again the Slobodeckij norm

$$\|u\|_{W^{s,p}(\Gamma)} := \left(\|u\|^p_{L^p(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(\mathbf{x}) - u(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{n-1+\sigma p}}\, ds(\mathbf{x})\, ds(\mathbf{y})\right)^{1/p},$$

ds denoting the surface measure on Γ.
For $1/p < s \leq 1$, the space $W^{s-1/p,p}(\Gamma)$ can also be seen as trace space of the space $W^{s,p}(\Omega)$. For a sufficiently smooth function u we can define boundary values of u on Γ, called the trace of u. Obviously every function $u \in C^\infty(\overline{\Omega})$ is sufficiently smooth to admit the evaluation of u on the boundary. Thus, it makes sense to define the trace operator γ_0 on $C^\infty(\overline{\Omega})$ by $\gamma_0(u) = u|_\Gamma$. The following theorem from [51, Thm. 3.9] shows that the trace operator can be extended to an operator on $W^{s,p}(\Omega)$.

Theorem 2.2. *Provided $1/p < s \leq 1$, the mapping γ_0 has a unique continuous extension as a linear operator from $W^{s,p}(\Omega)$ onto $W^{s-1/p,p}(\Gamma)$. Moreover,*

$$W^{1,p}_0(\Omega) = \{u \in W^{1,p}(\Omega) : \gamma_0(u) = 0\}.$$

Since the theorem gives a unique extension of γ_0 to Sobolev spaces, we will adopt the notation $u|_\Gamma$ and use it instead of $\gamma_0(u)$ for functions u that are sufficiently smooth to allow the evaluation on the boundary. Especially for $p = 2$ and $s = 1$, we get $H^{1/2}(\Gamma)$ as the trace space of $H^1(\Omega)$, and we will denote by $H^{-1/2}(\Gamma)$ its dual space.
Note that in [18, Lemma 3.6] it was shown that for $p = 2$ (and Ω a Lipschitz domain, as was assumed in this section), Theorem 2.2 holds for $1/2 < s < 3/2$.
Trace spaces for $s > 0$ can also be defined by restriction of more regular functions on Ω. We define

$$H^s(\Gamma) = \{u \in L^2(\Gamma) : u = U|_\Gamma \text{ for some } U \in H^{s+1/2}(\Omega)\}, \tag{2.6}$$

with natural norm

$$\|u\|_{H^s(\Gamma)} = \inf_{U \in H^{s+1/2}(\Omega),\, u = U|_\Gamma} \|U\|_{H^{s+1/2}(\Omega)}.$$

Note that the definition in (2.5) and (2.6) coincide for $s < 1$.
If we want to take zero boundary conditions into account, we use the following Sobolev spaces. We define $W^{s,p}_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in the $W^{s,p}(\Omega)$-norm. In the case $p = 2$ we write $H^s_0(\Omega) = W^{s,2}_0(\Omega)$ instead. If the boundary of our domain is sufficiently regular, the following theorem from [45, Thm. 3.40] enables us to handle these spaces in a more explicit way.

Theorem 2.3. *Assume that $\widetilde{\Omega} \subseteq \mathbb{R}^n$ is a $C^{k-1,1}$-domain.*

(i) If $0 \leq s \leq \frac{1}{2}$, then $H^s_0(\widetilde{\Omega}) = H^s(\widetilde{\Omega})$.

(ii) If $\frac{1}{2} < s \le k$, *then* $H_0^s(\widetilde{\Omega}) = \{u \in H^s(\widetilde{\Omega}) : \gamma_0(D^{\boldsymbol{\alpha}} u) = 0$ *for* $|\boldsymbol{\alpha}| < s - \frac{1}{2}\}$.

We now introduce another class of Sobolev spaces, by defining[2] the space $\widetilde{H}^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the $H^s(\mathbb{R}^n)$-norm, $s \in \mathbb{R}$.
Negative order Sobolev spaces are defined as topological dual spaces of Sobolev spaces with positive order. For $s > 0$ and $1/p + 1/q = 1$, we define $W^{-s,p}(\Omega) := (W_0^{s,q}(\Omega))'$ with the usual norm

$$\|u\|_{W^{-s,p}(\Omega)} = \sup_{v \in W_0^{s,q}(\Omega)} \frac{\langle u, v\rangle_\Omega}{\|v\|_{W^{s,q}(\Omega)}},$$

where $\langle \cdot, \cdot \rangle_\Omega$ denotes the duality product. Note that it holds $W^{-s,2}(\Omega) = H^{-s}(\Omega)$ for $s \ge 0$. By Theorem 3.30 of [45], we have $(H^s(\Omega))' = \widetilde{H}^{-s}(\Omega)$ and $(\widetilde{H}^s(\Omega))' = H^{-s}(\Omega)$ for $s \in \mathbb{R}$. For $s \ge 0$ with $s + 1/2 \notin \mathbb{N}$, it also holds by [45, Thm. 3.33] that $\widetilde{H}^s(\Omega) = H_0^s(\Omega)$, thus we can identify $H^{-s}(\Omega)$ as the dual space of $H_0^s(\Omega)$ for these values of s.
For functions $u \in L^2(\Omega)$, we can equip the negative order Sobolev spaces with the norms

$$\|u\|_{\widetilde{H}^{-s}(\Omega)} = \sup_{v \in H^s(\Omega)} \frac{(u, v)_{L^2(\Omega)}}{\|v\|_{H^s(\Omega)}}, \tag{2.7}$$

and ($s + 1/2 \notin \mathbb{N}$)

$$\|u\|_{H^{-s}(\Omega)} = \sup_{v \in H_0^s(\Omega)} \frac{(u, v)_{L^2(\Omega)}}{\|v\|_{H^s(\Omega)}}, \tag{2.8}$$

since for sufficiently regular functions, the duality product can be identified with the inner product in $L^2(\Omega)$.
If Ω is piecewise smooth (e.g. polygon, polyhedron), then we also need piecewise trace operators. This is justified by the following result, [59, Lemma 16.1] or [45, Thm. 3.37].

Lemma 2.4. *For* $s > 1/2$, *functions of* $H^s(\mathbb{R}^n)$ *have a trace on the hyperplane* $x_n = 0$, *belonging to* $H^{s-1/2}(\mathbb{R}^{n-1})$. *The mapping* γ_0 *is surjective from* $H^s(\mathbb{R}^n)$ *onto* $H^{s-1/2}(\mathbb{R}^{n-1})$.

By transformation and localization, this lemma yields for $s > 1/2$ and $u \in H^s(\Omega)$

$$\|u\|_{H^{s-1}(f)} \lesssim \|u\|_{H^s(\Omega)}$$

for each face f of Ω.

2.2 Interpolation spaces

In this section we give a short introduction to the theory of interpolation spaces. It turns out that the fractional order Sobolev spaces can be seen as interpolation spaces between integer order spaces. This fact is often helpful since the norms on integer order spaces are easier to work with than with the Slobodeckij norms. The main references for this section are [59] and [60].

[2] Although established in [45, Ch. 3, p. 77], the notation $\widetilde{H}^s(\Omega)$ is a bit misleading for $s < 0$, since this space does not consist of distributions defined on Ω, but of distributions on $\mathbb{R}^n$ with support in $\overline{\Omega}$, as described by the space $H^s_{\overline{\Omega}}$ in [45, Ch. 3, p. 76]. However, in [45, Th. 3.29] it is shown that both spaces coincide for domains Ω with at least C^0-boundary.

2.2.1 Basic interpolation results

Throughout this section, $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ denote normed spaces with a continuous imbedding $X_1 \subseteq X_0$.
One interpolation method is the K-method.

Definition 2.5. *The K-functional on the space X_0 is defined by*

$$K(t,u) := \inf_{v \in X_1} \|u - v\|_0 + t\|v\|_1.$$

For $0 < \theta < 1$, one then defines the interpolation space as

$$X_{\theta,q} := (X_0, X_1)_{\theta,q} := \{u \in X_0 : \|u\|_{\theta,q} < \infty\},$$

equipped with the norm

$$\|u\|_{\theta,q} := \begin{cases} \left(\int_0^\infty (t^{-\theta}K(t,u))^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty, \\ \sup_{t>0} t^{-\theta}K(t,u), & q = \infty, \end{cases}. \tag{2.9}$$

Lemma 2.6. *The following facts are well-known, cf. [59, Ch. 22], [60, Sec. 1.3.3].*

(i) The K-functional is monotone increasing in the variable t, and $t \mapsto t^{-1}K(t,u)$ is monotone decreasing.

(ii) For $\theta \in (0,1)$ and $q \in [1,\infty]$, it holds

$$t^{-\theta}K(t,u) \lesssim \|u\|_{\theta,\infty} \lesssim \|u\|_{\theta,q}.$$

(iii) The interpolation spaces are nested, i.e. for $\theta \in (0,1)$ and $1 \le q \le q' \le \infty$, we have

$$X_{\theta,1} \subseteq X_{\theta,q} \subseteq X_{\theta,q'} \subseteq X_{\theta,\infty}.$$

(iv) For $0 < \theta' < \theta < 1$ and arbitrary $q, q' \in [1,\infty]$, there holds

$$X_{\theta,q} \subseteq X_{\theta',q'}.$$

(v) Interpolation inequality: There exists a constant $C_{\theta,q}$ such that

$$\|u\|_{\theta,q} \le C_{\theta,q}\|u\|_0^{1-\theta}\|u\|_1^{\theta}$$

for all $u \in X_1$.

(vi) Interpolation of a space X with itself again gives X, i.e. $(X,X)_{\theta,q} = X$.

Indeed, integration to infinity in (2.9) is not necessary if the subspace X_1 is a proper subspace of X_0.

Lemma 2.7. *Let $X_1 \subsetneq X_0$ with a continuous embedding. Then for every $a > 0$, $\theta \in (0,1)$ and $q \in [1,\infty]$, there exists $C > 0$ such that*

$$\int_0^\infty |t^{-\theta} K(t,u)|^q \frac{dt}{t} \le C \int_0^a |t^{-\theta} K(t,u)|^q \frac{dt}{t}, \qquad q \in [1,\infty),$$
$$\sup_{t\in(0,\infty)} t^{-\theta} K(t,u) \le C \sup_{t\in(0,a)} t^{-\theta} K(t,u).$$

Proof. For arbitrary $g \in X_1$ we have

$$\begin{aligned} K(t,u) &= \inf_{v\in X_1} \|u-v\|_{X_0} + t\|v\|_{X_1} \le \|u\|_{X_0} \le \|u-g\|_{X_0} + \|g\|_{X_0} \\ &\le \|u-g\|_{X_0} + \left(C_{X_1\to X_0}\frac{2}{a}\right)\frac{a}{2}\|g\|_{X_1}. \end{aligned}$$

Infimizing over all $g \in X_1$ gives

$$K(t,u) \le \underbrace{\max\{1, 2a^{-1}C_{X_1\to X_0}\}}_{=:C_1} K(a/2,u).$$

For $q < \infty$, we now get

$$\int_a^\infty |t^{-\theta} K(t,u)|^q \frac{dt}{t} \le |K(a/2,u)|^q \underbrace{\int_a^\infty t^{-\theta q}\frac{dt}{t}}_{=:C_2}.$$

Since $t \mapsto K(t,u)$ is monotone increasing, it follows

$$\underbrace{\int_{a/2}^a t^{-\theta q}\frac{dt}{t}}_{=:C_3} |K(a/2,u)|^q \le \int_{a/2}^a |t^{-\theta}K(t,u)|^q \frac{dt}{t} \le \int_0^a |t^{-\theta}K(t,u)|^q \frac{dt}{t}.$$

Thus

$$\int_a^\infty |t^{-\theta}K(t,u)|^q \frac{dt}{t} \le \frac{C_2}{C_3}\int_0^a |t^{-\theta}K(t,u)|^q \frac{dt}{t}.$$

For $q = \infty$, the result follows in view of the monotonicity of $t \mapsto K(t,u)$. □

The usefulness of interpolation spaces comes from the following result, which can be found in [59, Lemma 22.3].

Lemma 2.8. *Let $X_1 \subseteq X_0$ and $Y_1 \subseteq Y_0$, and let $T : X_0 \to Y_0$ be a bounded linear operator that maps the subspace X_1 to Y_1. Then $T : X_{\theta,q} \to Y_{\theta,q}$ is also bounded and linear with the operator norm estimate*

$$\|T\|_{\theta,q} \lesssim \|T\|_{X_0\to Y_0}^{1-\theta}\|T\|_{X_1\to Y_1}^{\theta}.$$

Suitable interpolation between Sobolev spaces can result in another Sobolev space. In particular, it holds

$$(H^s(\Omega), H^k(\Omega))_{\theta,2} = H^{(1-\theta)s+\theta k}(\Omega)$$

with equivalent norms, cf. [59, Ch. 34]. In the case $p, q \ne 2$ we generally get Besov spaces.

Definition 2.9. *For $0 < s < 1$, the Besov space $B^s_{p,q}(\Omega)$ is defined by*

$$B^s_{p,q}(\Omega) := (L^p(\Omega), W^{1,p}(\Omega))_{s,q}.$$

Remark 2.10. More generally, the Besov space $B^{s_\theta}_{p,q}(\Omega)$ can also be written as

$$B^{s_\theta}_{p,q}(\Omega) = (W^{s_0,p}(\Omega), W^{s_1,p}(\Omega))_{\theta,q},$$

where $s_0, s_1 \geq 0$, $s_0 \neq s_1$, and $s_\theta = (1-\theta)s_0 + \theta s_1$, see [60, Sec. 2.4.2, Remark 4], [60, Sec. 4.3.1, Thm. 1], [61].

Since the interpolation technique provides another normed space together with a continuous imbedding, it is possible to interpolate between interpolation spaces. However, we do not get completely new spaces by this procedure, as the Reiteration Theorem [59, Thm. 26.3] shows.

Theorem 2.11 (Reiteration Theorem, [59, Thm. 26.3])**.** *For $q_1, q_2, q_3 \in [1, \infty]$, $0 \leq \theta_1 < \theta_2 \leq 1$ and $\theta_3 \in (0,1)$, there holds*

(i) $((X_0, X_1)_{\theta_1,q_1}, (X_0, X_1)_{\theta_2,q_2})_{\theta_3,q_3} = (X_0, X_1)_{(1-\theta_3)\theta_1+\theta_3\theta_2,q_3}$,

(ii) $(X_0, (X_0, X_1)_{\theta_2,q_2})_{\theta_3,q_3} = (X_0, X_1)_{\theta_2\theta_3,q_3}$.

The dual space of an interpolation space is again an interpolation space.

Lemma 2.12 ([60, Sec. 1.11.2])**.** *Let X_1 be a dense subspace of X_0. Then, for $1 \leq q < \infty$, it holds*

$$((X_0, X_1)_{\theta,q})^\star = (X_1', X_0')_{1-\theta,q'},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Interpolation between negative and positive order Sobolev spaces is also possible.

Lemma 2.13. *Let $r, s \geq 0$. Then it follows:*

(i) $(H^{-r}(\Omega), H^s(\Omega))_{\theta,2} = H^{-(1-\theta)r+\theta s}(\Omega)$,

(ii) $(\widetilde{H}^{-r}(\Omega), \widetilde{H}^s(\Omega))_{\theta,2} = \widetilde{H}^{-(1-\theta)r+\theta s}(\Omega)$,

(iii) $(\widetilde{H}^{-r}(\Omega), H^s(\Omega))_{\theta,2} = \begin{cases} \widetilde{H}^{-(1-\theta)r+\theta s}(\Omega), & -(1-\theta)r+\theta s < 0 \\ H^{-(1-\theta)r+\theta s}(\Omega), & -(1-\theta)r+\theta s \geq 0 \end{cases}$,

(iv) $(\widetilde{H}^{-r}(\Omega), H^s(\Omega))_{\theta,1} = B^{-(1-\theta)r+\theta s}_{2,1}(\Omega)$ *if* $-(1-\theta)r+\theta s > 0$,

(v) $(H^{-r}(\Omega), H^s(\Omega))_{\theta,1} = B^{-(1-\theta)r+\theta s}_{2,1}(\Omega)$ *if* $-(1-\theta)r+\theta s > 0$.

Proof. Statements (i) and (ii) are shown in [45, Thm. B.8, Thm. B.9].
We show (iii). From

$$L^2(\Omega) \stackrel{(ii)}{=} (\widetilde{H}^{-s}(\Omega), \widetilde{H}^s(\Omega))_{1/2,2} \subseteq (\widetilde{H}^{-s}(\Omega), H^s(\Omega))_{1/2,2} \subseteq (H^{-s}(\Omega), H^s(\Omega))_{1/2,2} \stackrel{(i)}{=} L^2(\Omega)$$

for $s > 0$, it follows equality in these expressions. It holds for $s > 0$ by the Reiteration theorem

$$\begin{aligned} H^{s\theta}(\Omega) &= (L^2(\Omega), H^s(\Omega))_{\theta,2} = ((\widetilde{H}^{-s}(\Omega), H^s(\Omega))_{1/2,2}, H^s(\Omega))_{\theta,2} \\ &= (H^s(\Omega), (H^s(\Omega), \widetilde{H}^{-s}(\Omega))_{1/2,2})_{1-\theta,2} = (H^s(\Omega), \widetilde{H}^{-s}(\Omega))_{\frac{1}{2}(1-\theta),2} \\ &= (\widetilde{H}^{-s}(\Omega), H^s(\Omega))_{\frac{1+\theta}{2},2}. \end{aligned} \tag{2.10}$$

Lemma 2.12 then gives

$$\widetilde{H}^{-s\theta}(\Omega) = H^{s\theta}(\Omega)^\star = (\widetilde{H}^{-s}(\Omega), H^s(\Omega))_{\frac{1-\theta}{2},2}. \tag{2.11}$$

Now assume $s > r$. Then (2.10) implies

$$\widetilde{H}^{-r}(\Omega) = (\widetilde{H}^{-s}(\Omega), H^s(\Omega))_{\frac{1-r/s}{2},2},$$

thus we obtain

$$\begin{aligned} (\widetilde{H}^{-r}(\Omega), H^s(\Omega))_{\theta,2} &= ((\widetilde{H}^{-s}(\Omega), H^s(\Omega))_{\frac{1-r/s}{2},2}, H^s(\Omega))_{\theta,2} \\ &= (H^s(\Omega), (H^s(\Omega), \widetilde{H}^{-s}(\Omega))_{1-\frac{1-r/s}{2},2})_{1-\theta,2} \\ &= (H^s(\Omega), \widetilde{H}^{-s}(\Omega))_{(1-\theta)(1-\frac{1-r/s}{2}),2} \\ &= (\widetilde{H}^{-s}(\Omega), H^s(\Omega))_{1-(1-\theta)(1-\frac{1-r/s}{2}),2}. \end{aligned} \tag{2.12}$$

If $-(1-\theta)r + \theta s > 0$, then

$$1 - (1-\theta)(1 - \frac{1-r/s}{2}) = \frac{1+\theta'}{2}$$

with $\theta' = 1 - (1-\theta)(1+r/s) \in (0,1)$. Thus (2.10) and (2.12) yield

$$(\widetilde{H}^{-r}(\Omega), H^s(\Omega))_{\theta,2} = H^{s(1-(1-\theta)(1+r/s))}(\Omega) = H^{-(1-\theta)r+\theta s}(\Omega).$$

If $-(1-\theta)r + \theta s < 0$, then

$$1 - (1-\theta)(1 - \frac{1-r/s}{2}) = \frac{1-\theta'}{2}$$

with $\theta' = -1 + (1-\theta)(1+r/s) \in (0,1)$, thus (2.11) and (2.12) yield

$$(\widetilde{H}^{-r}(\Omega), H^s(\Omega))_{\theta,2} = \widetilde{H}^{-s(-1+(1-\theta)(1+r/s))}(\Omega) = \widetilde{H}^{-(1-\theta)r+\theta s}(\Omega).$$

If $-(1-\theta)r+\theta s=0$, then $1-(1-\theta)(1-\frac{1-r/s}{2})=\frac{1}{2}$, thus we get from (2.12) that

$$(\widetilde{H}^{-r}(\Omega),H^s(\Omega))_{\theta,2}=L^2(\Omega).$$

The case $s<r$ is similar. Here we write

$$H^s(\Omega)=(\widetilde{H}^{-r}(\Omega),H^r(\Omega))_{\frac{1+s/r}{2},2}$$

and therefore obtain

$$\begin{aligned}(\widetilde{H}^{-r}(\Omega),H^s(\Omega))_{\theta,2}&=(\widetilde{H}^{-r}(\Omega),(\widetilde{H}^{-r}(\Omega),H^r(\Omega))_{\frac{1+s/r}{2},2})_{\theta,2}\\&=(\widetilde{H}^{-r}(\Omega),H^r(\Omega))_{\frac{1+s/r}{2}\theta,2}.\end{aligned}$$

Case by case analysis as above gives the desired result.
Statements (iv) and (v) are shown analogously to (iii). □

The next result shows how to control the decomposition given by the K-functional.

Lemma 2.14 (Bramble-Scott, [14, Lemma])**.** *Let $u\in X_0$, $v\in X_1$ and $t>0$ be such that*

$$\|u-v\|_0+t\|v\|_1\leq 2K(t,u).$$

Then, for any (θ,q) such that $u\in(X_0,X_1)_{\theta,q}$, we have

$$\|u-v\|_{\theta,q}\leq 3\|u\|_{\theta,q}.$$

In particular, there holds $\|v\|_{\theta,q}\leq 4\|u\|_{\theta,q}$.

Proof. We reproduce the proof for completeness' sake. For $s\geq t$, we have $K(s,u-v)\leq\|u-v\|_0\leq 2K(t,u)\leq 3K(\min(s,t),u)$. Now assume $s<t$, then we have for arbitrary $w\in X_1$

$$K(s,u-v)\leq\|u-v-(w-v)\|_0+s\|w-v\|_1\leq\|u-w\|_0+s\|w\|_1+s\|v\|_1$$

and therefore

$$\begin{aligned}K(s,u-v)&\leq\inf_{w\in X_1}(\|u-w\|_0+s\|w\|_1+s\|v\|_1)=K(s,u)+s\|v\|_1\\&\leq K(s,u)+2\frac{s}{t}K(t,u).\end{aligned}$$

As $s<t$ and the mapping $t\mapsto t^{-1}K(t,u)$ is decreasing, we obtain $K(s,u-v)\leq 3K(s,u)=3K(\min(s,t),u)$, thus we have

$$K(s,u-v)\leq 3K(\min(s,t),u)\tag{2.13}$$

for all $s>0$. Since $K(\cdot,u)$ is increasing, (2.13) implies $K(s,u-v)\leq 3K(s,u)$ and hence $\|u-v\|_{\theta,q}\leq 3\|u\|_{\theta,q}$. □

2.2.2 Interpolation results in sector domains

In this subsection, we give examples how the interpolation results can be applied. Some of the presented results will also play a role in Chapter 3.
We define the cone $\mathcal{C}$ by

$$\mathcal{C} := \{(r\cos\phi, r\sin\phi) : r > 0, \phi \in G := (0,\omega)\}, \tag{2.14}$$

with the angle $\omega \in (0, 2\pi)$. For $R > 0$ we use the abbreviation

$$\mathcal{C}_R := \mathcal{C} \cap B_R(0).$$

Obviously, functions in $\mathcal{C}_R$ are often written in polar coordinates. See also the beginning of Subsection 3.1.2 for a more detailed discussion.
In the following lemma, we give examples of functions in Besov spaces. This result will be useful later on, since these functions appear as parts of the singularity functions.

Lemma 2.15. *For $r > 0$ and $\phi \in (0,\omega)$, the following statements hold.*

(i) Let $\alpha \in \mathbb{R}^+\backslash\mathbb{N}$. Then the function $s^+(r,\phi) = r^\alpha \Phi(r,\phi)$ restricted to $\mathcal{C}_1$, where Φ is a smooth and bounded function, is in the space $B^{1+\alpha}_{2,\infty}(\mathcal{C}_1)$.

(ii) Let $\alpha \in (0,1)$. Then the function $s_0^-(r,\phi) = r^{-\alpha}\Phi_0^-(r,\phi)$, where Φ_0^- is smooth and zero on $\partial\mathcal{C}_1$, is in the space $\widetilde{B}^{1-\alpha}_{2,\infty}(\mathcal{C}_1) := (L^2(\mathcal{C}_1), H^1_0(\mathcal{C}_1))_{1-\alpha,\infty}$.

(iii) Let $\alpha \in (0,1)$. Then the function $s^-(r,\phi) = r^{-\alpha}\Phi^-(r,\phi)$, where Φ^- is a smooth function, is in the space $B^{1-\alpha}_{2,\infty}(\mathcal{C}_1) = (L^2(\mathcal{C}_1), H^1(\mathcal{C}_1))_{1-\alpha,\infty}$.

Proof. We first prove (i). We write the Besov space as the interpolation space

$$B^{1+\alpha}_{2,\infty}(\mathcal{C}_1) = (L^2(\mathcal{C}_1), H^{\lfloor\alpha\rfloor+2}(\mathcal{C}_1))_{\frac{1+\alpha}{\lfloor\alpha\rfloor+2},\infty}.$$

Next we select a smooth cut-off function χ that is zero on $B_{t^{\frac{1}{\lfloor\alpha\rfloor+2}}/2}(0)$, equals one on $B_1(0)\backslash B_{t^{\frac{1}{\lfloor\alpha\rfloor+2}}}(0)$, and whose derivatives satisfy $\|D^k\chi\|_{L^\infty(\mathcal{C}_1)} \lesssim t^{-\frac{k}{\lfloor\alpha\rfloor+2}}$. We then get

$$\|(1-\chi)s^+\|^2_{L^2(\mathcal{C}_1)} \lesssim \int_0^1 (1-\chi)^2 r^{2\alpha} r\,dr \lesssim \int_0^{t^{\frac{1}{\lfloor\alpha\rfloor+2}}} r^{2\alpha+1}\,dr \lesssim t^{\frac{2(\alpha+1)}{\lfloor\alpha\rfloor+2}}. \tag{2.15}$$

For the derivatives we obtain

$$\begin{aligned}
\|D^{\lfloor\alpha\rfloor+2}(\chi s^+)\|^2_{L^2(\mathcal{C}_1)} &\lesssim \int_0^1 \sum_{s=0}^{\lfloor\alpha\rfloor+2} (D^s\chi(r))^2 (D^{\lfloor\alpha\rfloor+2-s} r^\alpha)^2 r\,dr \\
&\lesssim \int_{t^{\frac{1}{\lfloor\alpha\rfloor+2}}/2}^1 r^{2(\alpha-\lfloor\alpha\rfloor-2)+1}\,dr + \sum_{s=1}^{\lfloor\alpha\rfloor+2} t^{-\frac{2s}{\lfloor\alpha\rfloor+2}} \left(t^{\frac{1}{\lfloor\alpha\rfloor+2}}\right)^{2(\alpha+s-\lfloor\alpha\rfloor-2)+2} \\
&\lesssim t^{\frac{2\alpha-2\lfloor\alpha\rfloor-2}{\lfloor\alpha\rfloor+2}} + \sum_{s=1}^{\lfloor\alpha\rfloor+2} t^{-\frac{2s}{\lfloor\alpha\rfloor+2}} \left(t^{\frac{1}{\lfloor\alpha\rfloor+2}}\right)^{2(\alpha+s-\lfloor\alpha\rfloor-2)+2} \\
&\lesssim t^{\frac{2(\alpha+1)}{\lfloor\alpha\rfloor+2}-2}.
\end{aligned} \tag{2.16}$$

The L^2-norm satisfies

$$\|\chi s^+\|^2_{L^2(\mathcal{C}_1)} \lesssim \int_{t^{\frac{1}{\lfloor\alpha\rfloor+2}}/2}^{1} r^{2\alpha+1}\,dr \lesssim t^{\frac{2(\alpha+1)}{\lfloor\alpha\rfloor+2}-2}, \tag{2.17}$$

since the integral is bounded and $\frac{2(\alpha+1)}{\lfloor\alpha\rfloor+2} - 2 < 0$. Lines (2.15), (2.16) and (2.17) imply

$$K(t, s^+) \lesssim t^{\frac{1+\alpha}{\lfloor\alpha\rfloor+2}},$$

thus $s^+ \in B^{1+\alpha}_{2,\infty}(\mathcal{C}_1)$.
The proofs of (ii) and (iii) follow in a similar way. We only want to mention that the assumptions on Φ_0^- imply the necessary boundary conditions. □

The following lemma gives an example how to apply the lemma about duality. Additionally, the functional S in Lemma 2.16 represent the singularity functions that prevent full regularity of the solutions, thus this result will be used in the proof of the shift theorem.

Lemma 2.16. *Let $0 < \alpha < 1$ and let s^- and s_0^- be defined as in Lemma 2.15. Then it follows:*

(i) The mapping

$$f \mapsto S(f) := \int_{\mathcal{C}_1} s_0^- f$$

is bounded and linear on $B^{\alpha-1}_{2,1}(\mathcal{C}_1) = (H^{-1}(\mathcal{C}_1), L^2(\mathcal{C}_1))_{\alpha,1}$.

(ii) The mapping

$$f \mapsto S(f) := \int_{\mathcal{C}_1} s^- f$$

is bounded and linear on $(\widetilde{H}^{-1}(\mathcal{C}_1), L^2(\mathcal{C}_1))_{\alpha,1}$.

Proof. Since by Lemma 2.12

$$s_0^- \in (L^2(\mathcal{C}_1), H^1_0(\mathcal{C}_1))_{1-\alpha,\infty} = \left((H^{-1}(\mathcal{C}_1), L^2(\mathcal{C}_1))_{\alpha,1}\right)^\star$$

and

$$s^- \in (L^2(\mathcal{C}_1), H^1(\mathcal{C}_1))_{1-\alpha,\infty} = \left((\widetilde{H}^{-1}(\mathcal{C}_1), L^2(\mathcal{C}_1))_{\alpha,1}\right)^\star,$$

the result is clear. □

Spaces of functions that vanish to certain order at $r = 0$ can be captured by interpolation spaces.

Proposition 2.17. *Let $k \in \mathbb{N}$. For $\theta \in (0,1)$, $\theta k \notin \mathbb{N}$, it holds*

$$(L^2(\mathcal{C}_1), {}_0H^k(\mathcal{C}_1))_{\theta,2} = {}_0H^{k\theta}(\mathcal{C}_1),$$

where

$${}_0H^s(\mathcal{C}_1) := \{u \in H^s(\mathcal{C}_1) : D^j u(0) = 0, j = 0, \dots, \lceil s \rceil - 2\}.$$

Remark 2.18. The notation ${}_0H^s(\mathcal{C}_1)$ is motivated by [43, Sec. 1.11], where ${}_0H^1(\Omega)$ was introduced.

Proof (of Prop. 2.17). We prove this result in several steps.
Step 1: Let $u \in {}_0H^{k\theta}(\mathcal{C}_1) \subseteq H^{k\theta}(\mathcal{C}_1) = (L^2(\mathcal{C}_1), H^k(\mathcal{C}_1))_{\theta,2}$. Then for every t, there is a function $v_t \in H^k(\mathcal{C}_1)$ such that

$$\|u - v_t\|_{L^2(\mathcal{C}_1)} + t\|v_t\|_{H^k(\mathcal{C}_1)} \leq 2K(t,u). \tag{2.18}$$

Now define $\widetilde{v} := \chi_{\sqrt[k]{t}} v_t$, where $\chi_{\sqrt[k]{t}}$ denotes a smooth cut-off-function satisfying $\chi_{\sqrt[k]{t}}(x) = 0$ for $|x| < \frac{\sqrt[k]{t}}{2}$ and $\chi_{\sqrt[k]{t}}(x) = 1$ for $|x| > \sqrt[k]{t}$ with $\|D^j \chi_{\sqrt[k]{t}}\|_{L^\infty(\mathcal{C}_1)} \lesssim (\sqrt[k]{t})^{-j}$. Note $\widetilde{v} \in {}_0H^k(\mathcal{C}_1)$, thus it is now sufficient to show

$$\int_0^1 \left(t^{-\theta}\left(\|u - \widetilde{v}\|_{L^2(\mathcal{C}_1)} + t\|\widetilde{v}\|_{H^k(\mathcal{C}_1)}\right)\right)^2 \frac{dt}{t} < \infty, \tag{2.19}$$

since (2.19) is an upper bound for $\|u\|^2_{\theta,2}$, cf. Lemma 2.7.
Step 2: We show estimates for the first norm in (2.19). We write

$$\|u - \widetilde{v}\|_{L^2(\mathcal{C}_1)} \leq \|u - v_t\|_{L^2(\mathcal{C}_1)} + \|(1 - \chi_{\sqrt[k]{t}})v_t\|_{L^2(\mathcal{C}_1)} \leq 2K(t,u) + \|v_t\|_{L^2(\mathcal{C}_1 \cap B_{\sqrt[k]{t}}(0))}$$

and get for the integral

$$\begin{aligned}\int_0^1 t^{-2\theta}\|v_t\|^2_{L^2(\mathcal{C}_1 \cap B_{\sqrt[k]{t}}(0))}\frac{dt}{t} &\lesssim \int_0^1 t^{-2\theta}\|u - v_t\|^2_{L^2(\mathcal{C}_1)}\frac{dt}{t} + \int_0^1 t^{-2\theta}\|u\|^2_{L^2(\mathcal{C}_1 \cap B_{\sqrt[k]{t}}(0))}\frac{dt}{t} \\ &\lesssim \|u\|^2_{\theta,2} + \int_0^1 t^{-2\theta}\|u\|^2_{L^2(\mathcal{C}_1 \cap B_{\sqrt[k]{t}}(0))}\frac{dt}{t}.\end{aligned} \tag{2.20}$$

We obtain with Lemma 3.31

$$\begin{aligned}\int_0^1 t^{-2\theta}\|u\|^2_{L^2(\mathcal{C}_1 \cap B_{\sqrt[k]{t}}(0))}\frac{dt}{t} &= \int_0^1 t^{-2\theta-1}\int_0^\omega \int_0^{\sqrt[k]{t}} u(r,\phi)^2 r\,dr\,d\phi\,dt \\ &= \int_0^\omega \int_0^1 \int_{r^k}^1 t^{-2\theta-1} u(r,\phi)^2 r\,dt\,dr\,d\phi = \int_0^\omega \frac{1}{2\theta}\int_0^1 (r^{-2k\theta} - 1)u(r,\phi)^2 r\,dr\,d\phi \\ &\lesssim \|r^{-k\theta}u\|^2_{L^2(\mathcal{C}_1)} \lesssim \|u\|^2_{H^{k\theta}(\mathcal{C}_1)}.\end{aligned}$$

Step 3: We show estimates for the second norm in (2.19). We have

$$t^2\|\widetilde{v}\|^2_{H^k(\mathcal{C}_1)} = t^2\sum_{j=0}^{k}|\widetilde{v}|^2_{H^j(\mathcal{C}_1)} \lesssim \sum_{j=0}^{k-1} t^{\frac{2j}{k}}|v_t|^2_{H^j(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))} + t^2\|v_t\|^2_{H^k(\mathcal{C}_1)}$$
$$\lesssim \sum_{j=0}^{k-1} t^{\frac{2j}{k}}|v_t|^2_{H^j(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))} + K(t,u)^2.$$

For each $0 < j < k$, scaling, with the scaled function $\widehat{v}_t(\widehat{x}) := v_t(\sqrt[k]{t}\widehat{x})$, the Gagliardo-Nirenberg inequality and Young's inequality give

$$\begin{aligned}
t^{\frac{2j}{k}}|v_t|^2_{H^j(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))} &\lesssim t^{\frac{2j}{k}}\left((t^{\frac{1}{k}})^{1-j}\right)^2|\widehat{v}_t|^2_{H^j(\mathcal{C}_1\cap B_1(0))}\\
&\lesssim t^{\frac{2}{k}}\left(|\widehat{v}_t|^{\frac{2j}{k}}_{H^k(\mathcal{C}_1\cap B_1(0))}\|\widehat{v}_t\|^{2(1-\frac{j}{k})}_{L^2(\mathcal{C}_1\cap B_1(0))} + \|\widehat{v}_t\|^2_{L^2(\mathcal{C}_1\cap B_1(0))}\right)\\
&\lesssim t^{\frac{2}{k}}\Bigg(\left((t^{\frac{1}{k}})^{-1+k}\right)^{\frac{2j}{k}}|v_t|^{\frac{2j}{k}}_{H^k(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))}\left((t^{\frac{1}{k}})^{-1}\right)^{2(1-\frac{j}{k})}\|v_t\|^{2(1-\frac{j}{k})}_{L^2(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))}\\
&\qquad + t^{-\frac{2}{k}}\|v_t\|^2_{L^2(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))}\Bigg)\\
&= \left(t^{\frac{1}{k}}\right)^{2j}\left(|v_t|^{\frac{2j}{k}}_{H^k(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))}\|v_t\|^{2(1-\frac{j}{k})}_{L^2(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))}\right) + \|v_t\|^2_{L^2(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))}\\
&\lesssim t^2|v_t|^2_{H^k(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))} + \|v_t\|^2_{L^2(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))}.
\end{aligned}$$

Thus we get

$$\int_0^1 t^{-2\theta}t^2\|\widetilde{v}\|^2_{H^k(\mathcal{C}_1)}\frac{dt}{t} \lesssim \|u\|^2_{\theta,2} + \int_0^1 t^{-2\theta}\|v_t\|^2_{L^2(\mathcal{C}_1\cap B_{\sqrt[k]{t}}(0))}\frac{dt}{t}, \tag{2.21}$$

where the remaining integral in (2.21) has already been handled in (2.20).
Step 4: We combine the results from the first three steps and obtain

$$\int_0^1\left(t^{-\theta}\left(\|u-\widetilde{v}\|_{L^2(\mathcal{C}_1)} + t\|\widetilde{v}\|_{H^k(\mathcal{C}_1)}\right)\right)^2\frac{dt}{t} \lesssim \|u\|^2_{H^{k\theta}(\mathcal{C}_1)} < \infty,$$

which shows the continuous embedding ${}_0H^{k\theta}(\mathcal{C}_1) \subseteq (L^2(\mathcal{C}_1), {}_0H^k(\mathcal{C}_1))_{\theta,2}$.
Step 5: To see the inclusion $(L^2(\mathcal{C}_1), {}_0H^k(\mathcal{C}_1))_{\theta,2} \subseteq {}_0H^{k\theta}(\mathcal{C}_1)$, let $u \in (L^2(\mathcal{C}_1), {}_0H^k(\mathcal{C}_1))_{\theta,2}$. It follows $u \in (L^2(\mathcal{C}_1), H^k(\mathcal{C}_1))_{\theta,2}$ and thus immediately $u \in H^{k\theta}(\mathcal{C}_1)$, so we must only show $D^ju(0) = 0$ for all $j = 0, \ldots, \lceil k\theta\rceil - 2$. In order to do this, we mention that for all $t > 0$ there exists $v_t \in {}_0H^k(\mathcal{C}_1)$ with the properties

(i) $\|u - v_t\|_{L^2(\mathcal{C}_1)} + t\|v_t\|_{H^k(\mathcal{C}_1)} \lesssim K(t,u)$,

(ii) $D^jv_t(0) = 0$ for all $j = 0, \ldots, k-2$,

(iii) $\|u - v_t\|_{H^{k\theta}(\mathcal{C}_1)} \leq 3\|u\|_{H^{k\theta}(\mathcal{C}_1)}$,

(iv) $K(t,u) \lesssim t^\theta |u|_{H^{k\theta}(\mathcal{C}_1)}$,

cf. Lemma 2.14 for (iii). For every $j = 0,\dots,\lceil k\theta\rceil - 2$, choose $\epsilon_j > 0$ such that $j+1 < j+1+\epsilon_j < k\theta$. Then we obtain via interpolation inequality

$$\begin{aligned}\|D^j u - D^j v_t\|_{H^{1+\epsilon_j}(\mathcal{C})} &\lesssim \|u - v_t\|_{H^{j+1+\epsilon_j}(\mathcal{C})} \lesssim \|u - v_t\|_{L^2(\mathcal{C})}^{1-\frac{j+1+\epsilon_j}{k\theta}} \|u - v_t\|_{H^{k\theta}(\mathcal{C})}^{\frac{j+1+\epsilon_j}{k\theta}} \\ &\overset{(i),(iii)}{\lesssim} K(t,u)^{1-\frac{j+1+\epsilon_j}{k\theta}} (3\|u\|_{H^{k\theta}(\mathcal{C})})^{\frac{j+1+\epsilon_j}{k\theta}} \overset{(iv)}{\lesssim} \left(t^\theta \|u\|_{H^{k\theta}(\mathcal{C})}\right)^{1-\frac{j+1+\epsilon_j}{k\theta}} \|u\|_{H^{k\theta}(\mathcal{C})}^{\frac{j+1+\epsilon_j}{k\theta}}\end{aligned}$$

for all $j = 0,\dots,\lceil k\theta\rceil - 2$, where the hidden constants are independent of t. Thus we get by a Sobolev embedding theorem

$$\|D^j u - D^j v_t\|_{L^\infty(\mathcal{C})} \lesssim \|D^j u - D^j v_t\|_{H^{1+\epsilon_j}(\mathcal{C})} \overset{t\to 0}{\longrightarrow} 0,$$

which implies $D^j u(0) = 0$ for all $j = 0,\dots,\lceil k\theta\rceil - 2$ with property (ii). This shows $(L^2(\mathcal{C}), {}_0H^k(\mathcal{C}))_{\theta,2} = {}_0H^{k\theta}(\mathcal{C})$. □

2.3 Vector-valued function spaces and the deRham-diagram

Aside from the usual gradient operator, the other well-known differential operators curl and divergence appear in many partial differential equations, e.g. Maxwell's equations. The natural functions spaces for a variational formulation containing these types of derivatives are $\mathbf{H}(\Omega, \mathbf{curl})$ and $\mathbf{H}(\Omega, \mathrm{div})$. After a short introduction of these spaces, as well as the more regular subspaces $\mathbf{H}^s(\Omega, \mathbf{curl})$ and $\mathbf{H}^s(\Omega, \mathrm{div})$, we consider how they are connected by differential operators, leading to the formulation of the deRham-diagram which will play an important role in our analysis of the interpolation operators. Most definitions and results in this section are standard, but can be found e.g. in [51, Chapter 3], [62, Chapter 3].
We start this section with stating the differential operators and the vector-valued function spaces. The gradient operator of a scalar function u is defined as

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)^T$$

and the divergence operator of a vector-valued function $\mathbf{u}$ as

$$\mathrm{div}\,\mathbf{u} := \sum_{j=1}^n \frac{\partial \mathbf{u}_i}{\partial x_n}.$$

For $n = 3$, the curl operator of a function $\mathbf{u}$ is defined as

$$\mathbf{curl}\,\mathbf{u} := \left(\frac{\partial \mathbf{u}_3}{\partial x_2} - \frac{\partial \mathbf{u}_2}{\partial x_3}, -\left(\frac{\partial \mathbf{u}_3}{\partial x_1} - \frac{\partial \mathbf{u}_1}{\partial x_3}\right), \frac{\partial \mathbf{u}_2}{\partial x_1} - \frac{\partial \mathbf{u}_1}{\partial x_2}\right)^T,$$

whereas for $n = 2$ we must distinguish between the scalar-valued curl operator of a function $\mathbf{u}$

$$\mathrm{curl}\,\mathbf{u} := \frac{\partial \mathbf{u}_2}{\partial x_1} - \frac{\partial \mathbf{u}_1}{\partial x_2}$$

and the vector-valued curl operator for a function u

$$\mathbf{curl}\, u := \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}\right)^T.$$

Similarly to (2.1), we can define weak versions of differential operators for functions not regular enough to admit derivatives in the classical sense.

Definition 2.19. *Let $u, \mathbf{u} \in L^2(\Omega)$.*

(i) We call $\mathbf{g} = \nabla u$ the (generalized) gradient of u if there holds

$$\int_\Omega \mathbf{g} \cdot \mathbf{v} = -\int_\Omega u \operatorname{div} \mathbf{v} \quad \forall \mathbf{v} \in C_0^\infty(\overline{\Omega}).$$

(ii) We call $\mathbf{c} = \mathbf{curl}\, \mathbf{u}$ the (generalized) curl of $\mathbf{u}$ if there holds

$$\int_\Omega \mathbf{c} \cdot \mathbf{v} = \int_\Omega \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} \quad \forall \mathbf{v} \in C_0^\infty(\overline{\Omega}).$$

(iii) We call $d = \operatorname{div} \mathbf{u}$ the (generalized) divergence of $\mathbf{u}$ if there holds

$$\int_\Omega d v = -\int_\Omega \mathbf{u} \cdot \nabla v \quad \forall v \in C_0^\infty(\overline{\Omega}).$$

The vector-valued function spaces corresponding to the various differential operators are the following. Let $s \geq 0$. The vector-valued Sobolev space $\mathbf{H}^s(\Omega)$ is defined to be $H^s(\Omega)$ componentwise, and the negative norm $\|\cdot\|_{\widetilde{\mathbf{H}}^{-s}(\Omega)}$ is defined analogously to (2.7). For $n = 3$ we set

$$\mathbf{H}^s(\Omega, \mathbf{curl}) := \{\mathbf{u} \in \mathbf{H}^s(\Omega) : \mathbf{curl}\, \mathbf{u} \in \mathbf{H}^s(\Omega)\}$$

and analogously

$$\mathbf{H}^s(\Omega, \operatorname{div}) := \{\mathbf{u} \in \mathbf{H}^s(\Omega) : \operatorname{div} \mathbf{u} \in H^s(\Omega)\}.$$

The corresponding norms are $\|\cdot\|^2_{\mathbf{H}^s(\Omega,\mathbf{curl})} := \|\cdot\|^2_{\mathbf{H}^s(\Omega)} + \|\,\mathbf{curl}\cdot\|^2_{\mathbf{H}^s(\Omega)}$ and $\|\cdot\|^2_{\mathbf{H}^s(\Omega,\operatorname{div})} := \|\cdot\|^2_{\mathbf{H}^s(\Omega)} + \|\operatorname{div}\cdot\|^2_{H^s(\Omega)}$, respectively. For $n = 2$, we have

$$\mathbf{H}^s(\Omega, \operatorname{curl}) := \{\mathbf{u} \in \mathbf{H}^s(\Omega) : \operatorname{curl} \mathbf{u} \in H^s(\Omega)\}$$

with norm $\|\cdot\|^2_{\mathbf{H}^s(\Omega,\operatorname{curl})} := \|\cdot\|^2_{\mathbf{H}^s(\Omega)} + \|\operatorname{curl}\cdot\|^2_{H^s(\Omega)}$. If $s = 0$, we simplify the notation by omitting the superscript s, which gives the spaces $\mathbf{H}(\Omega, \mathbf{curl})$, $\mathbf{H}(\Omega, \operatorname{div})$ and $\mathbf{H}(\Omega, \operatorname{curl})$. The tilde norms for these spaces are defined as

$$\|\cdot\|^2_{\widetilde{\mathbf{H}}^{-s}(\Omega,\mathbf{curl})} := \|\cdot\|^2_{\widetilde{\mathbf{H}}^{-s}(\Omega)} + \|\,\mathbf{curl}\cdot\|^2_{\widetilde{\mathbf{H}}^{-s}(\Omega)}$$

and analogously $\|\cdot\|_{\widetilde{\mathbf{H}}^{-s}(\Omega,\operatorname{div})}$ and $\|\cdot\|_{\widetilde{\mathbf{H}}^{-s}(\Omega,\operatorname{curl})}$.

For $\mathbf{u} \in C_0^\infty(\overline{\Omega})$, we denote the tangential component and tangential trace as $\Pi_\tau \mathbf{u} := \mathbf{n} \times (\mathbf{u}|_\Gamma \times \mathbf{n})$ and $\gamma_\tau := \mathbf{u}|_\Gamma \times \mathbf{n}$, where $\mathbf{n}$ denotes the outer normal vector of Ω. The operators Π_τ and γ_τ can then be extended to continuous and linear maps on $\mathbf{H}(\Omega, \mathbf{curl})$, cf. [62, Th. 3.6]. The analogous statement holds for the normal trace $\gamma_n \mathbf{u} := \mathbf{u} \cdot \mathbf{n}$ which can be extended to an operator on $\mathbf{H}(\Omega, \operatorname{div})$ ([62, Th. 3.9]).
The spaces $\mathbf{H}_0(\Omega, \mathbf{curl})$ and $\mathbf{H}_0(\Omega, \operatorname{div})$ are the subspaces of $\mathbf{H}(\Omega, \mathbf{curl})$ and $\mathbf{H}(\Omega, \operatorname{div})$ with vanishing tangential or normal trace, i.e.

$$\mathbf{H}_0(\Omega, \mathbf{curl}) := \{\mathbf{u} \in \mathbf{H}(\Omega, \mathbf{curl}) : \Pi_\tau \mathbf{u} = 0\}$$

and

$$\mathbf{H}_0(\Omega, \operatorname{div}) := \{\mathbf{u} \in \mathbf{H}(\Omega, \operatorname{div}) : \gamma_n \mathbf{u} = 0\},$$

We note that these spaces coincide with $\overline{C_0^\infty(\overline{\Omega})}^{\|\cdot\|_{\mathbf{H}(\Omega,\mathbf{curl})}}$ and $\overline{C_0^\infty(\overline{\Omega})}^{\|\cdot\|_{\mathbf{H}(\Omega,\operatorname{div})}}$, respectively, cf. [51, Thm. 3.25, Thm. 3.33].
If Ω is a convex domain, we have the continuous embeddings

$$\mathbf{H}_0(\Omega, \mathbf{curl}) \cap \mathbf{H}(\Omega, \operatorname{div}) \subset \mathbf{H}^1(\Omega) \tag{2.22}$$

and

$$\mathbf{H}(\Omega, \mathbf{curl}) \cap \mathbf{H}_0(\Omega, \operatorname{div}) \subset \mathbf{H}^1(\Omega), \tag{2.23}$$

cf. [11, 56], [3, Thm. 2.17], see also [51, Rem. 3.48]). If Ω is not convex, but a bounded Lipschitz polyhedron, the given function may not be as smooth as $\mathbf{H}^1(\Omega)$, but there is an $s \in (0, 1/2]$ such that (2.22) and (2.23) hold with $\mathbf{H}^s(\Omega)$ on the right-hand side, cf. [3, Prop. 3.7] or [51, Thm. 3.50].
We have the integration by parts formula (Gauss)

$$\int_\Omega \operatorname{div} \mathbf{u}\varphi \, d\mathbf{x} = -\int_\Omega \mathbf{u} \cdot \nabla\varphi \, d\mathbf{x} + \int_{\partial\Omega} \varphi \mathbf{u} \cdot \mathbf{n} \, ds(\mathbf{x}) \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega), \varphi \in H^1(\Omega)$$

which extends to $\mathbf{u} \in \mathbf{H}(\Omega, \operatorname{div})$. In this case, the boundary integral is replaced by a duality pairing, cf. [51, Th. 3.24]. Concerning the curl operator, we have the Stokes formula

$$\int_\Omega \mathbf{v} \cdot \mathbf{curl}\,\mathbf{u} \, d\mathbf{x} = \int_\Omega \mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} \, d\mathbf{x} - \int_{\partial\Omega} \Pi_\tau \mathbf{u} \cdot \gamma_\tau \mathbf{v} \, ds(\mathbf{x}) \qquad \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega) \tag{2.24}$$

which extends to $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\Omega, \mathbf{curl})$, cf. [51, Th. 3.29]. In two dimensions, the Stokes formula has the form

$$\int_\Omega \mathbf{u} \cdot \mathbf{curl}\, v \, d\mathbf{x} = \int_\Omega v \operatorname{curl} \mathbf{u} \, d\mathbf{x} - \int_{\partial\Omega} v \mathbf{u} \cdot \mathbf{t} \, ds(\mathbf{x}) \qquad \mathbf{u} \in \mathbf{H}^1(\Omega), v \in H^1(\Omega), \tag{2.25}$$

where the boundary (and therefore the tangential vectors $\mathbf{t}$) is oriented in the mathematically positive direction.
In the next part, we state the relations between the differential operators and the introduced function spaces. For the rest of the section, we need the following assumption.

Assumption 2.20. *Let the bounded Lipschitz domain Ω additionally be simply connected with only one boundary component.*

We now state the results about the existence of scalar (and vector) potentials in the case of curl-free (and divergence-free) vector fields.

Proposition 2.21 ([51, Th. 3.37])**.** *Let $\mathbf{u} \in L^2(\Omega)$. Then $\mathbf{curl}\,\mathbf{u} = 0$ in Ω if and only if there is a scalar potential $\phi \in H^1(\Omega)$ such that $\mathbf{u} = \nabla\phi$. The potential ϕ is unique up to an additive constant.*

An immediate consequence of Proposition 2.21 is the identity

$$\nabla H^1(\Omega) = \{\mathbf{v} \in \mathbf{H}(\Omega, \mathbf{curl}) : \mathbf{curl}\,\mathbf{v} = 0\}. \tag{2.26}$$

Proposition 2.22 ([36, Th. 3.4])**.** *Let $\mathbf{u} \in L^2(\Omega)$. Then $\operatorname{div}\mathbf{u} = 0$ if and only if*[3] *there is a vector potential $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{u} = \mathbf{curl}\,\mathbf{v}$.*

Since the well-known identity $\operatorname{div}\mathbf{curl}\,\mathbf{u} = 0$ holds, Proposition 2.22 implies

$$\mathbf{curl}\,\mathbf{H}(\Omega, \mathbf{curl}) = \{\mathbf{v} \in \mathbf{H}(\Omega, \operatorname{div}) : \operatorname{div}\mathbf{v} = 0\}. \tag{2.27}$$

For $f \in L^2(\Omega)$, we solve the Dirichlet problem

$$-(\nabla\psi, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

and get $\psi \in H_0^1(\Omega)$ as solution. Since $\mathbf{u} := \nabla\psi \in L^2(\Omega)$ satisfies $\operatorname{div}\mathbf{u} = f$, there holds $\mathbf{u} \in \mathbf{H}(\Omega, \operatorname{div})$, and we get

$$\operatorname{div}\mathbf{H}(\Omega, \operatorname{div}) = L^2(\Omega), \tag{2.28}$$

cf. [62, Lemma 3.15].
The relations between the function spaces as stated in (2.26), (2.27) and (2.28) can be presented in the deRham-diagram, which forms an exact sequence. That means that the range of each operator coincides with the kernel of the following operator. In three spatial dimensions, we have

$$\mathbb{R} \xrightarrow{\text{id}} H^1(\Omega) \xrightarrow{\nabla} \mathbf{H}(\Omega, \mathbf{curl}) \xrightarrow{\mathbf{curl}} \mathbf{H}(\Omega, \operatorname{div}) \xrightarrow{\operatorname{div}} L^2(\Omega) \xrightarrow{0} \{0\}, \tag{2.29}$$

cf. e.g. [51, eq. (3.59)]. If we also take boundary conditions into account, we obtain the sequence ([51, eq. (3.60)])

$$\{0\} \xrightarrow{\text{id}} H_0^1(\Omega) \xrightarrow{\nabla} \mathbf{H}_0(\Omega, \mathbf{curl}) \xrightarrow{\mathbf{curl}} \mathbf{H}_0(\Omega, \operatorname{div}) \xrightarrow{\operatorname{div}} L^2(\Omega)/\mathbb{R} \xrightarrow{0} \{0\}. \tag{2.30}$$

Similar sequences hold for functions with more regularity. For $s > 1$

$$\mathbb{R} \xrightarrow{\text{id}} H^s(\Omega) \xrightarrow{\nabla} \mathbf{H}^{s-1}(\Omega, \mathbf{curl}) \xrightarrow{\mathbf{curl}} \mathbf{H}^{s-1}(\Omega, \operatorname{div}) \xrightarrow{\operatorname{div}} H^{s-1}(\Omega) \xrightarrow{0} \{0\}.$$

[3] Note that for domains with more than one single boundary part Γ_i, we must additionally assume $\int_{\Gamma_i} \mathbf{u}\cdot\mathbf{n} = 0$ on each part in order to find such a vector potential.

In two dimensions, the situation is similar, however, there are two possible ways to build exact sequences due to the existence of two different curl-operators. The sequences here are

$$\mathbb{R} \xrightarrow{\text{id}} H^s(\Omega) \xrightarrow{\nabla} \mathbf{H}^{s-1}(\Omega, \text{curl}) \xrightarrow{\text{curl}} H^{s-1}(\Omega) \xrightarrow{0} \{0\}$$

and

$$\mathbb{R} \xrightarrow{\text{id}} H^s(\Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}^{s-1}(\Omega, \text{div}) \xrightarrow{\text{div}} H^{s-1}(\Omega) \xrightarrow{0} \{0\}.$$

We mention that we will only use the first one for the construction of the interpolation operators. The presented exact sequences are well-known, see e.g. [19] for a reference.
If Assumption 2.20 is violated, the sequences need not be exact. In this case the range of each operator is still contained in the kernel of the subsequent operator, but range and kernel do not coincide in general. The reason lies in the fact that there may be functions in the kernel which cannot be represented as gradient (or curl) of another function, cf. [38, Remark 5 and previous pages].

2.4 Finite element discretization

In this section, we recall the basics of the finite element method, especially for the p-version, since we construct p-version projection-based interpolation operators in Chapter 4. Although FE triangulation can have many different types of elements (triangles, squares in 2D; tetrahedra, hexahedra, wedges, pyramids in 3D), we will concentrate only on triangles and tetrahedra. For further reference to the finite element method, see e.g. [16], [51, Chapter 5 ff.], or [57] especially for the p-version.

2.4.1 Spaces on the reference element

Denote by $\widehat{K}$ the reference tetrahedron in 3D which is a fixed tetrahedron with diam $\widehat{K} = 1$. In 2D, we denote the reference triangle by $\widehat{f}$ which is a fixed triangle with diam $\widehat{f} = 1$. The reference element $\widehat{e}$ in 1D is the interval $\widehat{e} := (-1, 1)$.
We next introduce the p-version finite element spaces in the classical sense, see e.g. [38, 51, 54]. For the n-dimensional manifold $\nu \in \{\widehat{K}, \widehat{f}, \mathbb{R}^3, \mathbb{R}^2, \mathbb{R}\}$, we denote by $\mathcal{P}_p(\nu)$ the space of polynomials of degree p in n variables on ν. If we need spaces of polynomials on faces $f \in \mathcal{F}(\widehat{K})$ or on edges $e \in \mathcal{E}(\widehat{K})$, the spaces $\mathcal{P}_p(f)$ and $\mathcal{P}_p(e)$ are defined by identifying f or e with the reference elements $\widehat{f}$ or $\widehat{e}$ via an affine map.
On $\widehat{K}$, we define the discrete spaces of degree $p \geq 0$ by

$$W_p(\widehat{K}) := \mathcal{P}_p(\widehat{K}) := \text{span}\{x^\alpha : |\alpha| \leq p\}, \tag{2.31a}$$

$$\mathbf{Q}_p(\widehat{K}) := \mathcal{N}_p^I(\widehat{K}) := \{\mathbf{p}(\mathbf{x}) + \mathbf{x} \times \mathbf{q}(\mathbf{x}) : \mathbf{p}, \mathbf{q} \in (\mathcal{P}_p(\widehat{K}))^3\}, \tag{2.31b}$$

$$\mathbf{V}_p(\widehat{K}) := \mathbf{RT}_p(\widehat{K}) := \{\mathbf{p}(\mathbf{x}) + q(\mathbf{x})\mathbf{x} : \mathbf{p} \in (\mathcal{P}_p(\widehat{K}))^3, q \in \mathcal{P}_p(\widehat{K})\}. \tag{2.31c}$$

The spaces $\mathcal{N}_p^I(\widehat{K})$ and $\mathbf{RT}_p(\widehat{K})$ are the classical Nédélec type I and Raviart-Thomas elements.

The spaces in (2.31) are defined in such way that they satisfy an exact sequence property similar to the continuous case (2.29), namely

$$\mathbb{R} \xrightarrow{\text{id}} W_{p+1}(\widehat{K}) \xrightarrow{\nabla} \mathbf{Q}_p(\widehat{K}) \xrightarrow{\mathbf{curl}} \mathbf{V}_p(\widehat{K}) \xrightarrow{\text{div}} W_p(\widehat{K}) \xrightarrow{0} \{0\}, \tag{2.32}$$

see [26, (57)].
In 2D, the Nédélec type I elements on the reference triangle $\widehat{f}$ are defined by

$$\mathbf{Q}_p(\widehat{f}) := \mathcal{N}_p^I(\widehat{f}) := \{\mathbf{p}(\mathbf{x}) + q(\mathbf{x})(y,-x)^T : \mathbf{p} \in (\mathcal{P}_p(\widehat{f}))^2, q \in \widetilde{\mathcal{P}}_p(\widehat{f})\},$$

where $\widetilde{\mathcal{P}}_p(\widehat{f})$ denotes the homogeneous polynomials of degree p on $\widehat{f}$. Here we look at a shortened sequence of the form

$$\mathbb{R} \xrightarrow{\text{id}} W_{p+1}(\widehat{f}) \xrightarrow{\nabla} \mathbf{Q}_p(\widehat{f}) \xrightarrow{\text{curl}} W_p(\widehat{f}) \xrightarrow{0} \{0\}.$$

Since the interpolation operators will be built up by dimension, we need the trace spaces of the discrete spaces on various parts of the boundary. The trace spaces on faces $f \in \mathcal{F}(\widehat{K})$ are defined by

$$W_p(f) := W_p(\widehat{K})|_f, \quad \mathbf{Q}_p(f) := (\Pi_\tau \mathbf{Q}_p(\widehat{K}))|_f, \quad V_p(f) := \mathbf{V}_p(\widehat{K}) \cdot \mathbf{n}_f,$$

where $\mathbf{n}_f$ denotes the outer normal vector of f, which is fixed by coinciding with the outer normal vector of $\widehat{K}$ on f and thus also determines the orientation of f. These trace spaces can be identified with other already known spaces. After identifying the face f with the reference triangle $\widehat{f}$ using an affine bijection, the spaces $W_p(f)$ and $V_p(f)$ are both isomorphic to the polynomial space $P_p(\mathbb{R}^2)$, and the space $\mathbf{Q}_p(f)$ to $\mathbf{Q}_p(\widehat{f})$. On edges $e \in \mathcal{E}(\widehat{K})$, we set

$$W_p(e) := W_p(\widehat{K})|_e, \quad Q_p(e) := \mathbf{Q}_p(\widehat{K}) \cdot \mathbf{t}_e,$$

where $\mathbf{t}_e$ denotes the tangential vector of the edge e whose orientation is determined by the orientation of the face f (that must obviously satisfy $e \in \mathcal{E}(f)$). We proceed analogously to the faces above and identify e with $\widehat{e}$, which gives the identification of the spaces $W_p(e)$ and $Q_p(e)$ with the polynomial space $\mathcal{P}_p(\mathbb{R})$.

2.4.2 Discrete spaces with boundary conditions

We will need subspaces with vanishing boundary values also in the discrete setting. To this end we set

$$\mathring{W}_p(\widehat{K}) := W_p(\widehat{K}) \cap H_0^1(\widehat{K}), \qquad \mathring{\mathbf{Q}}_p(\widehat{K}) := \{\mathbf{u} \in \mathbf{Q}_p(\widehat{K}) : \Pi_\tau \mathbf{u} = 0\},$$
$$\mathring{\mathbf{V}}_p(\widehat{K}) := \{\mathbf{u} \in \mathbf{V}_p(\widehat{K}) : \gamma_n \mathbf{u} = 0\}, \qquad W_p^{aver}(\widehat{K}) := \{u \in W_p(\widehat{K}) : \int_{\widehat{K}} u = 0\}.$$

On the faces, the spaces look

$$\mathring{W}_p(f) := W_p(f) \cap H_0^1(f), \quad \mathring{\mathbf{Q}}_p(f) := \{\mathbf{u} \in \mathbf{Q}_p(f) : \Pi_{\tau,f} \mathbf{u} = 0\},$$
$$\mathring{V}_p(f) := \{u \in \mathbf{V}_p(f) : \int_f u = 0\}$$

for $f \in \mathcal{F}(\widehat{K})$, where $\Pi_{\mathcal{T},f}$ denotes the tangential trace on the boundary of f and is defined by $(\Pi_{\mathcal{T},f}\mathbf{u})|_e = \mathbf{u}|_e \cdot \mathbf{t}_e$ for each edge $e \in \mathbf{E}(f)$ and sufficiently regular functions $\mathbf{u}$. Lowering the dimension again, we get the trace spaces on the edges $e \in \mathcal{E}(\widehat{K})$

$$\mathring{W}_p(e) := W_p(e) \cap H_0^1(e), \quad \mathring{Q}_p(e) := \{u \in Q_p(e) : \int_e u = 0\}.$$

The exact sequences for the spaces with vanishing boundary values turn out to be

$$\begin{array}{ccccccccccc}
\{0\} & \xrightarrow{\mathrm{Id}} & \mathring{W}_{p+1}(\widehat{K}) & \xrightarrow{\nabla} & \mathring{\mathbf{Q}}_p(\widehat{K}) & \xrightarrow{\mathbf{curl}} & \mathring{\mathbf{V}}_p(\widehat{K}) & \xrightarrow{\mathrm{div}} & W_p^{aver}(\widehat{K}) & \xrightarrow{0} & \{0\} \\
\{0\} & \xrightarrow{\mathrm{Id}} & \mathring{W}_{p+1}(f) & \xrightarrow{\nabla_f} & \mathring{\mathbf{Q}}_p(f) & \xrightarrow{\mathrm{curl}_f} & \mathring{V}_p(f) & \xrightarrow{0} & \{0\} & & \\
\{0\} & \xrightarrow{\mathrm{Id}} & \mathring{W}_{p+1}(e) & \xrightarrow{\nabla_e} & \mathring{Q}_p(e) & \xrightarrow{0} & \{0\} & & & &
\end{array} \tag{2.33}$$

where ∇_f and ∇_e are the surface gradient on the face f and tangential differentiation along the edge e, respectively, and where curl_f denotes the surface curl on f, see [39, (4.16)]. Note that these differential operators are defined by

$$\nabla_e u = (\nabla u)|_e \cdot \mathbf{t}_e, \qquad \nabla_f u = (\nabla u)|_f - \mathbf{n}_f((\nabla u)|_f \cdot \mathbf{n}_f), \qquad \mathrm{curl}_f\,\mathbf{u} = (\mathbf{curl}\,\mathbf{u})|_f \cdot \mathbf{n}_f.$$

If we start in two spatial dimensions, we define on the reference triangle f

$$\mathring{W}_p(\widehat{f}) := W_p(\widehat{f}) \cap H_0^1(\widehat{f}), \quad \mathring{\mathbf{Q}}_p(\widehat{f}) := \{\mathbf{u} \in \mathbf{Q}_p(\widehat{f}) \,|\, \mathbf{u} \cdot \mathbf{t}_e = 0\, \forall e \in \mathcal{E}(\widehat{f})\}$$

and get the sequences from [26, (33)]

$$\begin{array}{ccccccccc}
\{0\} & \xrightarrow{\quad\mathrm{Id}\quad} & \mathring{W}_{p+1}(\widehat{f}) & \xrightarrow{\quad\nabla\quad} & \mathring{\mathbf{Q}}_p(\widehat{f}) & \xrightarrow{\quad\mathrm{curl}\quad} & \mathring{V}_p(\widehat{f}) & \xrightarrow{\quad 0\quad} & \{0\} \\
\{0\} & \xrightarrow{\quad\mathrm{Id}\quad} & \mathring{W}_{p+1}(e) & \xrightarrow{\quad\nabla_e\quad} & \mathring{Q}_p(e) & \xrightarrow{\quad 0\quad} & \{0\} & &
\end{array}. \tag{2.34}$$

2.4.3 Finite element spaces on meshes

In this subsection we assume that Ω is a bounded polygonal (in 2D) or polyhedral (in 3D) domain with Lipschitz boundary. We now introduce triangulations and FE-spaces for the global spaces $H^1(\Omega)$, $\mathbf{H}(\Omega, \mathbf{curl})$ and $\mathbf{H}(\Omega, \mathrm{div})$.
We use a regular, shape-regular triangulation $\mathcal{T}$ of Ω to define the discrete spaces. In 3D, such a triangulation is assumed to satisfy the following properties:

(i) The open elements $K \in \mathcal{T}$ are tetrahedra and cover Ω. That is $\overline{\Omega} = \bigcup_{K \in \mathcal{T}} \overline{K}$. Their intersection is either empty, a vertex, an edge, a face or they coincide. The sets of vertices, edges and faces of K are denoted $\mathcal{V}(K)$, $\mathcal{E}(K)$ and $\mathcal{F}(K)$, respectively.

(ii) Denote by $\widehat{K}$ the reference tetrahedron which is a fixed tetrahedron with $\mathrm{diam}\,\widehat{K} = 1$. Associated with each element is a C^1-diffeomorphism $F_K : \overline{\widehat{K}} \to \overline{K}$, called the element map.

(iii) There holds, with some shape-regularity constant γ,

$$(\mathrm{diam}\,K)^{-1}\|F_K'\|_{L^\infty(\widehat{K})} + (\mathrm{diam}\,K)\|(F_K')^{-1}\|_{L^\infty(\widehat{K})} \le \gamma.$$

(iv) The parametrization of edges and faces is compatible, i.e. if two elements $K, K' \in \mathcal{T}$ share an edge ($F_K(e) = F_{K'}(e')$ for $e, e' \in \mathcal{E}(\widehat{K})$) or a face ($F_K(f) = F_{K'}(f')$ for $f, f' \in \mathcal{F}(\widehat{K})$), then $F_K^{-1} \circ F_{K'} : f' \to f$ is an affine isomorphism.

The global finite element spaces on Ω are defined with the aid of the Piola transform. Note that we use the covariant transform for $\mathbf{H}(\Omega, \mathbf{curl})$ functions and the contravariant version for the space $\mathbf{H}(\Omega, \mathrm{div})$. We set

$$W_p(\mathcal{T}) := \{u \in H^1(\Omega) : u|_K \circ F_K \in \mathcal{P}_p(\widehat{K})\}, \tag{2.35a}$$

$$\mathbf{Q}_p(\mathcal{T}) := \{\mathbf{u} \in \mathbf{H}(\Omega, \mathbf{curl}) : (F_K')^T \mathbf{u}|_K \circ F_K \in \mathcal{N}_p^I(\widehat{K})\}, \tag{2.35b}$$

$$\mathbf{V}_p(\mathcal{T}) := \{\mathbf{u} \in \mathbf{H}(\Omega, \mathrm{div}) : (\det F_K')(F_K')^{-1} \mathbf{u}|_K \circ F_K \in \mathbf{RT}_p(\widehat{K})\}, \tag{2.35c}$$

cf. [51, (3.74), (3.76), (3.77)]. The covariant and contravariant transformations preserve tangential and normal traces, respectively. Moreover, differentiating the transformed functions is well-defined. For a function $\widehat{u} \in H^1(\widehat{K})$ that is transformed to a function u on K by $u \circ F_K = \widehat{u}$, it holds

$$\nabla u = (F_K')^{-T} \widehat{\nabla} \widehat{u},$$

where $\widehat{\nabla}$ denotes the gradient operator with respect to the coordinate system for $\widehat{K}$, cf. [51, (3.75)]. Similar results hold for the other differentiation operators, namely,

$$\mathbf{curl}\, \mathbf{u} \circ F_K = (\det F_K')^{-1} F_K' \widehat{\mathbf{curl}} \widehat{\mathbf{u}}$$

for $\widehat{\mathbf{u}} \in \mathbf{H}(\widehat{K}, \mathbf{curl})$ transformed by $\mathbf{u} \circ F_K = (F_K')^{-T} \widehat{\mathbf{u}}$, and

$$\mathrm{div}\, \mathbf{u} \circ F_K = (\det F_K')^{-1} \widehat{\mathrm{div}} \widehat{\mathbf{u}}$$

for $\widehat{\mathbf{u}} \in \mathbf{H}(\widehat{K}, \mathrm{div})$ transformed by $\mathbf{u} \circ F_K = (\det F_K')^{-1} F_K' \widehat{\mathbf{u}}$, where the Piola transformations as in (2.35b) and (2.35c) are used, see [51, Cor. 3.58, Lemma 3.59].

2.4.4 A p-version approximation result

In order to improve convergence in the p-version, one has to increase polynomial degrees in the discrete spaces (opposed to the h-version where convergence is achieved by reducing mesh size, or to the hp-version where both is done simultanously). An important role plays the following approximation result. For the one-dimensional case $n = 1$, see e.g. [46, Thm. 5.1], the higher-dimensional results then follow by applying the case $n = 1$ componentwise.

Lemma 2.23. *Let K be a fixed n-dimensional simplex in $\mathbb{R}^n$, $n \in \{1, 2, 3\}$. Fix $r \geq 0$. Then there are approximation operators $J_p : \mathbf{H}^r(K) \to (\mathcal{P}_p)^n$ such that*

$$\|\mathbf{u} - J_p \mathbf{u}\|_{\mathbf{H}^s(K)} \leq C(p+1)^{-(r-s)} \|\mathbf{u}\|_{\mathbf{H}^r(K)}, \qquad \forall p \in \mathbb{N}_0, \quad 0 \leq s \leq r.$$

2.5 Regularized right inverses and Helmholtz decompositions

Classical Helmholtz decompositions on bounded, simply connected domains Ω are well-known, where one decomposes a vector field $\mathbf{u} \in L^2(\Omega)$ into the sum of a gradient and a divergence-free part, see e.g. [36, Cor. 3.4], or also [51, Sec. 3.7, Sec. 4.4] for similar decompositions. However, what we need are decompositions for more regular functions in $\mathbf{H}^s(\Omega)$ for $s > 0$, which are constructed with the help of the following right inverses for the differential operators from [19].
We start with the 2D-case in the reference triangle $\widehat{f}$.

Lemma 2.24 ([19],[10, Sec. 2.3])**.** *Let $B \subseteq \widehat{f}$ be a ball, and let $\theta \in C_0^\infty(B)$ with $\int_B \theta = 1$. Define the operators*

$$R^{\mathrm{grad}}\mathbf{u}(\mathbf{x}) := \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^{1} \mathbf{u}(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))\, dt \cdot (\mathbf{x}-\mathbf{a})\, d\mathbf{a},$$
$$\mathbf{R}^{\mathrm{curl}}u(\mathbf{x}) := \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^{1} t u(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))\, dt \begin{pmatrix} -(\mathbf{x}_2 - \mathbf{a}_2) \\ \mathbf{x}_1 - \mathbf{a}_1 \end{pmatrix} d\mathbf{a}.$$

Then the following statements hold.

(i) For $u \in L^2(\widehat{f})$, there holds $\operatorname{curl}\mathbf{R}^{\mathrm{curl}}u = u$.

(ii) For $\mathbf{u}$ with $\operatorname{curl}\mathbf{u} = 0$, there holds $\nabla R^{\mathrm{grad}}\mathbf{u} = \mathbf{u}$.

(iii) If $\mathbf{u} \in \mathbf{Q}_p(\widehat{f})$, then $R^{\mathrm{grad}}\mathbf{u} \in W_{p+1}(\widehat{f})$.

(iv) If $u \in V_p(\widehat{f})$, then $\mathbf{R}^{\mathrm{curl}}u \in \mathbf{Q}_p(\widehat{f})$.

(v) For every $k \geq 0$, the operators R^{grad} and $\mathbf{R}^{\mathrm{curl}}$ are bounded linear operators $\mathbf{H}^k(\widehat{f}) \to H^{k+1}(\widehat{f})$ and $H^k(\widehat{f}) \to \mathbf{H}^{k+1}(\widehat{f})$, respectively.

These operators are now the main tool for the construction of regular decompositions.

Lemma 2.25. *Let $s \geq 0$. Then each $\mathbf{u} \in \mathbf{H}^s(\widehat{f}, \operatorname{curl})$ can be written as*

$$\mathbf{u} = \nabla\varphi + \mathbf{z},$$

where $\varphi \in H^{s+1}(\widehat{f})$ and $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{f})$.

Proof. Since Lemma 2.24, (i) shows $\operatorname{curl}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})) = 0$, there holds

$$\nabla R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})) = \mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})$$

by Lemma 2.24, (ii). Thus, we can write

$$\mathbf{u} = \nabla \underbrace{R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u}))}_{=:\varphi} + \underbrace{\mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})}_{=:\mathbf{z}}.$$

The mapping properties of $\mathbf{R}^{\mathrm{curl}}$ and R^{grad} of Lemma 2.24, (v) then give the result. $\square$

The following decomposition looks similar to Lemma 2.25, however, note that the proof does not rely on the right inverses, but uses the shift theorem as its main tool.

Lemma 2.26. *Let $s \in [1, \pi/\omega_{max})$, where ω_{max} denotes the maximal interior angle of $\widehat{f}$. Then each $\mathbf{u} \in \mathbf{H}^s(\widehat{f})$ can be written as*

$$\mathbf{u} = \nabla\varphi + \mathbf{curl}\, z, \tag{2.36}$$

where $\varphi \in H^{s+1}(\widehat{f}) \cap H^1_0(\widehat{f})$ and $z \in H^{s+1}(\widehat{f})$ with $(z, 1)_{L^2(\widehat{f})} = 0$. Additionally, the estimate

$$\|\varphi\|_{H^{s+1}(\widehat{f})} + \|z\|_{H^{s+1}(\widehat{f})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s(\widehat{f})}$$

holds.

Proof. We define $\varphi, z \in H^{s+1}(\widehat{f})$ as the solutions of the equations

$$\begin{aligned} -\Delta\varphi &= -\operatorname{div}\mathbf{u}, \\ \varphi &= 0 \quad \text{on } \partial\widehat{f} \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} &-\Delta z = \operatorname{curl}\mathbf{u}, \\ &\partial_n z = -\mathbf{t}\cdot(\mathbf{u} - \nabla\varphi) \quad \text{on } \partial\widehat{f}, \\ &\int_{\widehat{f}} z = 0. \end{aligned} \tag{2.38}$$

Note that $\mathbf{t}$ denotes the unit tangent vector on $\partial\widehat{f}$ and is oriented such that $\widehat{f}$ is "on the left". The Neumann problem (2.38) is solvable because the compatibility condition is satisfied, which is immediately seen after an integration by parts argument, cf. (2.25). Corollary 3.43 and Proposition 3.48 then yield the estimates

$$\|\varphi\|_{H^{s+1}(\widehat{f})} \lesssim \|\operatorname{div}\mathbf{u}\|_{H^{s-1}(\widehat{f})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s(\widehat{f})}$$

and

$$\|z\|_{H^{s+1}(\widehat{f})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s(\widehat{f})}.$$

We now show that the decomposition (2.36). The difference $\boldsymbol{\delta} := \mathbf{u} - (\nabla\varphi + \mathbf{curl}\, z)$ satisfies $\operatorname{div}\boldsymbol{\delta} = 0$ and $\operatorname{curl}\boldsymbol{\delta} = 0$. If we denote $\mathbf{t} = (t_1, t_2)^\top$ and $\mathbf{n} = (n_1, n_2)^\top$, it follows that $\mathbf{t} = (-n_2, n_1)^\top$, thus

$$\mathbf{t}\cdot\boldsymbol{\delta} = -\partial_n z + \mathbf{t}\cdot\mathbf{curl}\, z = 0.$$

This implies $\boldsymbol{\delta} = 0$, which yields the decomposition. □

In 3D, the definition of the right inverses and the resulting decompositions on the reference tetrahedron $\widehat{K}$ are similar.

Lemma 2.27 ([19], [39, Sec. 2]). *Let $B \subset \widehat{K}$ be a ball, and let $\theta \in C_0^\infty(B)$ with $\int_B \theta = 1$. Define the operators*

$$R^{\mathrm{grad}}\mathbf{u}(\mathbf{x}) := \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^{1} \mathbf{u}(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))\,dt \cdot (\mathbf{x}-\mathbf{a})\,d\mathbf{a},$$
$$\mathbf{R}^{\mathrm{curl}}\mathbf{u}(\mathbf{x}) := \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^{1} t\mathbf{u}(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))\,dt \times (\mathbf{x}-\mathbf{a})\,d\mathbf{a},$$
$$\mathbf{R}^{\mathrm{div}}u(\mathbf{x}) := \int_{\mathbf{a}\in B} \theta(\mathbf{a}) \int_{t=0}^{1} t^2 u(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))\,dt(\mathbf{x}-\mathbf{a})\,d\mathbf{a}.$$

Then the following statements hold.

(i) For $\mathbf{u}$ with $\operatorname{div}\mathbf{u} = 0$, there holds $\mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}\mathbf{u} = \mathbf{u}$.

(ii) For $\mathbf{u}$ with $\mathbf{curl}\,\mathbf{u} = 0$, there holds $\nabla R^{\mathrm{grad}}\mathbf{u} = \mathbf{u}$.

(iii) For $u \in L^2(\widehat{K})$, there holds $\operatorname{div}\mathbf{R}^{\mathrm{div}}u = u$.

(iv) If $\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$, then $R^{\mathrm{grad}}\mathbf{u} \in W_{p+1}(\widehat{K})$.

(v) If $\mathbf{u} \in \mathbf{V}_p(\widehat{K})$, then $\mathbf{R}^{\mathrm{curl}}\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$.

(vi) If $u \in W_p(\widehat{K})$, then $\mathbf{R}^{\mathrm{div}}u \in \mathbf{V}_p(\widehat{K})$.

(vii) For every $k \geq 0$, the operators R^{grad}, $\mathbf{R}^{\mathrm{curl}}$ and $\mathbf{R}^{\mathrm{div}}$ are bounded linear operators $\mathbf{H}^k(\widehat{K}) \to H^{k+1}(\widehat{K})$, $\mathbf{H}^k(\widehat{K}) \to \mathbf{H}^{k+1}(\widehat{K})$ and $H^k(\widehat{K}) \to \mathbf{H}^{k+1}(\widehat{K})$, respectively.

Lemma 2.28. *Let $s \geq 0$. Then each $\mathbf{u} \in \mathbf{H}^s(\widehat{K}, \mathbf{curl})$ can be written as*

$$\mathbf{u} = \nabla\varphi + \mathbf{z},$$

where $\varphi \in H^{s+1}(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{K})$, with the estimates

$$\|\varphi\|_{H^{s+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s(\widehat{K},\mathbf{curl})}, \qquad \|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^s(\widehat{K})}.$$

Proof. Since Lemma 2.27, (i) shows $\mathbf{curl}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})) = 0$, there holds

$$\nabla R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})) = \mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})$$

by Lemma 2.27, (ii). Thus, we can write

$$\mathbf{u} = \nabla \underbrace{R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u}))}_{=:\varphi} + \underbrace{\mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})}_{=:\mathbf{z}}.$$

The mapping properties of $\mathbf{R}^{\mathrm{curl}}$ and R^{grad} of Lemma 2.27, (vii) then imply the desired regularity of φ and $\mathbf{z}$.
The stability properties of the operators $\mathbf{R}^{\mathrm{curl}}$ and R^{grad} then give the estimates

$$\|\varphi\|^2_{H^{s+1}(\widehat{K})} \lesssim \|\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})\|^2_{\mathbf{H}^s(\widehat{K})} \lesssim \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K})} + \|\mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})\|^2_{\mathbf{H}^{s+1}(\widehat{K})}$$
$$\lesssim \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K})} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K})} = \|\mathbf{u}\|^2_{\mathbf{H}^s(\widehat{K},\mathbf{curl})}$$

and

$$\|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} = \|\mathbf{R}^{\mathrm{curl}}(\mathbf{curl}\,\mathbf{u})\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^{s}(\widehat{K})}.$$

□

Lemma 2.29. *Let $s \in [0,1]$. Then each $\mathbf{u} \in \mathbf{H}^s(\widehat{K})$ can be written as*

$$\mathbf{u} = \nabla\varphi + \mathbf{curl}\,\mathbf{z},$$

where $\varphi \in H^{s+1}(\widehat{K}) \cap H^1_0(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{K})$. Additionally, the estimate

$$\|\varphi\|_{H^{s+1}(\widehat{K})} + \|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^{s}(\widehat{K})}$$

holds.

Proof. We define $\varphi \in H^1_0(\widehat{K})$ as the solution of the problem

$$-\Delta\varphi = -\operatorname{div}\mathbf{u},$$
$$\varphi = 0 \text{ on } \partial\widehat{K}.$$

Since the differential operator div maps $H^1(\widehat{K}) \to L^2(\widehat{K})$ and $L^2(\widehat{K}) \to H^{-1}(\widehat{K})$, the convexity of $\widehat{K}$ gives $\varphi \in H^1_0(\widehat{K})$ if $s = 0$ and $\varphi \in H^2(\widehat{K}) \cap H^1_0(\widehat{K})$ if $s = 1$. Interpolation between $H^1(\widehat{K})$ and $H^2(\widehat{K})$ yields $\varphi \in H^{s+1}(\widehat{K}) \cap H^1_0(\widehat{K})$, and the estimate

$$\|\varphi\|_{H^{s+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^{s}(\widehat{K})}$$

holds. The mapping properties of Lemma 2.27, (vii) show $\mathbf{z} := \mathbf{R}^{\mathrm{curl}}(\mathbf{u} - \nabla\varphi) \in \mathbf{H}^{s+1}(\widehat{K})$. Note that $\operatorname{div}(\mathbf{u} - \nabla\varphi) = 0$, hence it follows $\mathbf{u} = \nabla\varphi + \mathbf{curl}\,\mathbf{z}$ by Lemma 2.27, (i). The stability property of $\mathbf{R}^{\mathrm{curl}}$ then gives the estimate

$$\|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} = \|\mathbf{R}^{\mathrm{curl}}(\mathbf{u} - \nabla\varphi)\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\mathbf{u} - \nabla\varphi\|_{\mathbf{H}^{s}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^{s}(\widehat{K})}.$$

□

We also need decompositions similar to Lemma 2.29, but with boundary conditions for the $\mathbf{H}(\mathbf{curl})$-functions. We start with a lemma.

Lemma 2.30. *The following regularity statements hold:*

(i) Let $v \in L^2(\widehat{K})$ and $g \in L^2(\partial\widehat{K})$ with $g|_f \in H^{1/2}(f)$ for each face $f \in \mathcal{F}(\widehat{K})$ satisfy the compatibility condition $\int_{\widehat{K}} v + \int_{\partial\widehat{K}} g = 0$. Then the solution $\varphi \in H^1(\widehat{K})$ of the Neumann problem

$$\begin{aligned} -\Delta\varphi &= v \quad \text{in } \widehat{K}, \\ \partial_n\varphi|_{\partial\widehat{K}} &= g \quad \text{on } \partial\widehat{K}, \end{aligned} \tag{2.39}$$

satisfies $\varphi \in H^2(\widehat{K})$ with the estimate

$$\|\varphi\|_{H^2(\widehat{K})} \lesssim \|v\|_{L^2(\widehat{K})} + \sum_{f\in\mathcal{F}(\widehat{K})} \|g\|_{H^{1/2}(f)}. \tag{2.40}$$

(ii) For $s \in [0, 1/2]$, let $v \in \widetilde{H}^{-1/2+s}(\widehat{K})$ and $g \in L^2(\partial\widehat{K})$ with $g|_f \in H^s(f)$ for each face $f \in \mathcal{F}(\widehat{K})$ satisfy the compatibility condition

$$\langle v, 1\rangle_{\widetilde{H}^{-1}(\widehat{K})\times H^1(\widehat{K})} + \int_{\partial\widehat{K}} g = 0. \tag{2.41}$$

Then the solution $\varphi \in H^1(\widehat{K})$ of the Neumann problem (2.39) *satisfies $\varphi \in H^{s+3/2}(\widehat{K})$ with the estimate*

$$\|\varphi\|_{H^{s+3/2}(\widehat{K})} \lesssim \|v\|_{\widetilde{H}^{-1/2+s}(\widehat{K})} + \sum_{f\in\mathcal{F}(\widehat{K})} \|g\|_{H^s(f)}.$$

Proof. We start with (i). Let $\boldsymbol{\sigma} \in \mathbf{H}^1(\widehat{K})$ be a vector field with the condition $\boldsymbol{\sigma}\cdot\mathbf{n} = g$ on $\partial\widehat{K}$. Such a vector field exists, since constructing such a vector field away from the vertices and edges is straightforward, and near the vertices and edges, the construction is reduced to one in an octant of $\mathbb{R}^3$ by an affine coordinate change. Each component of $\boldsymbol{\sigma}$ can there be constructed separately by lifting from one of the coordinate planes, since two components of $\mathbf{n}$ are always zero.
We now define $\mathbf{z} := \nabla\varphi - \boldsymbol{\sigma}$. Since we have

$$\operatorname{div}\mathbf{z} = \Delta\varphi - \operatorname{div}\boldsymbol{\sigma} = -v - \operatorname{div}\boldsymbol{\sigma} \in L^2(\widehat{K})$$

and

$$\mathbf{curl}\,\mathbf{z} = -\,\mathbf{curl}\,\boldsymbol{\sigma} \in L^2(\widehat{K})$$

together with $\mathbf{z}\cdot\mathbf{n} = 0$, it follows $\mathbf{z} \in \mathbf{H}^1(\widehat{K})$, cf. [3, Thm. 2.17], which implies $\varphi \in H^2(\widehat{K})$. The norm estimate follows directly from the continuity of the embedding in [3, Thm. 2.17] together with the continuity of the lifting.
We now show (ii). It is well-known that the solution $u \in H^1(\Omega)$ of the problem

$$\begin{aligned} -\Delta u &= -|\widehat{K}|^{-1}(g,1)_{L^2(\partial\widehat{K})} \in L^2(\widehat{K}),\\ \partial_n u|_{\partial\widehat{K}} &= g \in L^2(\partial\widehat{K}) \end{aligned} \tag{2.42}$$

lies in the space $H^{3/2}(\widehat{K})$, with the estimate

$$\|u\|_{H^{3/2}(\widehat{K})} \lesssim \left\| |\widehat{K}|^{-1}(g,1)_{L^2(\partial\widehat{K})} \right\|_{L^2(\widehat{K})} + \|g\|_{L^2(\partial\widehat{K})} \lesssim \|g\|_{L^2(\partial\widehat{K})},$$

cf. [51, Thm. 3.18], [40, Sec. 6]. It also holds that for $v \in \widetilde{H}^{-1/2}(\widehat{K})$, the solution $u \in H^1(\Omega)$ of the problem

$$\begin{aligned} -\Delta u &= v - |\widehat{K}|^{-1}\langle v, 1\rangle_{\widetilde{H}^{-1}(\widehat{K})\times H^1(\widehat{K})} \in \widetilde{H}^{-1/2}(\widehat{K}),\\ \partial_n u|_{\partial\widehat{K}} &= 0 \end{aligned} \tag{2.43}$$

satisfies $u \in H^{3/2}(\widehat{K})$, with the estimate

$$\|u\|_{H^{3/2}(\widehat{K})} \lesssim \|v\|_{\widetilde{H}^{-1/2}(\widehat{K})}.$$

This is seen by interpolation between the regularity results for $v \in \widetilde{H}^{-1}(\widehat{K})$ (cf. [51, Thm. 3.16]) and $v \in L^2(\widehat{K})$ (which has been shown in (i)). Adding both problems implies that the solution $u \in H^1(\widehat{K})$ of the problem

$$\begin{aligned} -\Delta u &= v - |\widehat{K}|^{-1} \langle v, 1\rangle_{\widetilde{H}^{-1}(\widehat{K})\times H^1(\widehat{K})} - |\widehat{K}|^{-1}(g,1)_{L^2(\partial\widehat{K})} \in \widetilde{H}^{-1/2}(\widehat{K}), \\ \partial_n u|_{\partial\widehat{K}} &= g \in L^2(\partial\widehat{K}) \end{aligned} \tag{2.44}$$

is in the space $H^{3/2}(\widehat{K})$. Note that the compatibility conditions for the problems (2.42), (2.43) and (2.44) are satisfied by construction.
Now assume the the compatibility condition $\langle v, 1\rangle_{\widetilde{H}^{-1}(\widehat{K})\times H^1(\widehat{K})} + \int_{\partial\widehat{K}} g = 0$ is satisfied. This implies that the Neumann problems (2.39) and (2.44) are equivalent, hence we obtain that the solution $u \in H^1(\widehat{K})$ of (2.39) for $v \in \widetilde{H}^{-1/2}(\widehat{K})$ and $g \in L^2(\partial\widehat{K})$ satisfies $u \in H^{3/2}(\widehat{K})$, with the estimate

$$\|u\|_{H^{3/2}(\widehat{K})} \lesssim \|v\|_{\widetilde{H}^{-1/2}(\widehat{K})} + \sum_{f\in\mathcal{F}(\widehat{K})} \|g\|_{L^2(f)}. \tag{2.45}$$

We now identify the sum of norms on the right-hand side of (2.45) with the product norm of the space

$$\widetilde{H}^{-1/2}(\widehat{K}) \times \prod_{f\in\mathcal{F}(\widehat{K})} L^2(f).$$

Analogously, the right-hand side of (2.40) can be seen as a function in

$$L^2(\widehat{K}) \times \prod_{f\in\mathcal{F}(\widehat{K})} H^{1/2}(f).$$

Since interpolation is compatible with generating the (finite) Cartesian product, the desired result follows from interpolation between (2.40) and (2.45). □

Lemma 2.31. *Any* $\mathbf{v} \in \mathbf{H}^1(\widehat{K})$ *can be written as*

$$\mathbf{v} = \nabla\varphi_0 + \mathbf{curl}\,\mathbf{curl}\,\mathbf{z}_0, \tag{2.46}$$
$$\mathbf{v} = \nabla\varphi_1 + \mathbf{curl}\,\mathbf{z}_1, \tag{2.47}$$

where $\varphi_0 \in H^2(\widehat{K}) \cap H^1_0(\widehat{K})$ *and* $\mathbf{z}_0 \in \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ *and where* $\varphi_1 \in H^2(\widehat{K})$ *and* $\mathbf{z}_1 \in \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ *together with the estimates*

$$\|\varphi_0\|_{H^2(\widehat{K})} + \|\mathbf{z}_0\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \le C\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})},$$
$$\|\varphi_1\|_{H^2(\widehat{K})} + \|\mathbf{z}_1\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \le C\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.$$

Proof. We proceed in several steps.

Step 1: We construct the decomposition (2.47).

We define $\varphi_1 \in H^2(\widehat{K})$ as the solution of the Neumann problem

$$\begin{aligned} -\Delta\varphi_1 &= -\operatorname{div}\mathbf{v} \quad \text{in } \widehat{K}, \\ \partial_n\varphi_1 &= \mathbf{n}\cdot\mathbf{v} \quad \text{on } \partial\widehat{K}, \end{aligned}$$

cf. Lemma 2.30. Note that the compatibility condition is clearly satisfied by an integration by parts argument. The function $\mathbf{z}_1$ is then defined by the following saddle point problem: Find $(\mathbf{z}_1, \psi) \in \mathbf{H}_0(\widehat{K}, \mathbf{curl}) \times H^1_0(\widehat{K})$ such that

$$(\mathbf{curl}\,\mathbf{z}_1, \mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})} - (\nabla\psi, \mathbf{w})_{L^2(\widehat{K})} = (\mathbf{curl}\,\mathbf{v}, \mathbf{w})_{L^2(\widehat{K})} \quad \forall \mathbf{w} \in \mathbf{H}_0(\widehat{K}, \mathbf{curl}), \tag{2.48a}$$

$$(\mathbf{z}_1, \nabla q)_{L^2(\widehat{K})} = 0 \quad \forall q \in H^1_0(\widehat{K}). \tag{2.48b}$$

We now show that the problem is uniquely solvable. We define the bilinear forms $a(\mathbf{w}, \mathbf{q}) := (\mathbf{curl}\,\mathbf{q}, \mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})}$ and $b(\mathbf{w}, \varphi) := (\mathbf{w}, \nabla\varphi)_{L^2(\widehat{K})}$ for $\mathbf{w}, \mathbf{q} \in \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ and $\varphi \in H^1_0(\widehat{K})$. Since the kernel of b equals

$$\ker b = \{\mathbf{q} \in \mathbf{H}_0(\widehat{K}, \mathbf{curl}) : (\mathbf{q}, \nabla\varphi)_{L^2(\widehat{K})} = 0 \quad \forall \varphi \in H^1_0(\widehat{K})\},$$

coercivity of a on $\ker b$ is a direct consequence of the Friedrichs inequality for the **curl**-operator (see e.g. [51, Cor. 3.51]) by

$$a(\mathbf{q}, \mathbf{q})_{L^2(\widehat{K})} = \|\,\mathbf{curl}\,\mathbf{q}\|^2_{L^2(\widehat{K})} \gtrsim \|\mathbf{q}\|^2_{L^2(\widehat{K})} + \|\,\mathbf{curl}\,\mathbf{q}\|^2_{L^2(\widehat{K})} \gtrsim \|\mathbf{q}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}$$

for all $\mathbf{q} \in \ker b$. The inf-sup-condition

$$\inf_{\varphi \in H^1_0(\widehat{K})} \sup_{\mathbf{w} \in \mathbf{H}_0(\widehat{K},\mathbf{curl})} \frac{b(\mathbf{w}, \varphi)}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\|\varphi\|_{H^1(\widehat{K})}} \geq C$$

is shown by choosing $\mathbf{w} = \nabla\varphi \in \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ for given $\varphi \in H^1_0(\widehat{K})$, which implies

$$\frac{b(\mathbf{w}, \varphi)}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\|\varphi\|_{H^1(\widehat{K})}} = \frac{\|\nabla\varphi\|^2_{L^2(\widehat{K})}}{\|\nabla\varphi\|_{L^2(\widehat{K})}\|\varphi\|_{H^1(\widehat{K})}} \geq C$$

by the standard Poincaré inequality. Thus, (2.48) has a unique solution. Choosing $\mathbf{w} = \nabla\psi$ as test function in (2.48a) shows $\psi = 0$. The solution $\mathbf{z}_1$ now satisfies the estimate

$$\|\mathbf{z}_1\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|f\|,$$

where $f(\mathbf{w}) = (\mathbf{curl}\,\mathbf{v}, \mathbf{w})_{L^2(\widehat{K})}$, and where $\|\cdot\|$ denotes the operator norm, cf. [12, Thm. 4.2.3]. Since

$$\|f\| = \sup_{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}} \left|(\mathbf{curl}\,\mathbf{v}, \mathbf{w})_{L^2(\widehat{K})}\right| \leq \|\,\mathbf{curl}\,\mathbf{v}\|_{L^2(\widehat{K})},$$

it follows

$$\|\mathbf{z}_1\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|\,\mathbf{curl}\,\mathbf{v}\|_{L^2(\widehat{K})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.$$

Equation (2.48b) yields $\operatorname{div} \mathbf{z}_1 = 0$, thus (2.22) implies $\|\mathbf{z}_1\|_{\mathbf{H}^1(\widehat{K})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}$. Note that the difference $\boldsymbol{\delta} := \mathbf{v} - \nabla\varphi_1 - \mathbf{curl}\,\mathbf{z}_1$ satisfies by construction $\operatorname{div}\boldsymbol{\delta} = 0$, $\mathbf{curl}\,\boldsymbol{\delta} = 0$, and

$$\mathbf{n}\cdot\boldsymbol{\delta} = (\mathbf{n}\cdot\mathbf{v} - \partial_n\varphi_1) - \mathbf{n}\cdot\mathbf{curl}\,\mathbf{z}_1 = 0 - \operatorname{curl}_{\partial\widehat{K}} \Pi_\tau \mathbf{z}_1 = 0 - 0 = 0$$

on the boundary, hence [51, Cor. 3.51] implies $\boldsymbol{\delta} = 0$. The regularity $\mathbf{curl}\,\mathbf{z}_1 \in \mathbf{H}^1(\widehat{K})$ now follows from $\mathbf{v} \in \mathbf{H}^1(\widehat{K})$, $\varphi_1 \in H^2(\widehat{K})$ and the representation (2.47).

Step 2: We construct the decomposition (2.46).
The proof uses similar arguments as above. We define $\varphi_0 \in H^2(\widehat{K}) \cap H^1_0(\widehat{K})$ as the solution of the Dirichlet problem

$$\begin{aligned} -\Delta\varphi_0 &= -\operatorname{div}\mathbf{v} \quad \text{in } \widehat{K}, \\ \varphi_0 &= 0 \quad \text{on } \partial\widehat{K}. \end{aligned}$$

Next, we define $(\mathbf{z}_0, \psi) \in \mathbf{H}_0(\widehat{K}, \mathbf{curl}) \times H^1_0(\widehat{K})$ as the solution of the saddle point problem

$$\begin{aligned} (\mathbf{curl}\,\mathbf{z}_0, \mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})} - (\nabla\psi, \mathbf{w})_{L^2(\widehat{K})} &= (\mathbf{v} - \nabla\varphi_0, \mathbf{w})_{L^2(\widehat{K})} \quad \forall \mathbf{w} \in \mathbf{H}_0(\widehat{K}, \mathbf{curl}), \\ (\mathbf{z}_0, \nabla q)_{L^2(\widehat{K})} &= 0 \qquad \forall q \in H^1_0(\widehat{K}), \end{aligned}$$

which is uniquely solvable with the same argumentation as for (2.48). Since $\operatorname{div}(\mathbf{v} - \nabla\varphi_0) = 0$, it follows again $\psi = 0$. We obtain the bound

$$\|\mathbf{z}_0\|_{\mathbf{H}(\mathbf{curl},\widehat{K})} \lesssim \|\mathbf{v} - \nabla\varphi_0\|_{L^2(\widehat{K})} \lesssim \|\mathbf{v}\|_{L^2(\widehat{K})},$$

which implies $\|\mathbf{z}_0\|_{\mathbf{H}^1(\widehat{K})} \lesssim \|\mathbf{v}\|_{L^2(\widehat{K})}$ by (2.22) using the observation $\operatorname{div}\mathbf{z}_0 = 0$. Representation (2.46) now follows by an integration by parts,

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{z}_0 = \mathbf{v} - \nabla\varphi_0.$$

□

If $\mathbf{v}$ is not a function in $\mathbf{H}^1(\widehat{K})$, but only in $\mathbf{H}^{1/2+\epsilon}(\widehat{K})$ for $\epsilon \in (0, 1/2)$, one can show a similar decomposition as in Lemma 2.31. Even the proof is quite similar.

Lemma 2.32. *For $\epsilon \in (0, 1/2)$, any $\mathbf{v} \in \mathbf{H}^{1/2+\epsilon}(\widehat{K})$ can be written as*

$$\mathbf{v} = \nabla\varphi_0 + \mathbf{curl}\,\mathbf{curl}\,\mathbf{z}_0, \tag{2.49}$$

$$\mathbf{v} = \nabla\varphi_1 + \mathbf{curl}\,\mathbf{z}_1, \tag{2.50}$$

where $\varphi_0 \in H^{3/2+\epsilon}(\widehat{K}) \cap H^1_0(\widehat{K})$ and $\mathbf{z}_0 \in \mathbf{H}^{1/2+\epsilon}(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ and where $\varphi_1 \in H^{3/2+\epsilon}(\widehat{K})$ and $\mathbf{z}_1 \in \mathbf{H}^{1/2+\epsilon}(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ together with the estimates

$$\begin{aligned} \|\varphi_0\|_{H^{3/2+\epsilon}(\widehat{K})} + \|\mathbf{z}_0\|_{\mathbf{H}^{1/2+\epsilon}(\widehat{K},\mathbf{curl})} &\le C\|\mathbf{v}\|_{\mathbf{H}^{1/2+\epsilon}(\widehat{K})}, \\ \|\varphi_1\|_{H^{3/2+\epsilon}(\widehat{K})} + \|\mathbf{z}_1\|_{\mathbf{H}^{1/2+\epsilon}(\widehat{K},\mathbf{curl})} &\le C\|\mathbf{v}\|_{\mathbf{H}^{1/2+\epsilon}(\widehat{K})}. \end{aligned}$$

Proof. We again start with decomposition (2.50).
We define $\varphi_1 \in H^{3/2+\epsilon}(\widehat{K})$ as the solution of the Neumann problem

$$-\Delta\varphi_1 = -\operatorname{div}\mathbf{v} \in \widetilde{H}^{-1/2+\epsilon}(\widehat{K}) \quad \text{in } \widehat{K}, \tag{2.51}$$

$$\partial_n\varphi_1 = \mathbf{n}\cdot\mathbf{v} \quad \text{on } \partial\widehat{K}, \tag{2.52}$$

cf. Lemma 2.30.
The contribution $\mathbf{z}_1$ is defined by the saddle point problem: Find $(\mathbf{z}_1, \psi) \in \mathbf{H}_0(\widehat{K}, \mathbf{curl}) \times H_0^1(\widehat{K})$ such that

$$\begin{aligned} (\mathbf{curl}\,\mathbf{z}_1, \mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})} - (\nabla\psi, \mathbf{w})_{L^2(\widehat{K})} &= (\mathbf{v}, \mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})} \quad \forall \mathbf{w} \in \mathbf{H}_0(\widehat{K}, \mathbf{curl}), \\ (\mathbf{z}_1, \nabla q)_{L^2(\widehat{K})} &= 0 \qquad \forall q \in H_0^1(\widehat{K}). \end{aligned} \tag{2.53}$$

By the same arguments as in Lemma 2.31, the coercivity on the kernel and the inf-sup-condition are satisfied, thus problem (2.53) is uniquely solvable with $\psi = 0$ (use the test function $\mathbf{w} = \nabla\psi$). Furthermore, we get the estimate

$$\|\mathbf{z}_1\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|\mathbf{v}\|_{L^2(\widehat{K})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^{1/2+\epsilon}(\widehat{K})},$$

and with (2.22) it follows $\|\mathbf{z}_1\|_{\mathbf{H}^1(\widehat{K})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^{1/2+\epsilon}(\widehat{K})}$. Writing $\boldsymbol{\delta} := \mathbf{v} - \nabla\varphi_1 - \mathbf{curl}\,\mathbf{z}_1$, we can conclude $\boldsymbol{\delta} = 0$ repeating the arguments in Lemma 2.31, which shows the desired decomposition.
The decomposition (2.46) is constructed analogously to Lemma 2.31 after obvious changes considering the regularity of $\mathbf{v}$. We only mention that the definition of $\varphi_0 \in H^{3/2+\epsilon}(\widehat{K}) \cap H_0^1(\widehat{K})$ as the solution of

$$\begin{aligned} -\Delta\varphi_0 &= -\operatorname{div}\mathbf{v} \in \widetilde{H}^{-1/2+\epsilon}(\widehat{K}) \quad \text{in } \widehat{K}, \\ \varphi_0 &= 0 \quad \text{on } \partial\widehat{K} \end{aligned}$$

is again meaningful by interpolation arguments. □

In 3D, we also need a decomposition for vector fields in the space $\mathbf{H}(\widehat{K}, \operatorname{div})$.

Lemma 2.33. *Let $s \geq 0$. Then each $\mathbf{u} \in \mathbf{H}^s(\widehat{K}, \operatorname{div})$ can be written as*

$$\mathbf{u} = \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z},$$

where $\boldsymbol{\varphi} \in \mathbf{H}^{s+1}(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{K})$. Additionally, the estimates

$$\|\boldsymbol{\varphi}\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s(\widehat{K},\operatorname{div})} \text{ and } \|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\operatorname{div}\mathbf{u}\|_{H^s(\widehat{K})}$$

hold.

Proof. Since Lemma 2.27, (iii) shows $\operatorname{div}(\mathbf{u} - \mathbf{R}^{\operatorname{div}}(\operatorname{div}\mathbf{u})) = 0$, there holds

$$\mathbf{curl}\,\mathbf{R}^{\operatorname{curl}}(\mathbf{u} - \mathbf{R}^{\operatorname{div}}(\operatorname{div}\mathbf{u})) = \mathbf{u} - \mathbf{R}^{\operatorname{div}}(\operatorname{div}\mathbf{u})$$

by Lemma 2.27, (i). Thus, we can write

$$\mathbf{u} = \mathbf{curl}\,\underbrace{\mathbf{R}^{\mathrm{curl}}(\mathbf{u} - \mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u}))}_{=:\boldsymbol{\varphi}} + \underbrace{\mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})}_{=:\mathbf{z}}.$$

The stability properties of $\mathbf{R}^{\mathrm{curl}}$ and $\mathbf{R}^{\mathrm{div}}$ of Lemma 2.27, (vii) then imply the estimates

$$\begin{aligned}\|\boldsymbol{\varphi}\|_{\mathbf{H}^{s+1}(\widehat{K})} &\lesssim \|\mathbf{u} - \mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})\|_{\mathbf{H}^{s}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^{s}(\widehat{K})} + \|\mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})\|_{H^{s+1}(\widehat{K})}\\ &\lesssim \|\mathbf{u}\|_{\mathbf{H}^{s}(\widehat{K})} + \|\operatorname{div}\mathbf{u}\|_{H^{s}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^{s}(\widehat{K},\mathrm{div})}\end{aligned}$$

and

$$\|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} = \|\mathbf{R}^{\mathrm{div}}(\operatorname{div}\mathbf{u})\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\operatorname{div}\mathbf{u}\|_{H^{s}(\widehat{K})}.$$

□

2.6 Discrete Friedrichs inequalities

The well-known Poincaré inequality states that the L^2-norm of a function $u \in H_0^1$ can be bounded by the H^1-seminorm. Similar results - called Friedrichs inequalities - hold true for functions in $\mathbf{H}(\mathbf{curl})$ and $\mathbf{H}(\mathrm{div})$, where the corresponding seminorms are defined naturally. However, just as the Poincaré inequality requires the function u to satisfy zero boundary conditions, additional conditions to the functions are needed for the Friedrichs inequalities. In this section, we now state several versions of the Friedrichs inequalities for discrete functions, first for 2D, then for 3D, which partially use the right inverses of Section 2.5 in the proofs.

Lemma 2.34 (discrete Friedrichs inequality in 2D)**.** *The estimate*

$$\|\mathbf{u}\|_{L^2(\widehat{f})} \lesssim \|\operatorname{curl}\mathbf{u}\|_{L^2(\widehat{f})}, \tag{2.54}$$

where the constant does not depend on p and $\mathbf{u}$, holds in the following two cases:

(i) $\mathbf{u} \in \mathbf{Q}_p(\widehat{f})$ satisfies $(\mathbf{u}, \nabla v)_{L^2(\widehat{f})} = 0$ for all $v \in W_{p+1}(\widehat{f})$.

(ii) $\mathbf{u} \in \mathring{\mathbf{Q}}_p(\widehat{f})$ satisfies $(\mathbf{u}, \nabla v)_{L^2(\widehat{f})} = 0$ for all $v \in \mathring{W}_{p+1}(\widehat{f})$.

Proof. A proof of (i) is found in [29, Lemma 6] or [26, Lemma 4.1]. In order to prove statement (ii), we use the operators R^{grad} and $\mathbf{R}^{\mathrm{curl}}$ of Lemma 2.24. We decompose $\mathbf{u} \in \mathring{\mathbf{Q}}_p(\widehat{f})$ as

$$\mathbf{u} = \nabla\,\underbrace{R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u}))}_{=:\psi} + \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u}).$$

Since $\mathbf{u} \in \mathbf{Q}_p(\widehat{f})$, it follows $\psi \in W_{p+1}(\widehat{f})$ with Lemma 2.24, (iii), (iv). The property $\mathbf{u} \in \mathring{\mathbf{Q}}_p(\widehat{f})$ implies

$$\mathbf{t}\cdot\nabla\psi = -\mathbf{t}\cdot\mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u}),$$

where $\mathbf{t}$ denotes the tangential vector on the boundary of $\widehat{f}$ oriented in the mathematical positive direction. Since ψ is continuous at the vertices of $\widehat{f}$, we obtain

$$\begin{aligned}|\psi|_{H^{1/2}(\partial\widehat{f})} \lesssim \|\nabla\psi\|_{L^2(\partial\widehat{f})} = \|\mathbf{t}\cdot\mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})\|_{L^2(\partial\widehat{f})} \leq \|\mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})\|_{L^2(\partial\widehat{f})}\\ \lesssim \|\mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})\|_{H^{1/2}(\partial\widehat{f})} \lesssim \|\mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})\|_{H^1(\widehat{f})} \lesssim \|\operatorname{curl}\mathbf{u}\|_{L^2(\widehat{f})}.\end{aligned} \tag{2.55}$$

With the lifting operator $\mathcal{L}^{\mathrm{grad},2d}: H^{1/2}(\partial\widehat{f}) \to H^1(\widehat{f})$ from [5], we define

$$\psi_0 := \psi - \mathcal{L}^{\mathrm{grad},2d}(\psi|_{\partial\widehat{f}}).$$

Since the lifting $\mathcal{L}^{\mathrm{grad},2d}$ has the property that $\psi \in W_{p+1}(\widehat{f})$ implies $\mathcal{L}^{\mathrm{grad},2d}(\psi|_{\partial\widehat{f}}) \in W_{p+1}(\widehat{f})$, it follows immediately $\psi_0 \in \mathring{W}_{p+1}(\widehat{f})$. Since furthermore $\mathcal{L}^{\mathrm{grad},2d}1 = 1$, we get

$$\|\nabla\mathcal{L}^{\mathrm{grad},2d}(\psi|_{\partial\widehat{f}})\|_{L^2(\widehat{f})} \lesssim |\psi|_{H^{1/2}(\partial\widehat{f})}, \tag{2.56}$$

which results in the estimate

$$\begin{aligned}\|\mathbf{u}\|^2_{L^2(\widehat{f})} &= (\mathbf{u}, \nabla\psi_0 + \nabla\mathcal{L}^{\mathrm{grad},2d}(\psi|_{\partial\widehat{f}}) + \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u}))_{L^2(\widehat{f})}\\ &= (\mathbf{u}, \nabla\mathcal{L}^{\mathrm{grad},2d}(\psi|_{\partial\widehat{f}}) + \mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u}))_{L^2(\widehat{f})}\\ &\leq \|\mathbf{u}\|_{L^2(\widehat{f})}\left\{\|\nabla\mathcal{L}^{\mathrm{grad},2d}(\psi|_{\partial\widehat{f}})\|_{L^2(\widehat{f})} + \|\mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})\|_{L^2(\widehat{f})}\right\}\\ &\overset{(2.56)}{\leq} \|\mathbf{u}\|_{L^2(\widehat{f})}\left\{|\psi|_{H^{1/2}(\partial\widehat{f})} + \|\mathbf{R}^{\mathrm{curl}}(\operatorname{curl}\mathbf{u})\|_{L^2(\widehat{f})}\right\} \overset{(2.55)}{\lesssim} \|\mathbf{u}\|_{L^2(\widehat{f})}\|\operatorname{curl}\mathbf{u}\|_{L^2(\widehat{f})},\end{aligned}$$

where we used the assumption $(\mathbf{u}, \nabla v)_{L^2(\widehat{f})} = 0$ for all $v \in \mathring{W}_{p+1}(\widehat{f})$ in the second equation. □

We present now the corresponding result in three dimensions, which is proved in [26, Lemma 5.1].

Lemma 2.35 (discrete Friedrichs inequality for $\mathbf{H}(\mathbf{curl})$ in 3D)**.** *The estimate*

$$\|\mathbf{u}\|_{L^2(\widehat{K})} \lesssim \|\mathbf{curl}\,\mathbf{u}\|_{L^2(\widehat{K})}, \tag{2.57}$$

where the constant is independent of p and $\mathbf{u}$, holds in the following cases:

(i) $\mathbf{u} \in \mathbf{Q}_{p,\perp}(\widehat{K}) := \{\mathbf{v} \in \mathbf{Q}_p(\widehat{K})\colon (\mathbf{v}, \nabla\psi)_{L^2(\widehat{K})} = 0 \quad \forall\psi \in W_{p+1}(\widehat{K})\}$,

(ii) $\mathbf{u} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K}) := \{\mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K})\colon (\mathbf{v}, \nabla\psi)_{L^2(\widehat{K})} = 0 \quad \forall\psi \in \mathring{W}_{p+1}(\widehat{K})\}$.

In three dimensions, we can also state a similar inequality for $\mathbf{H}(\mathrm{div})$-functions.

Lemma 2.36 (discrete Friedrichs inequality for $\mathbf{H}(\mathrm{div})$)**.** *With the notation of Lemma 2.35, the estimate*

$$\|\mathbf{u}\|_{L^2(\widehat{K})} \lesssim \|\operatorname{div}\mathbf{u}\|_{L^2(\widehat{K})}, \tag{2.58}$$

where the constant is independent of p and $\mathbf{u}$, holds in the following cases:

(i) $\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ *satisfies* $(\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0$ *for all* $\mathbf{v} \in \mathbf{Q}_p(\widehat{K})$,

(ii) $\mathbf{u} \in \mathring{\mathbf{V}}_p(\widehat{K})$ *satisfies* $(\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0$ *for all* $\mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K})$,

(iii) $\mathbf{u} \in \mathring{\mathbf{V}}_p(\widehat{K})$ *satisfies* $(\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0$ *for all* $\mathbf{v} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$.

Proof. The statements (i) and (ii) are shown in [26, Lemma 5.2]. In order to prove (iii), assume that $\mathbf{u}$ satisfies the given condition. We then write $\mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K})$ as

$$\mathbf{v} = \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v} + (\mathbf{v} - \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v}),$$

where $\Pi_{\nabla \mathring{W}_{p+1}}$ denotes the L^2-projection onto $\nabla \mathring{W}_{p+1}(\widehat{K}) \subseteq \mathring{\mathbf{Q}}_p(\widehat{K})$. Since it follows $\mathbf{v} - \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$, we obtain

$$(\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = (\mathbf{u}, \underbrace{\mathbf{curl}(\Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v})}_{=0})_{L^2(\widehat{K})} + \underbrace{(\mathbf{u}, \mathbf{curl}(\mathbf{v} - \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v}))_{L^2(\widehat{K})}}_{=0, \text{ since } \mathbf{v} - \Pi_{\nabla \mathring{W}_{p+1}} \mathbf{v} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})} = 0.$$

Thus, $\mathbf{u}$ in fact satisfies condition (ii), and the estimate follows. □

3 The shift theorem in 2D

This chapter is about the regularity of the solutions of the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u|_\Gamma &= 0 \quad \text{on } \Gamma := \partial\Omega \end{aligned} \tag{3.1}$$

and the Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \partial_n u|_\Gamma &= 0 \quad \text{on } \Gamma \end{aligned} \tag{3.2}$$

for given right-hand side f in a Sobolev or Besov space in a polygonal domain $\Omega \subseteq \mathbb{R}^2$. The goal is the statement of the shift theorem and its proof for the limit case in Besov spaces (from which the result for right-hand sides in Sobolev spaces will follow rather easily). Contrary to the multilevel techniques used by Bacuta [7] and Bacuta and Bramble [8, 9] which lead to regularity in non-standard Besov spaces, we obtain regularity results in standard spaces by the use of the Mellin calculus (provided $\frac{\pi}{\omega} \notin \mathbb{N}$).
The Mellin calculus is a well-known tool for dealing with regularity in Sobolev spaces, cf. [20, 21, 23, 37, 41], and we review the methods in Sections 3.1-3.3 since they allow us to show an explicit representation of the solution near the corners, cf. Lemma 2.15 and Lemma 2.16. This representation is then the key ingredient for the proof of the shift theorem, which is based on localization procedures and interpolation theory. Note that localization naturally leads to considerations about regularity in sectors, see Subsection 3.1.2. These first three sections are mainly guided by [24], see also [25].
Section 3.4 deals with more regular right-hand sides and provides more necessary tools needed for the shift theorem. Finally, in Section 3.5 we state the shift theorems for both Dirichlet (Theorem 3.44) and Neumann (Theorem 3.47) boundary conditions.

3.1 The Mellin transform in weighted Sobolev spaces

3.1.1 The Fourier and Mellin transforms

Although our goal is to examine regularity properties of the solution u of (3.1), we do not directly analyze this problem, but instead work with a transformed version. We first introduce the Fourier transform, since the Mellin transform - which turns out to be the convenient choice for transforming equation (3.1) - is just the Fourier transform of a special function.

Definition 3.1. *The Fourier transform for functions $f \in L^1(\mathbb{R})$ is defined by*

$$(\mathcal{F}f)(\xi) := \widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx$$

for $\xi \in \mathbb{R}$.

The Fourier transform is a well-known tool. We state some of its properties.

Lemma 3.2. *(i) For functions* $f \in L^2(\mathbb{R})$*, the Fourier transform is defined as*

$$(\mathcal{F}f)(\xi) := \widehat{f}(\xi) := \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-ix\xi} f(x)\, dx,$$

where the limit is understood in the $L^2(\mathbb{R})$ *sense.*

(ii) $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is an isometric isomorphism and it holds the Plancherel identity* $(f, g)_{L^2(\mathbb{R})} = (\widehat{f}, \widehat{g})_{L^2(\mathbb{R})}$.

(iii) The inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{f}(\xi)\, d\xi$$

holds.

(iv) If $f \in L^2(\mathbb{R})$ *has compact support* $\operatorname{supp} f \subseteq [-B, B]$*, then* $\widehat{f}$ *can be extended to an entire function with* $|\widehat{f}(z)| \le Ce^{B|\operatorname{Im} B|}$.

(v) If $f \in L^2(\mathbb{R})$ *is supported in* $[0, \infty)$*, then* $\widehat{f}$ *is holomorphic on* $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$.

For a function f to be in the space $L^2(\mathbb{R})$ means to decay at infinity in some sense. If we even control the exponential behavior of f at $\pm\infty$, we can specifically determine the domain of holomorphy of its Fourier transformation.

Theorem 3.3 (Paley-Wiener)**.** *(i) Let* $a < b$ *and* $e^{ax} f \in L^2(\mathbb{R})$ *and* $e^{bx} f \in L^2(\mathbb{R})$*. Then:*

(a) $\widehat{f}$ *is holomorphic on the strip* $\{z \in \mathbb{C} : a < \operatorname{Im} z < b\}$.

(b) The equations

$$\|e^{ax} f\|^2_{L^2(\mathbb{R})} = \|\widehat{f}(\cdot + ia)\|^2_{L^2(\mathbb{R})} = \lim_{\eta\to a+} \|\widehat{f}(\cdot + i\eta)\|^2_{L^2(\mathbb{R})}$$

and

$$\|e^{bx} f\|^2_{L^2(\mathbb{R})} = \|\widehat{f}(\cdot + ib)\|^2_{L^2(\mathbb{R})} = \lim_{\eta\to b-} \|\widehat{f}(\cdot + i\eta)\|^2_{L^2(\mathbb{R})}$$

hold.

(c) It holds $\lim_{\eta\to a} \widehat{f}(\cdot + i\eta) = \widehat{f}(\cdot + ia)$ *and* $\lim_{\eta\to b} \widehat{f}(\cdot + i\eta) = \widehat{f}(\cdot + ib)$ *in the* $L^2(\mathbb{R})$ *sense.*

(d) For $\theta \in [0, 1]$*, there holds*

$$\|\widehat{f}(\cdot + i(\theta a + (1-\theta)b))\|_{L^2(\mathbb{R})} \le \|f(\cdot + ia)\|^{\theta}_{L^2(\mathbb{R})} \|f(\cdot + ib)\|^{1-\theta}_{L^2(\mathbb{R})}.$$

(ii) If a function $\widehat{f}$ is holomorphic on $\{z \in \mathbb{C} : a < \operatorname{Im} z < b\}$ with the additional assumption $\sup_{a<\eta<b} \|\widehat{f}(\cdot + i\eta)\|_{L^2(\mathbb{R})} < \infty$, then the inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\operatorname{Im}\zeta=\eta} \widehat{f}(\zeta) e^{i\zeta x}\, d\zeta$$

holds for any $a < \eta < b$. In this case, we also have $e^{ax} f \in L^2(\mathbb{R})$ and $e^{bx} f \in L^2(\mathbb{R})$.

Proof. For the proof of (ia), we get for a compact set $K \subseteq \{z \in \mathbb{C} : a < \operatorname{Im} z < b\}$ and $z \in K$

$$\begin{aligned}\int_{-\infty}^{\infty} |e^{-ixz} f(x)|\, dx \le \int_{-\infty}^{\infty} |f(x)| e^{x \operatorname{Im} z}\, dx &= \int_{-\infty}^{0} |f(x)| e^{x \operatorname{Im} z}\, dx + \int_{0}^{\infty} |f(x)| e^{x \operatorname{Im} z}\, dx \\ &\le \int_{-\infty}^{0} |f(x)| e^{ax} e^{x(\operatorname{Im} z - a)}\, dx + \int_{0}^{\infty} |f(x)| e^{bx} e^{x(\operatorname{Im} z - b)}\, dx \\ &\lesssim \|e^{ax} f\|_{L^2(\mathbb{R})} + \|e^{bx} f\|_{L^2(\mathbb{R})}.\end{aligned}$$

Thus, the integral of the Fourier transformation is absolutely and uniformly convergent, and the result follows. Items (ib) and (ic) follows immediately by the Plancherel identity and the dominated convergence theorem, since $e^{\eta x} \le \max(e^{ax}, e^{bx})$. Statement (id) follows with the Plancherel identity and Hölder's inequality from

$$\begin{aligned}\|\widehat{f}(\cdot + i(\theta a + (1-\theta) b))\|_{L^2(\mathbb{R})}^2 &= \|e^{(\theta a + (1-\theta) b) x} f\|_{L^2(\mathbb{R})}^2 \\ &= \int_{-\infty}^{\infty} e^{2\theta a x} |f(x)|^{2\theta} e^{2(1-\theta) b x} |f(x)|^{2(1-\theta)}\, dx \\ &\le \left(\int_{-\infty}^{\infty} e^{2ax} |f(x)|^2\, dx \right)^{\theta} \left(\int_{-\infty}^{\infty} e^{2bx} |f(x)|^2\, dx \right)^{1-\theta}.\end{aligned}$$

We now show (ii). We fix $\eta \in (a, b)$ and define, using that by assumption $\widehat{f}(\cdot + i\eta) \in L^2(\mathbb{R})$,

$$e^{\eta x} f(x) := \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{ix\xi} \widehat{f}(\xi + i\eta)\, d\xi.$$

The idea is to show independence of η by the Cauchy integral formula. Take an arbitrary $\eta' \in (a, b)$ that does not equal η. By holomorphy of $\widehat{f}$ on the strip, the value of the integral is independent of the path of integration, thus we have

$$\int_{\Gamma_\eta} e^{ixz} \widehat{f}(z)\, dz = \int_{\Gamma_l} e^{ixz} \widehat{f}(z)\, dz + \int_{\Gamma_{\eta'}} e^{ixz} \widehat{f}(z)\, dz + \int_{\Gamma_r} e^{ixz} \widehat{f}(z)\, dz, \tag{3.3}$$

where the paths of integration are shown in Figure 3.1.
By assumption we have $\sup_{\eta \le y \le \eta'} \int_{-\infty}^{\infty} |\widehat{f}(\xi + iy)|^2\, d\xi < \infty$. Hence, using Fubini's theorem,

$$\infty > \int_{\eta}^{\eta'} \int_{-\infty}^{\infty} |\widehat{f}(\xi + iy)|^2\, d\xi\, dy = \int_{-\infty}^{\infty} \int_{\eta}^{\eta'} |\widehat{f}(\xi + iy)|^2\, dy\, d\xi. \tag{3.4}$$

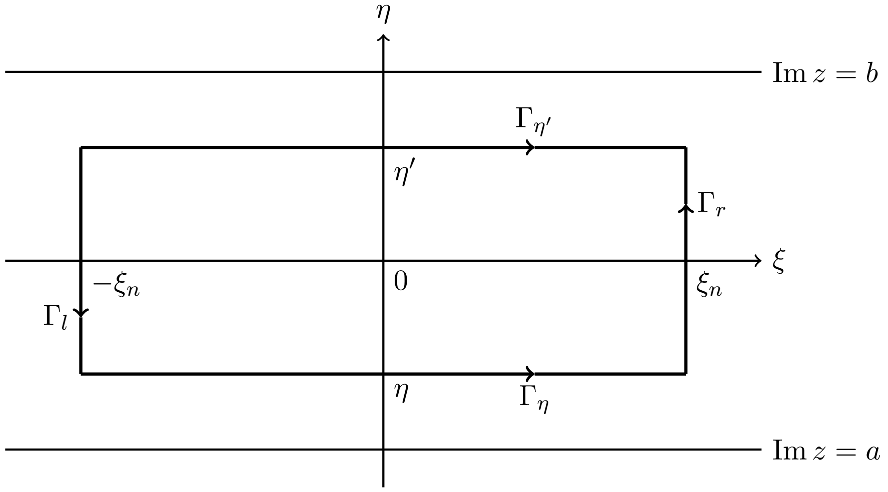

Figure 3.1: Paths of integration $(z = \xi + i\eta)$

With the definition

$$g(\xi) := \int_{\eta}^{\eta'} |\widehat{f}(\xi + iy)|^2 \, dy,$$

equality (3.4) implies the existence of a sequence $(\xi_n)_{n\in\mathbb{N}}$, $\xi_n \to \infty$, such that $g(\xi_n) \to 0$. From (3.3), we have

$$\lim_{n\to\infty} \int_{-\xi_n}^{\xi_n} e^{ix(\xi+i\eta)} \widehat{f}(\xi + i\eta) \, d\xi = \lim_{n\to\infty} \int_{-\xi_n}^{\xi_n} e^{ix(\xi+i\eta')} \widehat{f}(\xi + i\eta') \, d\xi. \tag{3.5}$$

Since $\widehat{f}(\cdot + i\eta)$ and $\widehat{f}(\cdot + i\eta')$ are both in $L^2(\mathbb{R})$, we get that the limits

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\xi_n}^{\xi_n} e^{ix\xi} \widehat{f}(\xi + i\eta) \, d\xi$$

and

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\xi_n}^{\xi_n} e^{ix\xi} \widehat{f}(\xi + i\eta') \, d\xi$$

exist in $L^2(\mathbb{R})$. Moreover,

$$f(x) = e^{-\eta x} \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\xi_n}^{\xi_n} e^{ix\xi} \widehat{f}(\xi + i\eta) \, d\xi = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\xi_n}^{\xi_n} e^{ix(\xi+i\eta)} \widehat{f}(\xi + i\eta) \, d\xi.$$

By passing, if necessary, to subsequences, the convergence is also pointwise a.e. We conclude with (3.5)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\operatorname{Im}\zeta=\eta} e^{ix\zeta} \widehat{f}(\zeta) \, d\zeta = \frac{1}{\sqrt{2\pi}} \int_{\operatorname{Im}\zeta=\eta'} e^{ix\zeta} \widehat{f}(\zeta) \, d\zeta,$$

where the convergence of both integrals is understood as Fourier transformation in $\xi \in \mathbb{R}$ ($\zeta = \xi + i\eta$). Thus the inversion formula is independent of the choice of the imaginary part. For the last part of the proof, note that we have for $\eta \in (a, b)$

$$\|e^{\eta x} f\|_{L^2(\mathbb{R})} = \|\widehat{f}(\cdot + i\eta)\|_{L^2(\mathbb{R})} \leq C$$

uniformly in η by Plancherel's identity. Hence for $x > 0$ and $\eta \to b-$, it follows $e^{\eta x}|f(x)| \nearrow e^{bx}|f(x)|$ and $\|e^{\eta x} f\|_{L^2(0,\infty)} \leq C$. The Monotone Convergence Theorem then implies $e^{bx} f \in L^2(0, \infty)$. The rest of the proof follows analogously. □

The following corollary is a simple application of the Paley-Wiener theorem by sending $a \to -\infty$ and taking $b = 0$.

Corollary 3.4. *(i) Let $f \in L^2(\mathbb{R})$ with $\operatorname{supp} f \subseteq [0, \infty)$. Then f is holomorphic on $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. Additionally, it holds*

$$\sup_{\eta<0} \|\widehat{f}(\cdot + i\eta)\|_{L^2(\mathbb{R})} \leq \|\widehat{f}\|_{L^2(\mathbb{R})}$$

and

$$\lim_{\eta\to 0} \|\widehat{f}(\cdot + i\eta) - \widehat{f}\|_{L^2(\mathbb{R})} = 0.$$

(ii) Conversely, a function $\widehat{f}$ that is holomorphic on $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ and satisfies $\sup_{\eta<0} \|\widehat{f}(\cdot + i\eta)\|_{L^2(\mathbb{R})} < \infty$, is the Fourier transformation of a function $f \in L^2(\mathbb{R})$ with $\operatorname{supp} f \subseteq [0, \infty)$.

The Mellin transform is closely related to the Fourier transform.

Definition 3.5. *Let $u = u(r) \in L^2(\mathbb{R}^+)$ be a function.*

(i) We define the function $\breve{u}$ by substitution $r = e^t$, i.e. $\breve{u}(t) := u(e^t)$.

(ii) The Mellin transformation of u is defined as the Fourier transformation of $\breve{u}$, i.e.

$$(\mathcal{M}u)(\xi) := (\mathcal{F}\breve{u})(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} \breve{u}(t)\, dt,$$

defined for all $\xi \in \mathbb{C}$ where the integral converges.

The Mellin transformation can also be written in other useful forms, i.e.

$$(\mathcal{M}u)(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-i\xi \ln r} u(r)\, \frac{dr}{r} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} r^{-i\xi} u(r)\, \frac{dr}{r}.$$

Results for the Fourier transform obviously carry over to the Mellin transform, e.g. the (formal) inverse of the Mellin transformation is given by

$$u(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{i\xi} (\mathcal{M}u)(\xi)\, d\xi.$$

Analogously to Corollary 3.4 one can show that for $u \in L^2(\mathbb{R}^+)$ with $u(r) = 0$ for $r \geq 1$, the Mellin transformation $\mathcal{M}u$ is holomorphic in the half-space $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. However, the domain of holomorphy can be determined more precisely, since we can apply the Paley-Wiener theorem to Mellin transformed functions.

Lemma 3.6. *For $u \in L^2(\mathbb{R}^+)$ and $\beta \in \mathbb{R}$ there holds the equivalence of norms*

$$\|r^\beta u\|_{L^2(\mathbb{R}^+)} \simeq \|(\mathcal{M}u)(\cdot + (\beta + \frac{1}{2})i)\|_{L^2(\mathbb{R})}.$$

If $\beta_1 < \beta_2$ and $r^{\beta_1} u \in L^2(\mathbb{R}^+)$ and $r^{\beta_2} u \in L^2(\mathbb{R}^+)$, then $\mathcal{M}u$ is holomorphic on the strip

$$\{z \in \mathbb{C} : \beta_1 + \frac{1}{2} < \operatorname{Im} z < \beta_2 + \frac{1}{2}\}.$$

Proof. Since $r^\beta u \in L^2(\mathbb{R}^+)$ implies

$$\infty > \int_0^\infty r^{2\beta} |u|^2 \, dr = \int_{-\infty}^\infty e^{2\beta t} |\check{u}|^2 e^t \, dt,$$

it follows $e^{\beta + \frac{1}{2} t} \check{u} \in L^2(\mathbb{R})$ and further that $(\mathcal{M}u)(\cdot + (\beta + \frac{1}{2})i) \in L^2(\mathbb{R})$ with equivalent norms by Plancherel. The other implication is analogous.
The result about holomorphy follows then as a direct consequence from Theorem 3.3. □

Sometimes, we can extend the Mellin transformation meromorphically into the lower half-plane. The extension is then holomorphic apart from simple poles that appear for negative integers on the imaginary axis.

Lemma 3.7. *Let $u \in C^{k,\sigma}([0, \infty))$ with $u(r) = 0$ for $r \geq 1$. Then $\mathcal{M}u$ has a meromorphic extension to $\{z \in \mathbb{C} : \operatorname{Im} z > -(k + \sigma)\}$ with simple poles at $\{-it : t = 0, \ldots, k\}$.*

Proof. We use a Taylor expansion to write

$$u(r) = \sum_{j=0}^k \frac{u^{(j)}(0)}{j!} r^j \chi_{(0,1)}(r) + R(r),$$

where the remainder can be estimated by

$$|R(r)| \leq \begin{cases} Cr^{k+\sigma}, & r \in (0, 1) \\ 0, & r \geq 1 \end{cases}.$$

Transformation leads to

$$(\mathcal{M}u)(z) = \sum_{j=0}^k \frac{u^{(j)}(0)}{j!} \mathcal{M}(r^j \chi_{(0,1)}(r))(z) + (\mathcal{M}R)(z).$$

Since $|\check{R}(t)| \leq Ce^{t(k+\sigma)}$ with $\operatorname{supp} \check{R} \subseteq (-\infty, 0]$, the Mellin transformation $\mathcal{M}R$ is holomorphic on $\{z \in \mathbb{C} : \operatorname{Im} z > -(k + \sigma)\}$, cf. Theorem 3.3. For the other term we can calculate

$$\mathcal{M}(r^j \chi_{(0,1)}(r))(z) = \frac{1}{\sqrt{2\pi}} \int_0^1 r^{j - iz} \frac{dr}{r} = \frac{1}{\sqrt{2\pi}} \frac{1}{j - iz}.$$

Thus we obtain

$$(\mathcal{M}u)(z) = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^k \frac{u^{(j)}(0)}{j!} \frac{1}{j - iz} + (\mathcal{M}R)(z).$$

The left-hand side is a priori defined for $\operatorname{Im} z > 0$. However, the right-hand side is meromorphic on $\{z \in \mathbb{C} : \operatorname{Im} z > -(k + \sigma)\}$ with simple poles. Thus $\mathcal{M}u$ has a meromorphic extension with the desired properties. □

3.1.2 The weighted spaces K^s_γ

For the rest of this chapter, our underlying domain is the cone $\mathcal{C}$, defined by

$$\mathcal{C} := \{(r\cos\phi, r\sin\phi) : r > 0, \phi \in G\}, \quad \text{where } G := (0,\omega) \tag{3.6}$$

with the angle $\omega \in (0, 2\pi)$. The boundary of $\mathcal{C}$ is denoted by Γ. The domain $\mathcal{C}$ has a corner at the origin, and one part of its boundary coincides with the positive x-axis. This choice may appear a little arbitrary, but helps reduce the complexity of calculations. Obviously, other cones can be considered after coordinate transformations.
Note that in the previous chapter, the boundary of the bounded domain Ω was also denoted by Γ, as is the usual notation. The meaning of Γ in every single case should be clear from the context without need of more precise distinction.
For $R > 0$ we use the abbreviation

$$\mathcal{C}_R := \mathcal{C} \cap B_R(0). \tag{3.7}$$

We also define for $0 < \rho < \sigma$ the annuli

$$A(\rho,\sigma) := \{x \in \mathcal{C} : \rho < |x| < \sigma\}. \tag{3.8}$$

We also need Sobolev spaces defined on the cone $\mathcal{C}$ vanishing on the boundary Γ. Thus, we introduce the space

$$H^1_{0,\Gamma}(\mathcal{C}) := \{u \text{ measurable} : u|_{A(R,2R)} \in H^1(A(R,2R)), u|_{\overline{A(R,2R)}\cap\Gamma} = 0 \,\forall R > 0\}.$$

The previous results, especially Lemma 3.6, suggest that it might be useful to consider functions in $L^2(\mathcal{C})$ with stronger decay properties near the corner. This can be established by considering functions in the following weighted subspace of $L^2(\mathcal{C})$.

Definition 3.8. *For $s \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$, we define*

$$K^s_\gamma(\mathcal{C}) := \{u \in L^2_{loc}(\mathcal{C}) : r^{|\alpha|-s+\gamma} D^\alpha u \in L^2(\mathcal{C}), |\alpha| \leq s\}$$

with the natural norm

$$\|u\|^2_{K^s_\gamma(\mathcal{C})} = \sum_{|\alpha|\leq s} \|r^{|\alpha|-s+\gamma} D^\alpha u\|^2_{L^2(\mathcal{C})}.$$

The spaces $K^s_\gamma(\mathcal{C}_R)$ are defined in the same way, just by replacing $\mathcal{C}$ by $\mathcal{C}_R$.

The following statements can be useful when dealing with this type of weighted L^2-spaces.

Lemma 3.9. *The following statements hold true.*

(i) For $|\alpha| \leq s$, the operator $D^\alpha : K^s_\gamma(\mathcal{C}) \to K^{s-|\alpha|}_\gamma(\mathcal{C})$ is linear and bounded.

(ii) The space $K^s_\gamma(\mathcal{C})$ is a subset of $K^0_{\gamma-s}(\mathcal{C})$.

(iii) *For $s' < s$, $\gamma' > \gamma$ and $R > 0$, the inclusions*

$$K^s_\gamma(\mathcal{C}_R) \subseteq K^{s'}_\gamma(\mathcal{C}_R), \quad K^s_\gamma(\mathcal{C}_R) \subseteq K^s_{\gamma'}(\mathcal{C}_R)$$

hold. Note that this is not true on $\mathcal{C}$!

(iv) *For $\gamma \le 0$, we have $K^s_\gamma(\mathcal{C}_R) \subseteq H^s(\mathcal{C}_R)$.*

(v) *For $\gamma \ge s$, we have $H^s(\mathcal{C}_R) \subseteq K^s_\gamma(\mathcal{C}_R)$.*

The following lemma is about a regularity estimate up to the boundary of the cone $\mathcal{C}$.

Lemma 3.10. *With the notation of (3.8), define the annuli $\widehat{A_1} := A(1/2, 2)$, $\widehat{A_2} := A(1/4, 4)$ and $\widehat{A_3} := A(1/8, 8)$. Then the following statements hold:*

(i) *Let $k \in \mathbb{N}_0$ and $\epsilon \in (0,1)$. Further let $f \in H^{k+\epsilon}(\widehat{A_3})$ and $u \in H^1(\widehat{A_3})$ satisfy $-\Delta u = f$ with zero Dirichlet boundary conditions on $\partial\widehat{A_3} \cap \partial\mathcal{C}$. Then it follows $u \in H^{k+2+\epsilon}(\widehat{A_1})$ with the regularity estimate*

$$|D^{k+2}u|_{H^\epsilon(\widehat{A_1})} \lesssim \|f\|_{H^{k+\epsilon}(\widehat{A_2})} + \|u\|_{H^{k+2}(\widehat{A_2})}.$$

(ii) *Let $k \in \mathbb{N}_0$ and $\epsilon \in (0,1)$. Further let $f \in H^{k+\epsilon}(\widehat{A_3})$ and $u \in H^1(\widehat{A_3})$ satisfy $-\Delta u = f$ with zero Neumann boundary conditions on $\partial\widehat{A_3} \cap \partial\mathcal{C}$. Then it follows $u \in H^{k+2+\epsilon}(\widehat{A_1})$ with the regularity estimate*

$$|D^{k+2}u|_{H^\epsilon(\widehat{A_1})} \lesssim \|f\|_{H^{k+\epsilon}(\widehat{A_2})} + \|u\|_{H^{k+2}(\widehat{A_2})}.$$

Proof. We start with (i). We first mention that the elliptic regularity estimate

$$\|u\|_{H^{k+2}(\widehat{A_2})} \lesssim \|f\|_{H^k(\widehat{A_3})} + \|u\|_{L^2(\widehat{A_3})} \tag{3.9}$$

(cf. [24, eq. (4.3)]) shows $u \in H^{k+2}(\widehat{A_2})$.
We select a smooth cut-off function χ with support on $A(3/8, 3)$ which equals one on $\widehat{A_1}$. The function χu then satisfies the equation

$$-\Delta(\chi u) = \chi f - 2\nabla\chi \cdot \nabla u - \Delta\chi u =: \widehat{f} \tag{3.10}$$

with the boundary condition $\chi u = 0$ on $\partial\widehat{A_2}$. Since we have

$$\begin{aligned} \|u\|_{H^{k+\epsilon}(\widehat{A_2})} + \|\nabla u\|_{\mathbf{H}^{k+\epsilon}(\widehat{A_2})} &\le \|u\|_{H^{k+2}(\widehat{A_2})} + \|\nabla u\|_{\mathbf{H}^{k+1}(\widehat{A_2})} \lesssim \|u\|_{H^{k+2}(\widehat{A_2})} \\ &\overset{(3.9)}{\lesssim} \|f\|_{H^k(\widehat{A_3})} + \|u\|_{L^2(\widehat{A_3})}, \end{aligned} \tag{3.11}$$

it follows $\widehat{f} \in H^{k+\epsilon}(\widehat{A_2})$. Note that we can smoothen the boundary of $\widehat{A_2}$ near the corners to get a C^∞-domain without changing the functions, since all functions in (3.10) are identically zero near the corners. This method allows us to use regularity results on smooth domains, and we obtain $\chi u \in H^{k+2+\epsilon}(\widehat{A_1})$ with the estimate

$$\|\chi u\|_{H^{k+2+\epsilon}(\widehat{A_1})} \lesssim \|\widehat{f}\|_{H^{k+\epsilon}(\widehat{A_2})}$$

which follows by interpolation. The proof is complete in view of

$$\begin{aligned}\|u\|_{H^{k+2+\epsilon}(\widehat{A_1})} &= \|\chi u\|_{H^{k+2+\epsilon}(\widehat{A_1})} \lesssim \|\widehat{f}\|_{H^{k+\epsilon}(\widehat{A_2})} \\ &\lesssim \|f\|_{H^{k+\epsilon}(\widehat{A_2})} + \|\nabla u\|_{\mathbf{H}^{k+\epsilon}(\widehat{A_2})} + \|u\|_{H^{k+\epsilon}(\widehat{A_2})} \\ &\overset{(3.11)}{\lesssim} \|f\|_{H^{k+\epsilon}(\widehat{A_2})} + \|u\|_{H^{k+2}(\widehat{A_2})}.\end{aligned}$$

The proof of (ii) is similar to (i). Here we use the elliptic regularity estimate from [45, Thm. 4.18],

$$\|u\|_{H^{k+2}(\widehat{A_2})} \lesssim \|f\|_{H^{k}(\widehat{A_3})} + \|u\|_{H^{1}(\widehat{A_3})}. \tag{3.12}$$

We now introduce the cut-off function χ which only has support on $A(3/8, 3)$, equals one in $\widehat{A_1}$ and additionally satisfies $\partial_n \chi = 0$ on $\partial\widehat{A_3}$. Such a function can be constructed as $\chi(r, \phi) = \chi_1(r)\chi_2(\phi)$ in polar coordinates where χ_2 is constant and χ_1 satisfies the desired properties. The function χu now satisfies the equation

$$-\Delta(\chi u) = \chi f - 2\nabla\chi \cdot \nabla u - \Delta\chi u =: \widehat{f}$$

with boundary condition $\partial_n(\chi u) = \chi\partial_n u + u\partial_n\chi = 0$. Since we have

$$\begin{aligned}\|u\|_{H^{k+\epsilon}(\widehat{A_2})} + \|\nabla u\|_{\mathbf{H}^{k+\epsilon}(\widehat{A_2})} &\leq \|u\|_{H^{k+2}(\widehat{A_2})} + \|\nabla u\|_{\mathbf{H}^{k+1}(\widehat{A_2})} \lesssim \|u\|_{H^{k+2}(\widehat{A_2})} \\ &\overset{(3.12)}{\lesssim} \|f\|_{H^{k}(\widehat{A_3})} + \|u\|_{H^{1}(\widehat{A_3})},\end{aligned} \tag{3.13}$$

it follows $\widehat{f} \in H^{k+\epsilon}(\widehat{A_2})$. The same smoothing process of the domain as above now gives $\chi u \in H^{k+2+\epsilon}(\widehat{A_1})$ with the estimate

$$\|\chi u\|_{H^{k+2+\epsilon}(\widehat{A_1})} \lesssim \|\widehat{f}\|_{H^{k+\epsilon}(\widehat{A_2})}.$$

The proof of (ii) is complete in view of

$$\begin{aligned}\|u\|_{H^{k+2+\epsilon}(\widehat{A_1})} &= \|\chi u\|_{H^{k+2+\epsilon}(\widehat{A_1})} \lesssim \|\widehat{f}\|_{H^{k+\epsilon}(\widehat{A_2})} \\ &\lesssim \|f\|_{H^{k+\epsilon}(\widehat{A_2})} + \|\nabla u\|_{\mathbf{H}^{k+\epsilon}(\widehat{A_2})} + \|u\|_{H^{k+\epsilon}(\widehat{A_2})} \\ &\overset{(3.13)}{\lesssim} \|f\|_{H^{k+\epsilon}(\widehat{A_2})} + \|u\|_{H^{k+2}(\widehat{A_2})}.\end{aligned}$$

□

The following lemma demonstrates a type of scaling arguments that we will sometimes use when dealing with the weighted spaces. We want to remind the reader of the notation introduced in (2.2).

Lemma 3.11. *Let $k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$ and $\epsilon \in (0,1)$. Further let $f \in L^2_{loc}(\mathcal{C})$ and $u \in H^1_{0,\Gamma}(\mathcal{C})$ satisfy $-\Delta u = f$ with zero Dirichlet boundary conditions.*

(i) For $f \in K^k_\gamma(\mathcal{C})$, $u \in K^0_{\gamma-k-2}(\mathcal{C})$ and $\beta \in (\mathbb{N}_0)^2$, $|\beta| \in [0, k+2]$, it holds

$$\|r^{-|\beta|+\gamma} D^{k+2-|\beta|} u\|_{L^2(\mathcal{C})} \lesssim \sum_{|\alpha|\leq k} \|r^{|\alpha|-k+\gamma} D^\alpha f\|_{L^2(\mathcal{C})} + \|r^{-k-2+\gamma} u\|_{L^2(\mathcal{C})}, \tag{3.14}$$

where $r = |\mathbf{x}|$.

(ii) Let $f \in H^{k+\epsilon}(\mathcal{C}_1)$ with compact support in $B_1(0)$. Further assume that there exists $R > 2$ such that u satisfies

$$\sum_{|\alpha|\leq k-2} \|r^{|\alpha|-k-2-\epsilon} D^\alpha u\|_{L^2(\mathcal{C}_R)} < \infty.$$

Then it holds

$$|D^{k+2} u|_{H^\epsilon(\mathcal{C}_1)} \lesssim |D^k f|_{H^\epsilon(\mathcal{C}_1)} + \sum_{|\alpha|\leq k} \|r^{|\alpha|-k-\epsilon} D^\alpha f\|_{L^2(\mathcal{C}_1)} + \sum_{|\alpha|\leq k-2} \|r^{|\alpha|-k-2-\epsilon} D^\alpha u\|_{L^2(\mathcal{C}_R)},$$

where $r = |\mathbf{x}|$ and $R > 2$.

Proof. Define the annuli $\widehat{A_1} := A(1/2, 2)$ and $\widehat{A_2} := A(1/4, 4)$, cf. (3.8). For $\rho > 0$ scaling yields for $A_{i,\rho} := \rho \widehat{A_i}$, $i = 1, 2$, $\widehat{u}(\xi) := u(\rho\xi)$ and $\widehat{f}(\xi) := f(\rho\xi)$ that $-\Delta\widehat{u} = \rho^2 \widehat{f}$ on $\widehat{A_2}$. We start the proof of (i) by noting the elliptic regularity estimate

$$\|\widehat{u}\|_{H^{k+2}(\widehat{A_1})} \lesssim \rho^2 \|\widehat{f}\|_{H^k(\widehat{A_2})} + \|\widehat{u}\|_{L^2(\widehat{A_2})}, \tag{3.15}$$

cf. [24, eq. (4.3)]. We multiply (3.15) by $\rho^{\gamma-k-2}$ and obtain after scaling

$$\sum_{|\alpha|\leq k+2} \rho^{2|\alpha|+2\gamma-2k-4} \|D^\alpha u\|^2_{L^2(A_{1,\rho})} \lesssim \sum_{|\alpha|\leq k} \rho^{2|\alpha|+2\gamma-2k} \|D^\alpha f\|^2_{L^2(A_{2,\rho})} + \rho^{2\gamma-2k-4} \|u\|^2_{L^2(A_{2,\rho})}.$$

The definition of the annuli implies $2^{-i}\rho < r < 2^i \rho$, $i = 1, 2$, on $A_{i,\rho}$, thus we get further

$$\sum_{|\alpha|\leq k+2} \|r^{|\alpha|+\gamma-k-2} D^\alpha u\|^2_{L^2(A_{1,\rho})} \lesssim \sum_{|\alpha|\leq k} \|r^{|\alpha|+\gamma-k} D^\alpha f\|^2_{L^2(A_{2,\rho})} + \|r^{\gamma-k-2} u\|^2_{L^2(A_{2,\rho})}.$$

We now cover $\mathcal{C}$ by annuli $A_{1,2^{-j}}$, $j \in \mathbb{Z}$. Since they only have finite overlap, we obtain

$$\begin{aligned}\|r^{-|\beta|+\gamma} D^{k+2-|\beta|} u\|^2_{L^2(\mathcal{C})} &\leq \sum_{|\alpha|\leq k+2} \|r^{|\alpha|+\gamma-k-2} D^\alpha u\|^2_{L^2(\mathcal{C})} \\ &\lesssim \sum_{|\alpha|\leq k} \|r^{|\alpha|+\gamma-k} D^\alpha f\|^2_{L^2(\mathcal{C})} + \|r^{\gamma-k-2} u\|^2_{L^2(\mathcal{C})}\end{aligned}$$

for $|\beta| \in [0, k+2]$, whereupon the result follows.

The proof of (ii) follows in a similar way with the regularity estimate

$$|D^{k+2}\widehat{u}|_{H^\epsilon(\widehat{A_1})} \lesssim \rho^2\|\widehat{f}\|_{H^{k+\epsilon}(\widehat{A_2})} + \|\widehat{u}\|_{H^{k+2}(\widehat{A_2})}, \tag{3.16}$$

which follows from Lemma 3.10 after scaling with ρ, cf. the lines before (3.15). Here the usual scaling arguments yield

$$\begin{aligned}
|D^{k+2}u|^2_{H^\epsilon(A_{1,\rho})} &\lesssim \rho^{2-2(k+2+\epsilon)}|D^{k+2}\widehat{u}|^2_{H^\epsilon(\widehat{A_1})} \\
&\lesssim \rho^{2-2(k+2+\epsilon)}\left(\rho^4\sum_{|\alpha|\le k}\|D^\alpha\widehat{f}\|^2_{L^2(\widehat{A_2})} + \rho^4|D^k\widehat{f}|^2_{H^\epsilon(\widehat{A_2})} + \sum_{|\alpha|\le k+2}\|D^\alpha\widehat{u}\|^2_{L^2(\widehat{A_2})}\right) \\
&\lesssim \rho^{2-2(k+2+\epsilon)}\Bigg(\rho^4\sum_{|\alpha|\le k}\rho^{-2+2|\alpha|}\|D^\alpha f\|^2_{L^2(A_{2,\rho})} + \rho^4\rho^{-2+2(k+\epsilon)}|D^k f|^2_{H^\epsilon(A_{2,\rho})} \\
&\qquad + \sum_{|\alpha|\le k+2}\rho^{-2+2|\alpha|}\|D^\alpha u\|^2_{L^2(A_{2,\rho})}\Bigg) \\
&\lesssim \sum_{|\alpha|\le k}\rho^{-2k-2\epsilon+2|\alpha|}\|D^\alpha f\|^2_{L^2(A_{2,\rho})} + |D^k f|^2_{H^\epsilon(A_{2,\rho})} + \sum_{|\alpha|\le k+2}\rho^{-2k-2\epsilon-4+2|\alpha|}\|D^\alpha u\|^2_{L^2(A_{2,\rho})}.
\end{aligned}$$

As in (i) we obtain by covering arguments

$$\begin{aligned}
|D^{k+2}u|^2_{H^\epsilon(\mathcal{C}_1)} \lesssim \sum_{|\alpha|\le k}\|r^{-k-\epsilon+|\alpha|}D^\alpha f\|^2_{L^2(\mathcal{C}_1)} + |D^k f|^2_{H^\epsilon(\mathcal{C}_1)} \\
+ \sum_{|\alpha|\le k+2}\|r^{-k-\epsilon-2+|\alpha|}D^\alpha u\|^2_{L^2(\mathcal{C}_R)}.
\end{aligned}$$

□

A simple application of the weighted spaces is the following sort of shift theorem. It is obtained as an immediate consequence of Lemma 3.11, using the same notation, by adding over $|\beta| \le k+2$ and suitable variable substitutions.

Lemma 3.12. *Let $s \in \mathbb{N}$, $s \ge 2$ and $\gamma \in \mathbb{R}$, and let $f \in K^{s-2}_\gamma(\mathcal{C})$ and $u \in H^1_{0,\Gamma}(\mathcal{C})$ satisfy $-\Delta u = f$ with zero Dirichlet boundary condition. Then the estimate*

$$\|u\|_{K^s_\gamma(\mathcal{C})} \lesssim \|f\|_{K^{s-2}_\gamma(\mathcal{C})} + \|u\|_{K^0_{\gamma-s}(\mathcal{C})}$$

holds in the sense, that if the right-hand side is finite, then $u \in K^s_\gamma(\mathcal{C})$.

Remark 3.13. Lemma 3.12 can be generalized to elliptic operators L of order $2m$: For $s \in \mathbb{N}$, $s \ge 2m$, let $f \in K^{s-2m}_\gamma(\mathcal{C})$ and $u \in H^m(\mathcal{C}_R)$ for all $R > 0$ satisfy $Lu = f$ with Dirichlet boundary conditions $\partial^j_n u = 0$, $j = 0, \dots, m-1$ on Γ. If $u \in K^0_{\gamma-s}(\mathcal{C})$, then $u \in K^s_\gamma(\mathcal{C})$, and the estimate

$$\|u\|_{K^s_\gamma(\mathcal{C})} \lesssim \|f\|_{K^{s-2m}_\gamma(\mathcal{C})} + \|u\|_{K^0_{\gamma-s}(\mathcal{C})}$$

holds.

Remark 3.14. Note that Lemma 3.12 and Remark 3.13 also hold for zero Neumann boundary conditions. Line (3.15) also holds in the Neumann setting, whereas the following estimates are only based on scaling arguments which are independent of the type of boundary conditions.

Lemma 3.15. *For every $R > 0$, it holds $H_0^1(\mathcal{C}_R) \subseteq K_0^1(\mathcal{C}_R)$ with the norm estimate $\|u\|_{K_0^1(\mathcal{C}_R)} \lesssim \|u\|_{H^1(\mathcal{C}_R)}$.*

Proof. We use the notation from the proof of Lemma 3.11. Let $u \in H_0^1(\mathcal{C}_R)$. The Poincaré inequality gives after scaling

$$\frac{1}{\rho}\|u\|_{L^2(A_{1,\rho})} \lesssim \|\nabla u\|_{L^2(A_{1,\rho})}.$$

Thus a covering argument implies

$$\left\|\frac{1}{r}u\right\|_{L^2(\mathcal{C}_R)} \lesssim \|\nabla u\|_{L^2(\mathcal{C}_R)}.$$

□

Notation 3.16. *We will use the following notation for working in polar coordinates: For $u = u(x)$, we write $u(x) = \widetilde{u}(r,\theta)$. After substitution of r with e^t as before, we write $\check{u}(t,\theta) = \widetilde{u}(e^t,\theta)$.*

The following lemma deals with transformation rules for differentiation operators. It shows how derivatives of u are related to $\widetilde{u}$ and $\check{u}$.

Lemma 3.17 ([24, Lemme 5.2])**.** *The following two statements hold.*

(i) For $\boldsymbol{\alpha} \in (\mathbb{N}_0)^2$, $\boldsymbol{\alpha} \neq 0$, there exist functions $d_{\boldsymbol{\alpha}\boldsymbol{\beta}} \in C^\infty(\partial B_1(0))$ such that

$$D_x^{\boldsymbol{\alpha}} u = \sum_{0<|\boldsymbol{\beta}|\leq|\boldsymbol{\alpha}|} r^{-|\boldsymbol{\alpha}|} d_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\theta)(r\partial_r)^{\beta_1} D_\theta^{\beta'}\widetilde{u},$$

where $\boldsymbol{\beta} = (\beta_1, \beta')$. In the same way,

$$D_x^{\boldsymbol{\alpha}} u = \sum_{0<|\boldsymbol{\beta}|\leq|\boldsymbol{\alpha}|} e^{-|\boldsymbol{\alpha}|t} d_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\theta) D_{(t,\theta)}^{\boldsymbol{\beta}}\check{u}.$$

(ii) Conversely, for $\boldsymbol{\beta} \in (\mathbb{N}_0)^2$, $\boldsymbol{\beta} \neq 0$, there exist functions $d^\star_{\boldsymbol{\alpha}\boldsymbol{\beta}} \in C^\infty(\partial B_1(0))$ such that

$$D_{(t,\theta)}^{\boldsymbol{\beta}}\check{u} = \sum_{0<|\boldsymbol{\alpha}|\leq|\boldsymbol{\beta}|} r^{|\boldsymbol{\alpha}|} d^\star_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\theta) D_x^{\boldsymbol{\alpha}} u.$$

A direct consequence of Lemma 3.17 is seen in the next lemma, which treats the transformation of the spaces $K_\gamma^s(\mathcal{C})$ under a change of variables.

Lemma 3.18 ([24, Lemme 5.3]). *For $\gamma \in \mathbb{R}$ and $s \in \mathbb{N}_0$, there holds an isomorphism between the space $K^s_\gamma(\mathcal{C})$ and the space*

$$\{\breve{u} \in L^2_{loc}(\mathbb{R} \times G) : e^{t(\gamma-s+1)}\breve{u} \in H^s(\mathbb{R} \times G)\}$$

by change of variables $x \leftrightarrow (t, \theta)$, cf. (3.6).

Proof. For $|\boldsymbol{\beta}| \le s$ and $u \in K^s_\gamma(\mathcal{C})$, we have with Lemma 3.17

$$|e^{t(\gamma-s+1)}D^{\boldsymbol{\beta}}\breve{u}| \le \sum_{|\boldsymbol{\alpha}|\le|\boldsymbol{\beta}|} r^{|\boldsymbol{\alpha}|}r^{\gamma-s+1}|d^\star_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\theta)||D^{\boldsymbol{\alpha}}_x u| = \sum_{|\boldsymbol{\alpha}|\le|\boldsymbol{\beta}|} r^{\gamma-s+|\boldsymbol{\alpha}|}|d^\star_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\theta)||D^{\boldsymbol{\alpha}}_x u|r.$$

Thus it follows

$$\begin{aligned}\|e^{t(\gamma-s+1)}D^{\boldsymbol{\beta}}\breve{u}\|^2_{L^2(\mathbb{R}\times G)} &\lesssim \sum_{|\boldsymbol{\alpha}|\le|\boldsymbol{\beta}|} \int_0^\infty \int_0^\omega r^{2(\gamma-s+|\boldsymbol{\alpha}|)}|D^{\boldsymbol{\alpha}}_x u|^2 r\, d\theta\, dr \\ &\lesssim \sum_{|\boldsymbol{\alpha}|\le|\boldsymbol{\beta}|} \int_{\mathcal{C}} r^{2(\gamma-s+|\boldsymbol{\alpha}|)}|D^{\boldsymbol{\alpha}}_x u|^2\, dx < \infty,\end{aligned}$$

since $u \in K^s_\gamma(\mathcal{C})$. This shows one implication, the other direction follows in a similar way. ☐

Next, we want to take Mellin transformations of functions $u \in K^s_\gamma(\mathcal{C})$ into account. Such Mellin transformations are to be seen as transformations with respect to the variable r, hence leaving the variable θ as parameter. This motivates the approach to consider the appearing functions not as functions in $H^s(\mathbb{R} \times G)$ as in the preceding lemma, but to view them in the space $L^2(\mathbb{R}; H^s(G))$, i.e. as functions with values in the Hilbert space $H^s(G)$. Note that the Paley-Wiener theorem and its corollaries also hold in this setting.

Theorem 3.19. *Let $s \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$, and set $\eta := s - \gamma - 1$.*

(i) Let $u \in K^s_\gamma(\mathcal{C})$. Then $e^{t(\gamma-s+1)}\breve{u} \in L^2(\mathbb{R}; H^s(G))$ and $\mathcal{M}u(\cdot - i\eta) \in L^2(\mathbb{R}; H^s(G))$. Additionally, the norms satisfy

$$\|u\|^2_{K^s_\gamma(\mathcal{C})} \simeq \int_{\xi\in\mathbb{R}} \|(\mathcal{M}u)(\xi - i\eta)\|^2_{H^s(G;|\xi|)}\, d\xi, \tag{3.17}$$

where the weighted norm is defined as

$$\|v\|^2_{H^s(G;\rho)} := \sum_{|\boldsymbol{\beta}|\le s} \rho^{2\beta_1}\|D^{\beta'}v\|^2_{L^2(G)} \simeq \sum_{\beta'\le s}(1+\rho)^{2(s-\beta')}\|D^{\beta'}v\|^2_{L^2(G)} \tag{3.18}$$

with $\boldsymbol{\beta} = (\beta_1, \beta')$.

(ii) Conversely, let $U(\cdot - i\eta) \in L^2(\mathbb{R}; H^s(G))$ with $\int_{\xi\in\mathbb{R}} \|U(\xi - i\eta)\|^2_{H^s(G;|\xi|)}\, d\xi < \infty$. Then $U(\cdot - i\eta)$ is the Mellin transformation of a function $u \in K^s_\gamma(\mathcal{C})$.

Proof. We show (i). Note that by Lemma 3.18, $u \in K^s_\gamma(\mathcal{C})$ implies $e^{-\eta t}\breve{u} \in H^s(\mathbb{R} \times G)$ and $e^{-\eta t}\breve{u} \in L^2(\mathbb{R}; H^s(G))$. Hence $(\mathcal{M}u)(\cdot - i\eta) \in L^2(\mathbb{R}; H^s(G))$, cf. Lemma 3.6. From Parseval, we get

$$\|\partial_t^{\beta_1}\partial_\theta^{\beta'}(e^{-\eta t}\breve{u})\|^2_{L^2(\mathbb{R})} \simeq \||\xi|^{\beta_1}\partial_\theta^{\beta'}(\mathcal{M}u)(\cdot - i\eta)\|^2_{L^2(\mathbb{R})}.$$

Thus it follows with $\boldsymbol{\beta} = (\beta_1, \beta')$

$$\begin{aligned}\|u\|^2_{K^s_\gamma(\mathcal{C})} \simeq \|e^{-\eta t}\breve{u}\|^2_{H^s(\mathbb{R}\times G)} &\simeq \sum_{|\boldsymbol{\beta}|\le s} \|\partial_t^{\beta_1}\partial_\theta^{\beta'}(e^{-\eta t}\breve{u})\|^2_{L^2(\mathbb{R}\times G)} \\ &\simeq \sum_{|\boldsymbol{\beta}|\le s} \int_{\xi\in\mathbb{R}} |\xi|^{2\beta_1}\|\partial_\theta^{\beta'}(\mathcal{M}u)(\xi - i\eta)\|^2_{L^2(G)}\, d\xi,\end{aligned}$$

which shows (3.17). The norm equivalence in (3.18) is seen by

$$\sum_{|\boldsymbol{\beta}|\le s} \rho^{2\beta_1}\|D^{\beta'}v\|^2_{L^2(G)} = \sum_{\beta'\le s}\|D^{\beta'}v\|^2_{L^2(G)} \sum_{\beta_1\le s-\beta'} \rho^{2\beta_1} \simeq \sum_{\beta'\le s}(1+\rho)^{2(s-\beta')}\|D^{\beta'}v\|^2_{L^2(G)}.$$

For the proof of (ii), there exists a function $e^{-\eta t}\breve{u} \in L^2(\mathbb{R}; H^s(G))$ that is the inverse Fourier transformation of $U(\cdot - i\eta)$. By Parseval, we have in fact

$$\|e^{-\eta t}\breve{u}\|^2_{H^s(\mathbb{R}\times G)} \simeq \int_{\xi\in\mathbb{R}} \|U(\cdot - i\eta)\|^2_{H^s(G;|\xi|)}\, d\xi,$$

and by the norm equivalence (3.17) we see $u \in K^s_\gamma(\mathcal{C})$. ☐

3.2 Decomposition of the solution - Dirichlet

For $s \ge 2$, we consider the following problem

$$\begin{aligned} -\Delta u &= f \in K^{s-2}_\gamma(\mathcal{C}) \\ u &= 0, \quad \phi \in \{0, \omega\}. \end{aligned} \tag{3.19}$$

Since we do not prescribe the behavior of the solution for $r \to 0$ and $r \to \infty$, we cannot expect uniqueness of the solution unless we specify the energy space. In fact, two solutions $u_\gamma \in K^s_\gamma(\mathcal{C})$ and $u_\delta \in K^s_\delta(\mathcal{C})$ may be different.
Our goal in this section is to rewrite equation (3.19) in polar coordinates and then apply the Mellin transform. Problem (3.19) written in polar coordinates gives

$$-\left((r\partial_r)^2\tilde{u} + \partial_\phi^2\tilde{u}\right) = r^2\tilde{f},$$

or, equivalently upon substitution of $r = e^t$ as before,

$$-\left(\partial_t^2\breve{u} + \partial_\phi^2\breve{u}\right) = e^{2t}\breve{f}$$

on $\mathbb{R} \times G$ with boundary conditions $\breve{u}(\cdot, 0) = \breve{u}(\cdot, \omega) = 0$. Writing $\breve{g} := e^{2t}\breve{f}$, Fourier transform in the variable t leads to

$$-\left(-\zeta^2 + \partial_\phi^2\right)\mathcal{M}u(\zeta) = \mathcal{M}g(\zeta) := (\mathcal{F}\breve{g})(\zeta) \tag{3.20}$$

with the boundary conditions $(\mathcal{M}u)(\zeta,0) = (\mathcal{M}u)(\zeta,\omega) = 0$. Upon introducing the operator ("Mellin Symbol") $\mathcal{L}(\zeta) := -\left(-\zeta^2 + \partial_\phi^2\right) : H^2(G) \cap H_0^1(G) \to L^2(G)$, we can write line (3.20) in the more compact form

$$\begin{aligned} \mathcal{L}(\zeta)(\mathcal{M}u)(\zeta) &= \mathcal{M}g(\zeta), \\ (\mathcal{M}u)(\zeta,0) &= 0, \\ (\mathcal{M}u)(\zeta,\omega) &= 0. \end{aligned} \tag{3.21}$$

The following equivalences are immediate consequences of the results in Section 3.1 and show the interconnection of functions $f \in K_\gamma^s(\mathcal{C})$ with their counterparts $\check{f}$ and $\check{g}$ and suitable Mellin transforms.

Remark 3.20. If $\check{g} := e^{2t}\check{f}$, the following statements are equivalent:

(i) $f \in K_\gamma^{s-2}(\mathcal{C})$,

(ii) $e^{t(\gamma-s+3)}\check{f} \in H^{s-2}(\mathbb{R}\times G)$,

(iii) $e^{t(\gamma-s+1)}\check{g} \in H^{s-2}(\mathbb{R}\times G)$,

(iv) $(\mathcal{M}g)(\cdot - i\eta) \in L^2(\mathbb{R}; H^{s-2}(G))$ with $\eta = -(\gamma - s + 1)$ and

$$\int_{\xi\in\mathbb{R}} \|\mathcal{M}g(\xi - i\eta)\|^2_{H^{s-2}(G;|\xi|)}\, d\xi < \infty.$$

The next lemma is the key result for solving (3.21).

Lemma 3.21. *Consider the problem*

$$\begin{aligned} \left(-\partial_\phi^2 + \zeta^2\right)\widehat{u} &= F \in L^2(G), \\ \widehat{u}(0) &= 0, \\ \widehat{u}(\omega) &= 0. \end{aligned} \tag{3.22}$$

Set $\sigma := \{\lambda_n := \frac{\pi}{\omega}n : n \in \mathbb{N}\}$. Then it follows:

(i) For $\zeta \in \mathbb{C}\setminus \pm i\sigma$ problem (3.22) *has a unique solution $\widehat{u_\zeta} \in H^2(G) \cap H_0^1(G)$.*

(ii) Assume $\{\xi + i\eta : \xi \in \mathbb{R}\} \cap \pm i\sigma = \emptyset$. Then there exists a constant $C = C(\eta)$ such that for all $\zeta \in \{\xi + i\eta : \xi \in \mathbb{R}\}$, the solution $\widehat{u_\zeta}$ of (3.22) *satisfies*

$$\|\widehat{u_\zeta}\|^2_{H^2(G;|\xi|)} = (1+|\xi|^2)^2\|\widehat{u_\zeta}\|^2_{L^2(G)} + (1+|\xi|^2)|\widehat{u_\zeta}|^2_{H^1(G)} + |\widehat{u_\zeta}|^2_{H^2(G)} \le C\|F\|^2_{L^2(G)}.$$

Proof. We first show (i). Since problem (3.22) is a boundary value problem with constant coefficients, for $\zeta \neq 0$ a fundamental system is given by the functions $\widehat{u_1}(t) = e^{\zeta t}$ and $\widehat{u_2}(t) = e^{-\zeta t}$. The problem then has a unique solution if the matrix

$$\begin{pmatrix} \widehat{u_1}(0) & \widehat{u_2}(0) \\ \widehat{u_1}(\omega) & \widehat{u_2}(\omega) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ e^{\zeta\omega} & e^{-\zeta\omega} \end{pmatrix}$$

is regular, which is true exactly for $\zeta \in \mathbb{C} \setminus \pm i\sigma$. If $\zeta = 0$, the fundamental system consists of the functions $\widehat{u_1}(t) = 1$ and $\widehat{u_2}(t) = t$ with corresponding regular matrix

$$\begin{pmatrix} \widehat{u_1}(0) & \widehat{u_2}(0) \\ \widehat{u_1}(\omega) & \widehat{u_2}(\omega) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \omega \end{pmatrix}.$$

We now show (ii). We write the function F in terms of its Fourier series expansion $F = \sqrt{\frac{2}{\omega}} \sum_{n=1}^{\infty} F_n \sin(\lambda_n \phi)$ with Fourier coefficients F_n and use the solution formula to obtain

$$u_\zeta = \sqrt{\frac{2}{\omega}} \sum_{n=1}^{\infty} \frac{F_n}{\lambda_n^2 + \zeta^2} \sin(\lambda_n \phi).$$

Since $\int_0^\omega \sin^2(\lambda_n \phi)\, d\phi = \frac{\omega}{2}$, a simple calculation yields $\|F\|_{L^2(G)}^2 \simeq \sum_{n=1}^{\infty} |F_n|^2$. Additionally, we get

$$\|u_\zeta\|_{L^2(G)}^2 \simeq \sum_{n=1}^{\infty} \frac{|F_n|^2}{|\zeta^2 + \lambda_n^2|^2},$$

$$|u_\zeta|_{H^1(G)}^2 \simeq \sum_{n=1}^{\infty} \frac{|F_n|^2}{|\zeta^2 + \lambda_n^2|^2} \lambda_n^2,$$

$$|u_\zeta|_{H^2(G)}^2 \simeq \sum_{n=1}^{\infty} \frac{|F_n|^2}{|\zeta^2 + \lambda_n^2|^2} \lambda_n^4.$$

The condition $\{\zeta \in \mathbb{C} : \operatorname{Im} \zeta = -\eta\} \cap \pm i\sigma = \emptyset$ implies for fixed η

$$\begin{aligned} \frac{1}{|\zeta^2 + \lambda_n^2|^2} &= \frac{1}{|\zeta - i\lambda_n|^2} \frac{1}{|\zeta + i\lambda_n|^2} = \frac{1}{|\xi|^2 + |\lambda_n + \eta|^2} \frac{1}{|\xi|^2 + |\lambda_n - \eta|^2} \\ &\leq \begin{cases} \frac{1}{|\xi|^2} \frac{1}{|\xi|^2} = |\xi|^{-4} \leq \min(|\xi|^{-4}, \lambda_n^{-4}), & |\lambda_n| \leq |\xi| \\ C|\lambda_n|^{-4} \leq C \min(|\xi|^{-4}, \lambda_n^{-4}), & |\lambda_n| > |\xi| \end{cases}, \end{aligned} \tag{3.23}$$

where the last estimate for $|\lambda_n| > |\xi|$ is seen by case by case analysis in the following way:

- Assume $\lambda_n > \eta$. In the case $\lambda_n \geq 2\eta$ we get $\lambda_n - \eta = \frac{1}{2}\lambda_n + \frac{1}{2}\lambda_n - \eta \geq \frac{1}{2}\lambda_n$, and in the case $\lambda_n \leq 2\eta$ we have, defining $\delta := \min_{n \in \mathbb{N}}(\lambda_n - \eta)$,

$$\lambda_n - \eta \geq \delta = \frac{\delta}{\lambda_n} \lambda_n \geq \frac{\delta}{2\eta} \lambda_n.$$

- Assume $\lambda_n < \eta$. In the case $\eta \geq 2\lambda_n$ we get $\eta - \lambda_n = \frac{1}{2}\eta + \frac{1}{2}\eta - \lambda_n \geq \frac{1}{2}\eta \geq \frac{1}{2}\lambda_n$, and in the case $\eta \leq 2\lambda_n$ we have, defining $\delta := \min_{n \in \mathbb{N}}(\eta - \lambda_n)$,

$$\eta - \lambda_n \geq \delta = \frac{\delta}{\lambda_n} \lambda_n \geq \frac{\delta}{\eta} \lambda_n.$$

This case analysis together with basic estimates shows (3.23). The line

$$(1 + |\xi|^2)^2 \|\widehat{u_\zeta}\|_{L^2(G)}^2 + (1 + |\xi|^2) |\widehat{u_\zeta}|_{H^1(G)}^2 + |\widehat{u_\zeta}|_{H^2(G)}^2 \lesssim \sum_{n=1}^{\infty} |F_n|^2 \simeq \|F\|_{L^2(G)}^2$$

finishes the proof. □

We can now apply Lemma 3.21 to solve (3.20). Assume $s = 2$ and $\eta = -(\gamma - 2 + 1)$ such that $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta = -\eta\} \cap \pm i\sigma = \emptyset$. Then the solution of (3.20) satisfies

$$\|\mathcal{M}u(\xi - i\eta)\|_{H^2(G;|\xi|)} \lesssim \|\mathcal{M}g(\xi - i\eta)\|_{L^2(G)}$$

uniformly in $\xi \in \mathbb{R}$. Hence we get with Theorem 3.19

$$\begin{aligned} \|u\|^2_{K^2_\gamma(\mathcal{C})} &\lesssim \int_{\xi\in\mathbb{R}} \|\mathcal{M}u(\xi - i\eta)\|^2_{H^2(G;|\xi|)}\, d\xi \lesssim \int_{\xi\in\mathbb{R}} \|\mathcal{M}g(\xi - i\eta)\|^2_{L^2(G)}\, d\xi \\ &\lesssim \|e^{t(\gamma-1)}\check{g}\|^2_{L^2(\mathbb{R}\times G)} = \|e^{t(\gamma+1)}\check{f}\|^2_{L^2(\mathbb{R}\times G)} \lesssim \|f\|^2_{K^0_\gamma(\mathcal{C})}. \end{aligned} \tag{3.24}$$

Thus, for $s = 2$, we have shown existence and uniqueness of a solution in $K^s_\gamma(\mathcal{C})$ of problem (3.19) together with a regularity estimate by using the Mellin transform and solving the transformed problem. For general $s \geq 2$, we have the following result.

Theorem 3.22. *Let $s \in \mathbb{N}$, $s \geq 2$ and $\gamma \in \mathbb{R}$ such that $\gamma - s + 1 \notin \pm\sigma$. Then, for every $f \in K^{s-2}_\gamma(\mathcal{C})$, problem* (3.19) *has a unique solution $u \in K^s_\gamma(\mathcal{C})$ with the a priori estimate*

$$\|u\|_{K^s_\gamma(\mathcal{C})} \lesssim \|f\|_{K^{s-2}_\gamma(\mathcal{C})}.$$

Proof. Since $f \in K^{s-2}_\gamma(\mathcal{C}) \subseteq K^0_{\gamma-s+2}(\mathcal{C})$, we get a solution $u \in K^2_{\gamma-s+2}(\mathcal{C})$ with

$$\|u\|_{K^2_{\gamma-s+2}(\mathcal{C})} \lesssim \|f\|_{K^0_{\gamma-s+2}(\mathcal{C})}$$

from the above calculations. For this solution $u \in K^2_{\gamma-s+2}(\mathcal{C}) \subseteq K^0_{\gamma-s}(\mathcal{C})$ we can apply Lemma 3.12 to get $u \in K^s_\gamma(\mathcal{C})$ with the estimate

$$\|u\|_{K^s_\gamma(\mathcal{C})} \lesssim \|f\|_{K^{s-2}_\gamma(\mathcal{C})} + \|u\|_{K^0_{\gamma-s}(\mathcal{C})} \lesssim \|f\|_{K^{s-2}_\gamma(\mathcal{C})} + \|u\|_{K^2_{\gamma-s+2}(\mathcal{C})} \lesssim \|f\|_{K^{s-2}_\gamma(\mathcal{C})}.$$

□

We consider solutions $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$ of

$$\begin{aligned} -\Delta u_1 &= f \in L^2(\mathcal{C}), \\ u_1 &= 0, \quad \phi \in \{0, \omega\}. \end{aligned} \tag{3.25}$$

Note that this implies $\operatorname{supp} f \subseteq B_1(0)$. It follows $f \in K^0_0(\mathcal{C})$ and $u_1 \in K^1_0(\mathcal{C}) \cap K^2_1(\mathcal{C})$, cf. Lemma 3.9, Lemma 3.12 and Lemma 3.15. By Theorem 3.22 there also exists a function $u_0 \in K^2_0(\mathcal{C})$ that solves (3.19). Note that both the energy solution u_1 and the solution u_0 can be different, since they lie in different spaces. However, it will turn out that the functions do not differ widely. We now demonstrate how the functions u_0 and u_1 are related.
Using the Mellin transform and considering the function spaces of u_0 and u_1, we get

$$(-\partial^2_\phi + \zeta^2)\mathcal{M}u_1 = \mathcal{M}g \quad \text{on } \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$$

and

$$(-\partial^2_\phi + \zeta^2)\mathcal{M}u_0 = \mathcal{M}g \quad \text{on } \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta = -1\}.$$

Since $f \in K_0^0(\mathcal{C})$, it follows $e^{-t}\breve{g} \in L^2(\mathbb{R} \times G)$, and thus $\mathcal{M}g$ is holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > -1\}$ with values in $L^2(G)$, cf. Remark 3.20 and Theorem 3.3. We get that $\mathcal{M}u_1$ is holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$ with values in $H^2(G)$ the same way. From Theorem 3.19 we obtain

$$\mathcal{M}g(\cdot - i) \in L^2(\mathbb{R}; L^2(G)), \qquad \mathcal{M}u_1 \in L^2(\mathbb{R}; H^2(G))$$

and

$$\mathcal{M}u_0(\cdot - i) \in L^2(\mathbb{R}; H^2(G)).$$

Now note that the operator $(\mathcal{L}(\zeta))^{-1}$ is meromorphic on $\mathbb{C}$ with poles at $\pm i\lambda_n$, where $\lambda_n = \frac{\pi}{\omega}n$ as before. From the observation

$$\mathcal{L}(\zeta)\mathcal{M}u_1 = \mathcal{M}g,$$

where $\mathcal{M}u_1$ is holomorphic for $\operatorname{Im}\zeta > 0$ and $\mathcal{M}g$ is holomorphic for $\operatorname{Im}\zeta > -1$, we get that $\mathcal{M}u_1$ can be extended meromorphically to $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > -1\}$ by

$$U(\zeta) := \mathcal{M}u_1(\zeta) := (\mathcal{L}(\zeta))^{-1}\,\mathcal{M}g(\zeta).$$

Let us mention that $U(\zeta)$ and $\mathcal{M}u_0(\zeta)$ coincide on ζ with $\operatorname{Im}\zeta = -1$. The next theorem states the relation of u_0 and u_1.

Theorem 3.23. *It holds*

$$u_0 - u_1 = \sum_{\substack{\zeta \in -i\sigma \\ \operatorname{Im}\zeta \in (-1,0)}} \frac{2\pi i}{\sqrt{2\pi}} \operatorname{Res}_{\zeta}\left(r^{i\zeta}\,(\mathcal{L}(\zeta))^{-1}\,\mathcal{M}g(\zeta)\right),$$

where $\sigma = \{\lambda_n = \frac{\pi}{\omega}n : n \in \mathbb{N}\}$.

Proof. We define $\breve{u}_0$ and $\breve{u}_1$ by the inverse Fourier transformation of U by

$$\widetilde{u_0}(t,\phi) := \frac{1}{\sqrt{2\pi}}\int_{\operatorname{Im}(\zeta)=-1} e^{i\zeta t}U(\zeta)\,d\zeta, \quad \widetilde{u_1}(t,\phi) := \frac{1}{\sqrt{2\pi}}\int_{\operatorname{Im}(\zeta)=0} e^{i\zeta t}U(\zeta)\,d\zeta.$$

Note that by definition, the function U coincides with u_0 on the line with imaginary part -1 (and with u_1 at the horizontal axis, $\operatorname{Im} z = 0$, respectively). Thus, both functions are indeed the inverse transforms of u_0 and u_1. We now proceed with similar arguments as in the proof of Theorem 3.3, (ii), and use the residue theorem. Note that we again write $\zeta = \xi + i\eta$. The observation

$$\begin{aligned}
\int_{\operatorname{Im}\zeta=0} \|e^{i\zeta t}U(\zeta)\|_{L^2(G)}\,d\zeta &\le \left(\int_{\operatorname{Im}\zeta=0} \frac{1}{(1+|\xi|^2)^2}\,d\zeta\right)^{\frac12}\left(\int_{\operatorname{Im}\zeta=0}(1+|\xi|^2)^2\|U(\zeta)\|^2_{L^2(G)}\,d\zeta\right)^{\frac12}\\
&\lesssim \left(\int_{\operatorname{Im}\zeta=0}\|\mathcal{L}(\zeta)^{-1}\mathcal{M}g(\zeta)\|^2_{H^2(G;|\xi|)}\,d\zeta\right)^{\frac12}\\
&\overset{\text{Lemma 3.21}}{\lesssim} \left(\int_{\operatorname{Im}\zeta=0}\|\mathcal{M}g(\zeta)\|^2_{L^2(G)}\,d\zeta\right)^{\frac12}\\
&\overset{(3.24)}{\lesssim} \|f\|_{K_1^0(\mathcal{C})} \lesssim \|f\|_{L^2(\mathcal{C})} < +\infty
\end{aligned} \tag{3.26}$$

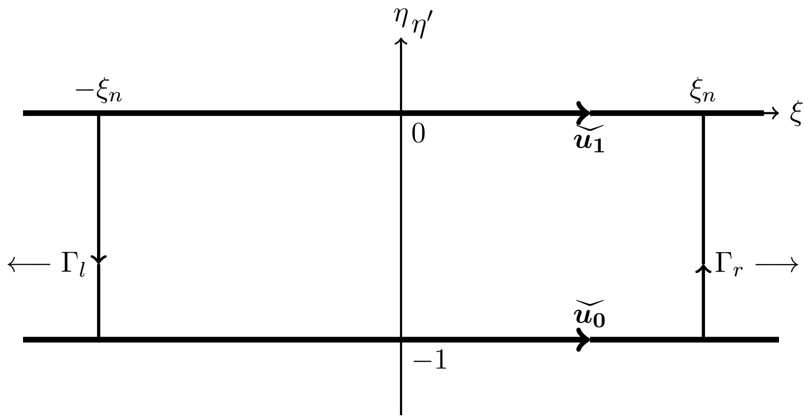

Figure 3.2: Paths of integration ($\zeta = \xi + i\eta$)

shows absolute convergence of the integral used for the definition of $\widetilde{u_1}$ with values in $L^2(G)$. The same holds for $\widetilde{u_0}$.

The next step of the proof is similar to the proof of Theorem 3.3 and is motivated by [24, Sec. 7.2]. We define for fixed t

$$I(\xi) := \int_{-1}^{0} \|e^{i\zeta t} U(\zeta)\|_{L^2(G)}\, d\eta$$

and get

$$I(\xi)^2 \lesssim \int_{-1}^{0} \|U(\zeta)\|^2_{L^2(G)}\, d\eta.$$

After fixing $\xi_0 > 0$, we integrate over ξ and obtain

$$\int_{|\xi|>\xi_0} I(\xi)^2\, d\xi \lesssim \int_{-1}^{0}\int_{|\xi|>\xi_0} \|\mathcal{L}(\zeta)^{-1}\mathcal{M}g(\zeta)\|^2_{L^2(G)}\, d\xi\, d\eta \lesssim \int_{-1}^{0}\int_{|\xi|>\xi_0} \|\mathcal{M}g(\zeta)\|^2_{L^2(G)}\, d\xi\, d\eta$$
$$\lesssim \int_{-1}^{0} \|f\|^2_{K^0_{\eta+1}(\mathcal{C})}\, d\eta \overset{\operatorname{supp} f\subseteq B_1(0)}{\lesssim} \|f\|^2_{K^0_0(\mathcal{C})} = \|f\|^2_{L^2(\mathcal{C})} < +\infty.$$

Thus there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty}\xi_n = \infty$ such that

$$\lim_{n\to\infty} I(\xi_n) = 0 \text{ and } \lim_{n\to\infty} I(-\xi_n) = 0. \tag{3.27}$$

Define $\mathcal{R}_{\xi_n}$ as the path of integration formed by the boundary of the rectangle defined by $-\xi_n < \operatorname{Re}\zeta < \xi_n$, $-1 < \operatorname{Im}\zeta < 0$, see Figure 3.2. It then follows together with (3.26) and (3.27)

$$\lim_{n\to\infty} \left\| (\widetilde{u_0} - \widetilde{u_1}) - \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}_{\xi_n}} e^{i\zeta t} U(\zeta)\, d\zeta \right\|_{L^2(G)} = 0.$$

That shows that $\breve{u_0} - \breve{u_1}$ can be approximated by closed curves, thus we can use the residue theorem to obtain

$$\breve{u_0} - \breve{u_1} = \frac{1}{\sqrt{2\pi}} 2\pi i \operatorname*{Res}_{\zeta=-i\lambda_1} e^{i\zeta t} U(\zeta) = \frac{2\pi i}{\sqrt{2\pi}} \operatorname*{Res}_{\zeta=-i\lambda_1} r^{i\zeta} U(\zeta) \tag{3.28}$$

in the case of exactly one pole of $e^{i\zeta t}U(\zeta)$, namely at $-i\lambda_1$, in the strip $\{\zeta \in \mathbb{C} : \operatorname{Im}(\zeta) \in (-1,0)\}$. Changing the variables on the left side of (3.28) finishes the proof. □

Remark 3.24. We consider different values of λ_1 in regard to Theorem 3.23:

- If $\lambda_1 > 1$, which appears for $\omega < \pi$, then there is no pole in the strip. This implies that the sum in Theorem 3.23 is zero, and thus $u_0 = u_1$.
- If $\lambda_1 < 1$, which appears for $\omega > \pi$, then the sum has exactly one term. In this case, the functions u_0 and u_1 are indeed different.
- If $\lambda_1 = 1$, we can't directly use Theorem 3.23 since we would need to integrate over the pole. However, this is only the case for $\omega = \pi$, thus we have to deal with a half-space problem, where explicit solutions are known, cf. [35, Section 2.2.4].

Considering Theorem 3.23, it is useful to evaluate the residue at the pole. For this purpose we take the equation $\mathcal{L}(\zeta)U(\zeta) = \mathcal{M}g(\zeta)$ and develop the right-hand side in its Fourier expansion in the variable ϕ

$$\mathcal{M}g(\zeta) = \sum_{n=1}^{\infty} F_n(\zeta) \sqrt{\frac{2}{\omega}} \sin(\lambda_n \phi)$$

where $F_n(\zeta)$ denote the Fourier coefficients. We also take the solution formula

$$U(\zeta) = \sum_{n=1}^{\infty} \frac{F_n(\zeta)}{\zeta^2 + \lambda_n^2} \sqrt{\frac{2}{\omega}} \sin(\lambda_n \phi)$$

and rearrange it as

$$U(\zeta) = \frac{F_1(\zeta)}{2i\lambda_1} \left(\frac{1}{\zeta - i\lambda_1} - \frac{1}{\zeta + i\lambda_1} \right) \sqrt{\frac{2}{\omega}} \sin(\lambda_1 \phi) + \sum_{n=2}^{\infty} \frac{F_n(\zeta)}{\zeta^2 + \lambda_n^2} \sqrt{\frac{2}{\omega}} \sin(\lambda_n \phi).$$

Since the sum is holomorphic in $\{\zeta \in \mathbb{C} : \operatorname{Im} \zeta \in (-1,0)\}$, as well as the function r^{ζ} in a neighborhood of $\zeta = -i\lambda_1$, the desired residue can be easily determined and equals

$$\operatorname*{Res}_{\zeta=-i\lambda_1} \left(r^{i\zeta} \left(\mathcal{L}(\zeta) \right)^{-1} \mathcal{M}g(\zeta) \right) = \frac{F_1(-i\lambda_1)}{2i\lambda_1} r^{i(-i\lambda_1)} \sqrt{\frac{2}{\omega}} \sin(\lambda_1 \phi).$$

The next step is to evaluate the Fourier coefficient $F_1(-i\lambda_1)$. We have

$$\begin{aligned} F_1(-i\lambda_1) &= \sqrt{\frac{2}{\omega}} \int_G \sin(\lambda_1\varphi)\mathcal{F}\left(e^{2t} f(e^t,\varphi)\right)|_{\xi=-i\lambda_1}\, d\varphi \\ &= \sqrt{\frac{2}{\omega}} \int_0^\omega \sin(\lambda_1\varphi)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2t} f(e^t,\varphi) e^{-i(-i\lambda_1)t}\, dt\, d\varphi \\ &= \sqrt{\frac{2}{\omega}} \frac{1}{\sqrt{2\pi}} \int_0^\omega \sin(\lambda_1\varphi) \int_0^\infty r f(r,\varphi) r^{-\lambda_1}\, dr\, d\varphi \\ &= \sqrt{\frac{2}{\omega}} \frac{1}{\sqrt{2\pi}} \int_{\mathcal{C}} r^{-\lambda_1} \sin(\lambda_1\varphi) f(x)\, dx. \end{aligned}$$

Thus, by Theorem 3.23, we obtain

$$\begin{aligned} u_1 &= u_0 - \frac{2\pi i}{\sqrt{2\pi}} \frac{1}{2i\lambda_1} r^{\lambda_1} \sqrt{\frac{2}{\omega}} \frac{1}{\sqrt{2\pi}} \left(\int_{\mathcal{C}} r^{-\lambda_1} \sin(\lambda_1\varphi) f(x)\, dx\right) \sqrt{\frac{2}{\omega}} \sin(\lambda_1\phi) \\ &= u_0 - \frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1} \sin(\lambda_1\varphi) f(x)\, dx\right) r^{\lambda_1} \sin(\lambda_1\phi) \end{aligned} \tag{3.29}$$

with $u_0 \in H^2(\mathcal{C}_R)$ for any $R > 0$. We wish to recall that even $u_0 \in K_0^2(\mathcal{C})$.
The decomposition of the solution of the Dirichlet problem on a polygon Ω instead of the cone $\mathcal{C}$ is now a simple consequence of (3.29) together with localization near the vertices of Ω.

Corollary 3.25. *Let $\Omega \subseteq \mathbb{R}^2$ be a polygon with vertices A_i, $i = 1, \dots, J$ and interior angles $\omega_i \neq \pi$. Let $f \in L^2(\Omega)$, and let $u \in H_0^1(\Omega)$ solve the problem $-\Delta u = f$. Then u can be written as*

$$u = u_{H^2} + \sum_{i=1}^{J} c_i \chi_i r_i^{\frac{\pi}{\omega_i}} \sin\left(\frac{\pi}{\omega_i}\phi_i\right),$$

where $u_{H^2} \in H^2(\Omega) \cap H_0^1(\Omega)$, and where the functions χ_i are smooth cut-off functions with $\chi_i \equiv 1$ near A_i satisfying $\sum_{i=1}^{J} \chi_i = 1$. The functions c_i equal zero if $\omega_i < \pi$, and

$$c_i(f) = -\frac{1}{\pi} \int_\Omega f(s_i\eta_i) + u\Delta(s_i\eta_i)$$

if $\omega_i > \pi$, where η_i is also a smooth cut-off function and

$$s_i = r_i^{-\frac{\pi}{\omega_i}} \sin\left(\frac{\pi}{\omega_i}\phi\right).$$

Furthermore, the estimate $\|u_{H^2}\|_{H^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$ holds.

Proof. The idea is the localization near the vertices A_i by using the cut-off functions η_i. According to (3.29), each function $\eta_i u$ can be written as

$$\begin{aligned} \eta_i u &= u_{0,i} - r^{\lambda_1} \sin(\lambda_1\phi)\frac{1}{\pi}\left(\int_\Omega r^{-\lambda_1} \sin(\lambda_1\varphi)\left(-\Delta(\eta_i u)\right)\, dx\right) \\ &= \underbrace{u_{0,i}}_{\in H^2} - r^{\lambda_1} \sin(\lambda_1\phi)\frac{1}{\pi}\left(\int_\Omega s_i\left(-\Delta(\eta_i u)\right)\, dx\right), \end{aligned}$$

where only the H^2-term remains in the case $\omega_i < \pi$, cf. Remark 3.24. Since $\Delta s_i = 0$, we get

$$\begin{aligned}\int_\Omega -\Delta(\eta_i u)s_i\,dx &= \int_\Omega -\left(\Delta\eta_i u + 2\nabla\eta_i\nabla u + \eta_i\Delta u\right)s_i\,dx\\ &= \int_\Omega u\Delta\eta_i s_i - 2(\nabla\eta_i\nabla u + u\Delta\eta_i)s_i + f\eta_i s_i\,dx\\ &= \int_\Omega u\Delta\eta_i s_i + 2(u\nabla\eta_i)\nabla s_i + f\eta_i s_i\,dx\\ &= \int_\Omega u\Delta(\eta_i s_i) + f\eta_i s_i\,dx.\end{aligned}$$

The norm estimate is a direct consequence of Theorem 3.22. □

3.3 Decomposition of the solution - Neumann

In the case of Neumann boundary conditions, the situation is handled similarly to Section 3.2. For $\omega \neq \pi$ we consider here solutions $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$ of

$$\begin{aligned}-\Delta u_1 &= f \in L^2(\mathcal{C})\\ \partial_n u_1 &= 0, \quad \phi \in \{0,\omega\},\end{aligned} \tag{3.30}$$

This implies $\operatorname{supp} f \subseteq B_1(0)$ as well, and it again follows $f \in K^0_0(\mathcal{C})$. Note that $u_1 \in K^1_\delta(\mathcal{C})$ for every $\delta > 0$, since it is by definition equivalent to $r^\delta \nabla u_1 \in L^2(\mathcal{C})$ (which is obvious since $u_1 \in H^1(\mathcal{C})$) and $r^{\delta-1}u_1 \in L^2(\mathcal{C})$, which follows from a Hardy inequality, cf. [48, Lemma A.1.6, Lemma A.1.7]. From Lemma 3.9, Lemma 3.12 and Lemma 3.15 we then get $u_1 \in K^1_\delta(\mathcal{C}) \cap K^2_{1+\delta}(\mathcal{C})$.
Using the Mellin transform leads to

$$(-\partial^2_\phi + \zeta^2)\mathcal{M}u_1 = \mathcal{M}g \quad \text{on } \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > \delta\}$$

with boundary conditions $\partial_\phi(\mathcal{M}u_1)(0) = \partial_\phi(\mathcal{M}u_1)(\omega) = 0$. As in the Dirichlet case, one can show with Remark 3.20 and Theorem 3.3 that $\mathcal{M}g$ is holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > -1\}$ and that $\mathcal{M}u_1$ is holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > \delta\}$. We again extend $\mathcal{M}u_1$ meromorphically to $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > -1\}$ by

$$U(\zeta) := \mathcal{M}u_1(\zeta) := (\mathcal{L}(\zeta))^{-1}\,\mathcal{M}g(\zeta).$$

The following lemma is the Neumann analog to Lemma 3.21.

Lemma 3.26. *Consider the problem*

$$\begin{aligned}\left(-\partial^2_\phi + \zeta^2\right)\widehat{u} &= F \in L^2(G)\\ \widehat{u}'(0) &= 0\\ \widehat{u}'(\omega) &= 0.\end{aligned} \tag{3.31}$$

Set $\sigma^N := \{\lambda_n := \frac{\pi}{\omega}n : n \in \mathbb{N}_0\}$. *Then it follows:*

(i) For $\zeta \in \mathbb{C} \backslash \pm i\sigma^N$ *problem* (3.31) *has a unique solution* $\widehat{u_\zeta} \in H^2(G)$.

(ii) Assume $\{\xi + i\eta : \xi \in \mathbb{R}\} \cap \pm i\sigma^N = \emptyset$. *Then there exists a constant* $C = C(\eta)$ *such that for all* $\zeta \in \{\xi + i\eta : \xi \in \mathbb{R}\}$, *the solution* $\widehat{u_\zeta}$ *of* (3.31) *satisfies*

$$\|\widehat{u_\zeta}\|^2_{H^2(G;|\xi|)} = (1+|\xi|^2)^2\|\widehat{u_\zeta}\|^2_{L^2(G)} + (1+|\xi|^2)|\widehat{u_\zeta}|^2_{H^1(G)} + |\widehat{u_\zeta}|^2_{H^2(G)} \le C\|F\|^2_{L^2(G)}.$$

Proof. The proof follows the same lines as the proof of Lemma 3.21. The only change is the Fourier transformation used for F which has now the form

$$F = \sqrt{\frac{1}{\omega}}F_0 + \sqrt{\frac{2}{\omega}}\sum_{n=1}^{\infty} F_n \cos(\lambda_n \phi),$$

together with the appropriate solution formula for $\widehat{u_\zeta}$. □

Similar to Theorem 3.22 for the Dirichlet case, we have the following result for Neumann boundary conditions. The proof follows analogous lines, but is based on Lemma 3.26, cf. Remark 3.14.

Proposition 3.27. *Let* $s \in \mathbb{N}$, $s \ge 2$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma - s + 1 \notin \pm\sigma^N$. *Then, for every* $f \in K^{s-2}_\gamma(\mathcal{C})$, *problem*

$$\begin{aligned} -\Delta u &= f \in K^{s-2}_\gamma(\mathcal{C}) \\ \partial_n u &= 0, \quad \phi \in \{0, \omega\} \end{aligned} \tag{3.32}$$

has a unique solution $u \in K^s_\gamma(\mathcal{C})$ *with the a priori estimate*

$$\|u\|_{K^s_\gamma(\mathcal{C})} \lesssim \|f\|_{K^{s-2}_\gamma(\mathcal{C})}.$$

Proposition 3.27 shows that for $f \in K^0_0(\mathcal{C})$ there exists a unique solution $u_0 \in K^2_0(\mathcal{C})$ of problem (3.32). One can also prove analogously to Theorem 3.23 that

$$u_0 - u_1 = \sum_{\substack{\zeta \in -i\sigma^N \\ \operatorname{Im}\zeta \in (-1,\delta)}} \frac{2\pi i}{\sqrt{2\pi}} \operatorname*{Res}_{\zeta}\left(r^{i\zeta}\left(\mathcal{L}(\zeta)\right)^{-1}\mathcal{M}g(\zeta)\right)$$

by obvious changes concerning the path of integration. However, the evaluation of the residue is different since the set σ^N has an additional pole in $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \in (-1,\delta)\}$.
For the evaluation of the residue at the poles, we again use Fourier expansions in the variable ϕ. We get with $F_n(\zeta)$ denoting the Fourier coefficients

$$\mathcal{M}g(\zeta) = \sqrt{\frac{1}{\omega}}F_0(\zeta) + \sum_{n=1}^{\infty} F_n(\zeta)\sqrt{\frac{2}{\omega}}\cos(\lambda_n\phi)$$

and

$$\begin{aligned} U(\zeta) &= \frac{F_0(\zeta)}{\zeta^2}\sqrt{\frac{1}{\omega}} + \sum_{n=1}^{\infty}\frac{F_n(\zeta)}{\zeta^2+\lambda_n^2}\sqrt{\frac{2}{\omega}}\cos(\lambda_n\phi) \\ &= \frac{F_0(\zeta)}{\zeta^2}\sqrt{\frac{1}{\omega}} + \frac{F_1(\zeta)}{2i\lambda_1}\left(\frac{1}{\zeta - i\lambda_1} - \frac{1}{\zeta + i\lambda_1}\right)\sqrt{\frac{2}{\omega}}\cos(\lambda_1\phi) + \sum_{n=2}^{\infty}\frac{F_n(\zeta)}{\zeta^2+\lambda_n^2}\sqrt{\frac{2}{\omega}}\cos(\lambda_n\phi) \end{aligned}$$

after rearranging the terms. The first term and the sum are obviously holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \in (-1,0)\}$ and thus in a neighborhood of $-i\lambda_1 = -i\frac{\pi}{\omega}$. It follows that the residue at the pole $-i\lambda_1$ can be calculated similar to the previous section, and we obtain

$$\operatorname*{Res}_{\zeta=-i\lambda_1}\left(r^{i\zeta}\left(\mathcal{L}(\zeta)\right)^{-1}\mathcal{M}g(\zeta)\right) = \frac{F_1(-i\lambda_1)}{2i\lambda_1} r^{i(-i\lambda_1)}\sqrt{\frac{2}{\omega}}\cos(\lambda_1\phi),$$

where the Fourier coefficient $F_1(-i\lambda_1)$ equals

$$\begin{aligned}
F_1(-i\lambda_1) &= \sqrt{\frac{2}{\omega}}\int_G \cos(\lambda_1\varphi)\mathcal{F}\left(e^{2t}f(e^t,\varphi)\right)|_{\xi=-i\lambda_1}\,d\varphi \\
&= \sqrt{\frac{2}{\omega}}\int_0^\omega \cos(\lambda_1\varphi)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2t}f(e^t,\varphi)e^{-i(-i\lambda_1)t}\,dt\,d\varphi \\
&= \sqrt{\frac{2}{\omega}}\frac{1}{\sqrt{2\pi}}\int_0^\omega \cos(\lambda_1\varphi)\int_0^\infty r f(r,\varphi)r^{-\lambda_1}\,dr\,d\varphi \\
&= \sqrt{\frac{2}{\omega}}\frac{1}{\sqrt{2\pi}}\int_{\mathcal{C}} r^{-\lambda_1}\cos(\lambda_1\varphi)f(x)\,dx.
\end{aligned}$$

The pole at $\zeta = 0$ is a double pole of $\mathcal{L}(\zeta)^{-1}$. As before, we write

$$U(\zeta) = \frac{F_0(\zeta)}{\zeta^2}\sqrt{\frac{1}{\omega}} + \sum_{n=1}^{\infty}\frac{F_n(\zeta)}{\zeta^2+\lambda_n^2}\sqrt{\frac{2}{\omega}}\cos(\lambda_n\phi)$$

and note that the sum is holomorphic in a neighbourhood of zero. In order to get the residue, we use Taylor expansion to get

$$\begin{aligned}
r^{i\zeta}\frac{F_0(\zeta)}{\zeta^2}\sqrt{\frac{1}{\omega}} &= \sqrt{\frac{1}{\omega}}F_0(\zeta)\frac{1}{\zeta^2}e^{i\zeta\ln r} \\
&= \sqrt{\frac{1}{\omega}}F_0(\zeta)\left(\frac{1}{\zeta^2}+\frac{i\ln r}{\zeta}+\frac{(i\ln r)^2}{2}+\dots\right),
\end{aligned}$$

hence

$$\begin{aligned}
\operatorname*{Res}_{\zeta=0}\left(r^{i\zeta}\frac{F_0(\zeta)}{\zeta^2}\sqrt{\frac{1}{\omega}}\right) &= \sqrt{\frac{1}{\omega}}F_0(0)i\ln r + \sqrt{\frac{1}{\omega}}F_0'(0) \\
&= \sqrt{\frac{1}{\omega}}i\ln r\sqrt{\frac{1}{\omega}}\int_G \mathcal{F}(e^{2t}f(e^t,\phi))|_{\xi=0}\,d\phi + \sqrt{\frac{1}{\omega}}F_0'(0) \\
&= \frac{1}{\omega}i\ln r\int_0^\omega \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2t}f(e^t,\phi)\,dt\,d\phi + \sqrt{\frac{1}{\omega}}F_0'(0) \\
&= \frac{1}{\omega}\frac{1}{\sqrt{2\pi}}i\ln r\int_{\mathcal{C}} f(x)\,dx + \sqrt{\frac{1}{\omega}}F_0'(0).
\end{aligned}$$

Since

$$F_0(\zeta) = \sqrt{\frac{1}{\omega}}\int_G \mathcal{F}(e^{2t}f(e^t,\phi))|_\zeta\,d\phi = \sqrt{\frac{1}{\omega}}\int_0^\omega \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2t}f(e^t,\phi)e^{-i\zeta t}\,dt\,d\phi,$$

we get

$$F_0'(\zeta) = \sqrt{\frac{1}{\omega}} \int_0^\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -ite^{2t} f(e^t, \phi) e^{-i\zeta t} \, dt \, d\phi$$

and further

$$\begin{aligned} F_0'(0) &= \sqrt{\frac{1}{\omega}} \int_0^\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -ite^{2t} f(e^t, \phi) \, dt \, d\phi \\ &= \sqrt{\frac{1}{\omega}} \int_0^\omega \frac{1}{\sqrt{2\pi}} \int_0^1 -ir \ln r \, f(r, \phi) \, dr \, d\phi, \end{aligned}$$

where existence of the integrals is based on the compact support of f. We obtain that the expression $C(f) := \sqrt{\frac{1}{\omega}} F_0'(0)$ is just a constant depending on the function f and thus in H^2 in a bounded domain, with the estimate

$$\begin{aligned} |C(f)| &\lesssim \int_0^\omega \int_0^1 |r \ln r \, f(r)| \, dr \, d\phi \\ &\lesssim \left(\int_0^1 r (\ln r)^2 \, dr \right)^{1/2} \left(\int_0^\omega \int_0^1 r |f(r)|^2 \, dr \, d\phi \right)^{1/2} \lesssim \|f\|_{L^2(\mathcal{C})}. \end{aligned} \tag{3.33}$$

We get

$$\begin{aligned} u_1 &= u_0 - C(f) - \frac{2\pi i}{\sqrt{2\pi}} \frac{1}{2i\lambda_1} r^{\lambda_1} \sqrt{\frac{2}{\omega}} \frac{1}{\sqrt{2\pi}} \left(\int_{\mathcal{C}} r^{-\lambda_1} \cos(\lambda_1 \varphi) f(x) \, dx \right) \sqrt{\frac{2}{\omega}} \cos(\lambda_1 \phi) \\ &\quad - \frac{2\pi i}{\sqrt{2\pi}} \frac{1}{\omega} \frac{1}{\sqrt{2\pi}} i \ln r \left(\int_{\mathcal{C}} f(x) \, dx \right) \\ &= u_0 - C(f) - \frac{1}{\pi} \left(\int_{\mathcal{C}} r^{-\lambda_1} \cos(\lambda_1 \varphi) f(x) \, dx \right) r^{\lambda_1} \cos(\lambda_1 \phi) + \frac{\ln r}{\omega} \left(\int_{\mathcal{C}} f(x) \, dx \right). \end{aligned}$$

Since we know $u_1 \in H^1(\mathcal{C})$, we see that $\frac{\ln r}{\omega} \left(\int_{\mathcal{C}} f(x) \, dx \right)$ must equal zero, thus we conclude

$$u_1 = u_0 - C(f) - \frac{1}{\pi} \left(\int_{\mathcal{C}} r^{-\lambda_1} \cos(\lambda_1 \varphi) f(x) \, dx \right) r^{\lambda_1} \cos(\lambda_1 \phi). \tag{3.34}$$

The proof of the following result now follows the same lines as the proof of Corollary 3.25.

Corollary 3.28. *Let $\Omega \subseteq \mathbb{R}^2$ be a polygon with vertices A_i, $i = 1, \dots, J$ and interior angles $\omega_i \neq \pi$. Let $f \in L^2(\Omega)$, and let $u \in H^1(\Omega)$ solve $-\Delta u = f$ with boundary conditions $\partial_n u = 0$. Then u can be written as*

$$u = u_{H^2} + \sum_{i=1}^{J} c_i \chi_i r_i^{\frac{\pi}{\omega_i}} \cos\left(\frac{\pi}{\omega_i} \phi_i \right),$$

where $u_{H^2} \in H^2(\Omega)$, and where the functions χ_i are smooth cut-off functions with $\chi_i \equiv 1$ near A_i. The functions c_i equal zero if $\omega_i < \pi$, and

$$c_i(f) = -\frac{1}{\pi} \int_\Omega f(s_i \eta_i) + u \Delta(s_i \eta_i)$$

if $\omega_i > \pi$*, where* η_i *is also a smooth cut-off function and*

$$s_i = r_i^{-\frac{\pi}{\omega_i}} \cos\left(\frac{\pi}{\omega_i}\phi\right).$$

Furthermore, the estimate $\|u_{H^2}\|_{H^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$ *holds.*

3.4 Higher regularity decompositions

In the Corollaries 3.25 and 3.28, we always made the assumption $f \in L^2(\Omega)$. This led to decompositions of the solution u into a more regular $H^2(\Omega)$-part and the singularity function. In this section f will be assumed to be of higher regularity, i.e. $f \in H^s(\Omega)$ with $s > 0$, and we generalize the results of Sections 3.2 and 3.3. It turns out that the solution u can then be written as the sum of an $H^{2+s}(\Omega)$-function and a (slightly different) singularity function.

We start with an inequality of the Ehrling's lemma type.

Lemma 3.29. *Let* $0 < \rho < \rho'$*, and let* $u \in H^{k+\epsilon}(A(\rho,\rho'))$ *for* $k \in \mathbb{N}$*,* $\epsilon \in (0,1)$*, cf.* (3.8)*. Denote by* Γ_1 *and* Γ_2 *the two straight-lined parts of the boundary of* $A(\rho,\rho')$*. Assume that the direction vectors of* Γ_1 *and* Γ_2 *are linearly independent. Then the inequality*

$$\|u\|_{H^{k+\epsilon}(A(\rho,\rho'))} \le C(|D^k u|_{H^\epsilon(A(\rho,\rho'))} + \sum_{l=1}^{2}\sum_{j=0}^{k-1} \|D^j u\|_{L^2(\Gamma_l)})$$

holds with a constant C *dependent on* k*,* ϵ *and the domain. For* $u \in H^\epsilon(A(\rho,\rho'))$ *with* $\epsilon \in (1/2,1)$*, we have the inequality*

$$\|u\|_{H^\epsilon(A(\rho,\rho'))} \le C(|u|_{H^\epsilon(A(\rho,\rho'))} + \sum_{l=1}^{2} \|u\|_{L^2(\Gamma_l)}).$$

Proof. Suppose the statement is false. Then for $u \in H^{k+\epsilon}(A(\rho,\rho'))$ with $\epsilon \in (0,1)$, there is a function $u_n \in H^{k+\epsilon}(A(\rho,\rho'))$ for every $n \in \mathbb{N}$ such that without loss of generality

$$1 = \|u_n\|_{H^{k+\epsilon}(A(\rho,\rho'))} \ge n(|D^k u_n|_{H^\epsilon(A(\rho,\rho'))} + \sum_{l=1}^{2}\sum_{j=0}^{k-1} \|D^j u_n\|_{L^2(\Gamma_l)}). \tag{3.35}$$

As a bounded sequence in $H^{k+\epsilon}(A(\rho,\rho'))$, $(u_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence, again denoted by $(u_n)_{n\in\mathbb{N}}$, with limit u_∞ in $H^{k+\epsilon}(A(\rho,\rho'))$. Observation (3.35) implies $|D^k u_\infty|_{H^\epsilon(A(\rho,\rho'))} = 0$, and thus $D^k u_\infty$ is constant which implies that u_∞ is a polynomial in $\mathcal{P}_k$. The compact embedding of $H^{k+\epsilon}(A(\rho,\rho'))$ in $H^k(A(\rho,\rho'))$ allows for the extraction of another subsequence of $(u_n)_{n\in\mathbb{N}}$ satisfying $u_n \to u_\infty$ in $H^k(A(\rho,\rho'))$. By the continuity of the trace operator, this means $(D^j u_n)|_{\Gamma_l} \to (D^j u_\infty)|_{\Gamma_l}$ in $L^2(\Gamma_l)$ for $j = 0,\ldots,k-1$, $l = 1,2$, and by (3.35) we get $D^j u_\infty = 0$ for $j = 0,\ldots,k-1$ on the boundary parts Γ_l. Now assume w.l.o.g. that Γ_1 coincides with a part of the x-axis, and write $u_\infty = \sum_{0\le i+j\le k} a_{i,j} x^i y^j$. On Γ_1, u_∞ equals zero, i.e. $\sum_{i=0}^{k} a_{i,0} x^i = 0$ for all x in a suitable interval, thus we get $a_{i,0} = 0$

for $i = 0, \dots, k$. Using the same arguments for the y-derivatives $D_y^1 u_\infty, \dots, D_y^{k-1} u_\infty$, we obtain inductively $a_{i,j} = 0$ for $i = 0, \dots, k$, $j = 0, \dots, k-1$. Thus $u_\infty = a_{0,k} y^k$, and since the direction vector of Γ_2 is linearly independent from the direction vector of Γ_1 (thus the y-coordinate of the other part cannot be constant), we can conclude $a_{0,k} = 0$. Therefore, $u_\infty = 0$ which contradicts $\|u_n\|_{H^{k+\epsilon}(A(\rho,\rho'))} = 1$ for all $n \in \mathbb{N}$.
For $k = 0$ the proof follows similar lines. Suppose that for every $n \in \mathbb{N}$ there is a function u_n such that

$$\|u_n\|_{H^\epsilon(A(\rho,\rho'))} \geq n(|u_n|_{H^\epsilon(A(\rho,\rho'))} + \sum_{l=1}^{2} \|u_n\|_{L^2(\Gamma_l)}). \tag{3.36}$$

Without loss of generality, we again assume $\|u_n\|_{H^\epsilon(A(\rho,\rho'))} = 1$ for all $n \in \mathbb{N}$. It follows $|u_n|_{H^\epsilon(A(\rho,\rho'))} \leq 1/n$, thus $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^\epsilon(A(\rho,\rho'))$ which then has a subsequence, again denoted by $(u_n)_{n\in\mathbb{N}}$, that converges weakly to a function u_∞ in $H^\epsilon(A(\rho,\rho'))$. From inequality (3.36), we get that u_∞ is a constant function. Since $H^\epsilon(A(\rho,\rho'))$ is compactly embedded in $H^{s'}(A(\rho,\rho'))$ for $1/2 < s' < \epsilon$, the sequence $(u_n)_{n\in\mathbb{N}}$ has another subsequence, denoted by $(u_n)_{n\in\mathbb{N}}$, that converges to u_∞ in $H^{s'}(A(\rho,\rho'))$. The trace operator is continuous, thus $\|u_n\|_{L^2(\Gamma_l)} \to \|u_\infty\|_{L^2(\Gamma_l)} = 0$, $l = 1, 2$, by (3.36) and $u_\infty = 0$, which is a contradiction to the assumption $\|u_n\|_{H^\epsilon(A(\rho,\rho'))} = 1$. □

The next results deal with the fact that fractional order Sobolev spaces can be seen as subspaces of suitable weighted Sobolev spaces of the K_γ^s-type. It is however necessary to ensure decay at zero.

Lemma 3.30. *Let f be a function with compact support in $B_1(0)$. Then the following assertions hold.*

(i) Let $f \in H^\epsilon(\mathcal{C}_1)$ for $\epsilon \in (0,1)$. Then $f \in K_{-\epsilon}^0(\mathcal{C})$.

(ii) Let $f \in H^{1+\epsilon}(\mathcal{C}_1)$ for $\epsilon \in (0,1)$ with $f(0) = 0$. Then $f \in K_{-\epsilon}^1(\mathcal{C})$.

(iii) Let $f \in H^{2+\epsilon}(\mathcal{C}_1)$ for $\epsilon \in (0,1)$ with $f(0) = 0$ and $\nabla f(0) = 0$. Then $f \in K_{-\epsilon}^2(\mathcal{C})$.

Proof. We start with (i). For $\epsilon \in (0, 1/2)$, the result can be found in [37, Theorem 1.4.4.3]. Now assume $\epsilon \in (1/2, 1)$. We take for $d > 0$ the domains $A(d, 2d)$ and denote by Γ_1^d and Γ_2^d the parts of $\partial A(d, 2d)$ that coincide with $\partial\mathcal{C}$. We write $\widehat{f}$ for the function f scaled to the reference element $A(1, 2)$. We now get by scaling

$$\int_{A(d,2d)} r^{-2\epsilon} f^2 \lesssim d^{2-2\epsilon} \int_{A(1,2)} \widehat{f}^2 \overset{\text{Lemma 3.29}}{\lesssim} d^{2(1-\epsilon)} \left(|\widehat{f}|^2_{H^\epsilon(A(1,2))} + \sum_{l=1}^{2} \|\widehat{f}\|^2_{L^2(\Gamma_l^1)} \right)$$
$$\lesssim d^{2(1-\epsilon)} \left(d^{-2+2\epsilon} |f|^2_{H^\epsilon(A(d,2d))} + d^{-1} \sum_{l=1}^{2} \|f\|^2_{L^2(\Gamma_l^d)} \right).$$

Covering $\mathcal{C}$ with annuli of the form $A(d, 2d)$, we obtain

$$\int_{\mathcal{C}} r^{-2\epsilon} f^2 \lesssim |f|^2_{H^\epsilon(\mathcal{C}_1)} + \sum_{l=1}^{2} \|r^{1/2-\epsilon} f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})}, \tag{3.37}$$

where $\Gamma_l^{\mathcal{C}_1}$, $l = 1, 2$, denote the straight-lined parts of $\partial\mathcal{C}_1$. The first term on the right hand side of (3.37) is finite by assumption on the regularity of f, and for the second term note that for $f \in H^\epsilon(\mathcal{C}_1)$, it follows $f|_{\Gamma_l^{\mathcal{C}_1}} \in H^{\epsilon-1/2}(\Gamma_l^{\mathcal{C}_1})$, and so $\|r^{1/2-\epsilon} f\|_{L^2(\Gamma_l^{\mathcal{C}_1})} < \infty$ again by [37, Theorem 1.4.4.3]. Thus the second term is also finite, and (3.37) shows $f \in K^0_{-\epsilon}(\mathcal{C})$. The case $\epsilon = 1/2$ is seen by interpolation arguments, cf. [59, Ch. 23].
We now prove (ii) for $\epsilon \in (0, 1/2)$. We get with Lemma 3.29

$$\begin{aligned}\int_{A(d,2d)} r^{-2-2\epsilon} f^2 &\lesssim d^{-2\epsilon} \int_{A(1,2)} \widehat{f}^2 \lesssim d^{-2\epsilon} \left(|\nabla \widehat{f}|^2_{H^\epsilon(A(1,2))} + \sum_{l=1}^{2} \|\widehat{f}\|^2_{L^2(\Gamma_l^1)} \right) \\ &\lesssim d^{-2\epsilon} \left(d^{-2+2(1+\epsilon)} |\nabla f|^2_{H^\epsilon(A(d,2d))} + d^{-1} \sum_{l=1}^{2} \|f\|^2_{L^2(\Gamma_l^d)} \right).\end{aligned}$$

Covering $\mathcal{C}$ with annuli of the form $A(d, 2d)$, we obtain

$$\int_{\mathcal{C}} r^{-2-2\epsilon} f^2 \lesssim |\nabla f|^2_{H^\epsilon(\mathcal{C}_1)} + \sum_{l=1}^{2} \int_{\Gamma_l^{\mathcal{C}_1}} r^{-1-2\epsilon} f^2(x)\, ds(x). \tag{3.38}$$

The first term is finite by assumption on the regularity of f. Now note that $f \in H^{1+\epsilon}(\mathcal{C}_1)$ implies $f|_{\Gamma_l^{\mathcal{C}_1}} \in H^{1/2+\epsilon}(\Gamma_l^{\mathcal{C}_1})$ by trace estimates. Thus we get

$$\int_{\Gamma_l^{\mathcal{C}_1}} r^{-1-2\epsilon} f^2 \lesssim \|r^{-(1/2+\epsilon)} f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})} < +\infty$$

by [37, Theorem 1.4.4.3], since $f|_{\Gamma_l^{\mathcal{C}_1}}$ vanishes at the origin. To estimate the missing part of the $K^1_{-\epsilon}(\mathcal{C})$-norm, we get again with Lemma 3.29

$$\begin{aligned}\int_{A(d,2d)} r^{-2\epsilon} |\nabla f|^2 &\lesssim d^{-2\epsilon} \int_{A(1,2)} |\hat{\nabla} \widehat{f}|^2 \lesssim d^{-2\epsilon} \left(|\widehat{f}|^2_{H^{1+\epsilon}(A(1,2))} + \sum_{l=1}^{2} \|\widehat{f}\|^2_{L^2(\Gamma_l^1)} \right) \\ &\lesssim d^{-2\epsilon} \left(d^{-2+2(1+\epsilon)} |f|^2_{H^{1+\epsilon}(A(d,2d))} + d^{-1} \sum_{l=1}^{2} \|f\|^2_{L^2(\Gamma_l^d)} \right).\end{aligned}$$

Covering $\mathcal{C}$ with annuli of the form $A(d, 2d)$, we obtain

$$\int_{\mathcal{C}} r^{-2\epsilon} |\nabla f|^2 \lesssim |f|^2_{H^{1+\epsilon}(\mathcal{C}_1)} + \sum_{l=1}^{2} \|r^{-(1/2+\epsilon)} f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})} < \infty$$

by [37, Theorem 1.4.4.3], which shows $f \in K^1_{-\epsilon}(\mathcal{C})$. For $\epsilon \in [1/2, 1)$, the proof follows analogously, using [37, Theorem 1.4.4.4]. We mention that the assumptions of this result are satisfied since $f|_{\Gamma_l^{\mathcal{C}_1}} \in H_0^{1/2+\epsilon}(\Gamma_l^{\mathcal{C}_1})$ by Theorem 2.3, cf. also [45, Thm. 3.40].

We now show (iii) for $\epsilon \in (0, 1/2)$. We obtain by Lemma 3.29

$$\int_{A(d,2d)} r^{-4-2\epsilon} f^2 \lesssim d^{-2-2\epsilon} \int_{A(1,2)} \widehat{f}^2 \lesssim d^{-2-2\epsilon} \left(|D^2 \widehat{f}|^2_{H^\epsilon(A(1,2))} + \sum_{j=0}^{1} \sum_{l=1}^{2} \|D^j \widehat{f}\|^2_{L^2(\Gamma_l^1)} \right)$$
$$\lesssim d^{-2-2\epsilon} \left(d^{-2+2(2+\epsilon)} |D^2 f|^2_{H^\epsilon(A(d,2d))} + d^{-1+2} \sum_{l=1}^{2} \|\nabla f\|^2_{L^2(\Gamma_l^d)} + d^{-1} \sum_{l=1}^{2} \|f\|^2_{L^2(\Gamma_l^d)} \right). \tag{3.39}$$

Covering $\mathcal{C}$ with annuli of the form $A(d, 2d)$, we obtain

$$\int_{\mathcal{C}} r^{-4-2\epsilon} f^2 \lesssim |D^2 f|^2_{H^\epsilon(\mathcal{C}_1)} + \sum_{l=1}^{2} \left(\int_{\Gamma_l^{\mathcal{C}_1}} r^{-1-2\epsilon} (\nabla f)^2(x)\, ds(x) + \int_{\Gamma_l^{\mathcal{C}_1}} r^{-3-2\epsilon} f^2(x)\, ds(x) \right). \tag{3.40}$$

The first term is finite by assumption on the regularity of f. The second term satisfies

$$\begin{aligned} \int_{\Gamma_l^{\mathcal{C}_1}} r^{-1-2\epsilon} (\nabla f)^2(x)\, ds(x) &\lesssim \|r^{-(1/2+\epsilon)} \nabla f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})} \\ &\lesssim \|r^{-(1/2+\epsilon)} \partial_x f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})} + \|r^{-(1/2+\epsilon)} \partial_y f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})}. \end{aligned} \tag{3.41}$$

Since $f \in H^{2+\epsilon}(\mathcal{C}_1)$ implies $\partial_x f|_{\Gamma_l^{\mathcal{C}_1}} \in H^{1/2+\epsilon}(\Gamma_l^{\mathcal{C}_1})$, the right hand side of (3.41) is finite by [37, Theorem 1.4.4.3], since $\partial_x f(0) = 0$ and $\partial_y f(0) = 0$ by assumption. The third term of (3.40) is handled by

$$\int_{\Gamma_l^{\mathcal{C}_1}} r^{-3-2\epsilon} f^2(x)\, ds(x) = \|r^{-(3/2+\epsilon)} f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})} < +\infty$$

and [37, Theorem 1.4.4.4], since $f(0) = 0$, $\nabla f(0) = 0$ and $f \in H^{2+\epsilon}(\mathcal{C}_1)$ imply $f|_{\Gamma_l^{\mathcal{C}_1}} \in H_0^{3/2+\epsilon}(\Gamma_l^{\mathcal{C}_1})$, see again Theorem 2.3. Since scaling shows

$$\int_{A(d,2d)} r^{-2-2\epsilon} |\nabla f|^2 \lesssim d^{-2-2\epsilon} \int_{A(1,2)} |\hat{\nabla} \widehat{f}|^2 = d^{-2-2\epsilon} |\widehat{f}|^2_{H^1(A(1,2))}$$

and

$$\int_{A(d,2d)} r^{-2\epsilon} |D^2 f|^2 \lesssim d^{-2-2\epsilon} \int_{A(1,2)} |\hat{D}^2 \widehat{f}|^2 = d^{-2-2\epsilon} |\widehat{f}|^2_{H^2(A(1,2))},$$

$f \in K^2_{-\epsilon}(\mathcal{C})$ follows with Lemma 3.29 together with (3.39). The case $\epsilon \in [1/2, 1)$ is seen analogously using [37, Theorem 1.4.4.4]. □

Lemma 3.30 can even be generalized to higher Sobolev orders. The proofs still rely on Lemma 3.29 and remain largely similar.

Lemma 3.31. *Let $f \in H^{k+\epsilon}(\mathcal{C}_1)$ for $k \in \mathbb{N}_0$ and $\epsilon \in (0,1)$ such that $D^j f(0) = 0$ for $j = 0, \dots, k-1$. Additionally assume that f is compactly supported in $B_1(0)$. Then it follows $f \in K^k_{-\epsilon}(\mathcal{C})$ with the norm estimate*

$$\|f\|_{K^k_{-\epsilon}(\mathcal{C})} \lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}.$$

Proof. The cases $k \le 2$ have already been treated in Lemma 3.30, thus we only consider $k > 2$. Using the same notation as in the proof of Lemma 3.30, we get by scaling

$$\begin{aligned} \int_{A(d,2d)} r^{-2k-2\epsilon}|f|^2 &\lesssim d^{-2k-2\epsilon+2}\|\widehat{f}\|^2_{L^2(A(1,2))} \\ &\lesssim d^{-2k-2\epsilon+2}\left(|D^k\widehat{f}|^2_{H^\epsilon(A(1,2))} + \sum_{j=0}^{k-1}\sum_{l=1}^{2}\|D^j\widehat{f}\|^2_{L^2(\Gamma_l^1)}\right) \\ &\lesssim d^{-2k-2\epsilon+2}\left(d^{-2+2(k+\epsilon)}|D^k f|^2_{H^\epsilon(A(d,2d))} + \sum_{j=0}^{k-1} d^{-1+2j}\sum_{l=1}^{2}\|D^j f\|^2_{L^2(\Gamma_l^d)}\right) \\ &\lesssim |D^k f|^2_{H^\epsilon(A(d,2d))} + \sum_{j=0}^{k-1}\sum_{l=1}^{2}\|r^{-(k+\epsilon-(1/2+j))}D^j f\|^2_{L^2(\Gamma_l^d)}. \end{aligned} \tag{3.42}$$

Covering $\mathcal{C}$ with annuli of the form $A(d, 2d)$, we obtain

$$\int_{\mathcal{C}} r^{-2k-2\epsilon}|f|^2 \lesssim |D^k f|^2_{H^\epsilon(\mathcal{C}_1)} + \sum_{j=0}^{k-1}\sum_{l=1}^{2}\|r^{-(k+\epsilon-(1/2+j))}D^j f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})}.$$

As $f \in H^{k+\epsilon}(\mathcal{C}_1)$, the trace theorem gives $f|_{\Gamma_l^{\mathcal{C}_1}} \in H^{k+\epsilon-1/2}(\Gamma_l^{\mathcal{C}_1})$. Since $D^j f(0) = 0$ for $j = 0, \dots, k-1$, we even have $f|_{\Gamma_l^{\mathcal{C}_1}} \in H_0^{k+\epsilon-1/2}(\Gamma_l^{\mathcal{C}_1})$, cf. [45, Thm. 3.40]. It follows

$$\sum_{j=0}^{k-1}\|r^{-(k+\epsilon-(1/2+j))}D^j f\|^2_{L^2(\Gamma_l^{\mathcal{C}_1})} \lesssim \|f\|^2_{H^{k+\epsilon-1/2}(\Gamma_l^{\mathcal{C}_1})} \lesssim \|f\|^2_{H^{k+\epsilon}(\mathcal{C}_1)} \tag{3.43}$$

by [37, Theorem 1.4.4.4]. Higher derivatives of f satisfy the inequalities

$$\int_{A(d,2d)} r^{-2k-2\epsilon+2j}|D^j f|^2 \lesssim d^{-2k-2\epsilon+2}\|D^j\widehat{f}\|^2_{L^2(A(1,2))}$$

for $j = 1, \dots, k$. Proceeding as in (3.42) via Lemma 3.29, covering $\mathcal{C}$ with annuli and using (3.43) shows $f \in K^k_{-\epsilon}(\mathcal{C})$. □

Solutions of the Poisson problem in weighted spaces are locally members of Sobolev spaces, as the next proposition shows.

Proposition 3.32. *Let $k \in \mathbb{N}_0$ and $\epsilon \in (0,1)$, let $\omega < \frac{2\pi}{k+1+\epsilon}$, $\omega \neq \frac{\pi}{k+1+\epsilon}$, and let $f \in H^{k+\epsilon}(\mathcal{C}_1)$ satisfy $D^j f(0) = 0$ for $j = 0, \dots, k-1$ and be compactly supported in $B_1(0)$. Further let $u \in K^{k+2}_{-\epsilon}(\mathcal{C})$ solve the problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \mathcal{C}, \\ u &= 0, \quad \phi \in \{0, \omega\}. \end{aligned}$$

Then $u \in H^{k+2+\epsilon}(\mathcal{C}_R)$ for $R > 0$ together with the estimate

$$\|u\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}.$$

Proof. The function $f \in H^{k+\epsilon}(\mathcal{C}_1)$ is in $K^k_{-\epsilon}(\mathcal{C})$ by Lemma 3.31 if extended to $\mathcal{C}$ by zero. Note that the assumptions of Theorem 3.22 are satisfied since $\omega \neq \frac{\pi}{k+1+\epsilon}$, hence it follows $u \in K^{k+2}_{-\epsilon}(\mathcal{C})$. We see that the problem is well-posed and $u \in H^{k+2}(\mathcal{C}_R)$, and we have

$$\|u\|_{H^{k+2}(\mathcal{C}_R)} \lesssim \|u\|_{K^{k+2}_{-\epsilon}(\mathcal{C})} \lesssim \|f\|_{K^k_{-\epsilon}(\mathcal{C})} \lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}.$$

Lemma 3.11 then yields

$$\begin{aligned} |D^{k+2}u|_{H^\epsilon(\mathcal{C}_R)} &\lesssim |D^k f|_{H^\epsilon(\mathcal{C}_1)} + \|f\|_{K^k_{-\epsilon}(\mathcal{C})} + \|u\|_{K^{k+2}_{-\epsilon}(\mathcal{C})} \\ &\lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C}_1)} + \|f\|_{K^k_{-\epsilon}(\mathcal{C})} \lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}. \end{aligned}$$

Hence we get $u \in H^{k+2+\epsilon}(\mathcal{C} \cap B_R(0))$, and the desired estimate holds. □

Remark 3.33. Proposition 3.32 also holds in the Neumann setting, which follows after replacing Theorem 3.22 with Proposition 3.27. The condition $\gamma - s + 1 \notin \pm\sigma^N$ which arises from Proposition 3.27 is satisfied by the same arguments as in the Dirichlet case. The additional pole in σ^N at zero is no problem since $\gamma - s + 1 = -\epsilon - (k+2) + 1 = -\epsilon - k - 1 \neq 0$ for all $k \in \mathbb{N}_0$ and $\epsilon \in (0,1)$.

We have now collected the necessary tools for proving decompositions similar to those in Sections 3.2 and 3.3.

Proposition 3.34. *For $k \in \mathbb{N}_0$ and $\epsilon \in (0,1)$, let $\omega < \frac{2\pi}{k+1+\epsilon}$, $\omega \neq \frac{\pi}{k+1+\epsilon}$, and $f \in H^{k+\epsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$. Further assume $D^j f(0) = 0$ for $j = 0, \dots, k-1$. Then $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$ solving*

$$\begin{aligned} -\Delta u_1 &= f \in H^{k+\epsilon}(\mathcal{C}) \\ u_1 &= 0, \quad \phi \in \{0, \omega\} \end{aligned} \tag{3.44}$$

satisfies

$$u_1 = \begin{cases} u_0, & k + \epsilon + 1 < \frac{\pi}{\omega} \\ u_0 - \frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1} \sin(\lambda_1\varphi) f(x)\, dx\right) r^{\lambda_1} \sin(\lambda_1 \phi), & k + \epsilon + 1 > \frac{\pi}{\omega} \end{cases}$$

for a $u_0 \in H^{k+2+\epsilon}(\mathcal{C} \cap B_R(0))$ for $R > 0$, with the estimate

$$\|u_0\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}.$$

Proof. By Lemma 3.31 the function f is in the space $K^k_{-\epsilon}(\mathcal{C})$. By Theorem 3.22 there exists a function $u_0 \in K^{k+2}_{-\epsilon}(\mathcal{C})$ that solves

$$\begin{aligned} -\Delta u_0 &= f \in K^k_{-\epsilon}(\mathcal{C}) \\ u_0 &= 0, \quad \phi \in \{0, \omega\}. \end{aligned}$$

In fact, $u_0 \in H^{k+2+\epsilon}(\mathcal{C}_R)$ by Proposition 3.32. As in Section 3.2, both the energy solution u_1 and the solution u_0 will be different, since we specify different conditions at infinity. We now demonstrate how the functions u_0 and u_1 are related, using the ideas of Section3.2. Using the Mellin transform and considering the function spaces of u_0 and u_1 as well as the definition of u_0 by the proof of Theorem 3.22, we get

$$(-\partial_\phi^2 + \zeta^2)\mathcal{M}u_1 = \mathcal{M}g \quad \text{on } \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$$

and

$$(-\partial_\phi^2 + \zeta^2)\mathcal{M}u_0 = \mathcal{M}g \quad \text{on } \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta = -1 - k - \epsilon\}$$

with the usual definition of $\check{g} := e^{2t}\check{f}$. Since $f \in K^k_{-\epsilon}(\mathcal{C})$, it follows by Lemma 3.18 that

$$e^{-t(k+\epsilon+1)}\check{g} \in H^k(\mathbb{R} \times G),$$

and thus $\mathcal{M}g$ is holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > -1 - k - \epsilon\}$ with values in $H^k(G)$, cf. Remark 3.20 and Theorem 3.3. We get that $\mathcal{M}u_1$ is holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$ with values in $H^2(G)$ the same way. From Theorem 3.19 we obtain

$$\mathcal{M}g(\cdot - (1 + k + \epsilon)i) \in L^2(\mathbb{R}; H^k(G)), \quad \mathcal{M}u_1 \in L^2(\mathbb{R}; H^2(G))$$

and

$$\mathcal{M}u_0(\cdot - (1 + k + \epsilon)i) \in L^2(\mathbb{R}; H^{k+2}(G)).$$

Now note that the operator $(\mathcal{L}(\zeta))^{-1}$ is meromorphic on $\mathbb{C}$ with poles at $\pm i\lambda_n$, where $\lambda_n = \frac{\pi}{\omega}n$ as before. From the equation

$$\mathcal{L}(\zeta)\mathcal{M}u_1 = \mathcal{M}g(\zeta),$$

where $\mathcal{M}u_1$ is holomorphic for $\operatorname{Im}\zeta > 0$ and $\mathcal{M}g$ is holomorphic for $\operatorname{Im}\zeta > -1 - k - \epsilon$, we observe that $\mathcal{M}u_1$ can be meromorphically extended to $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > -1 - k - \epsilon\}$ by

$$U(\zeta) := \mathcal{M}u_1(\zeta) := (\mathcal{L}(\zeta))^{-1}\mathcal{M}g(\zeta).$$

Let us mention that by definition, $U(\zeta)$ and $\mathcal{M}u_0(\zeta)$ coincide on ζ with imaginary part $-1 - k - \epsilon$. Since the proof of Theorem 3.23 still holds analogously with one inverse transform defined at $\operatorname{Im}\zeta = -1 - k - \epsilon$ instead of $\operatorname{Im}\zeta = -1$, we get

$$u_0 - u_1 = \sum_{\substack{\zeta \in -i\sigma \\ \operatorname{Im}\zeta \in (-1-k-\epsilon, 0)}} \frac{2\pi i}{\sqrt{2\pi}} \operatorname*{Res}_{\zeta}\left(r^{i\zeta}(\mathcal{L}(\zeta))^{-1}\mathcal{M}g(\zeta)\right), \tag{3.45}$$

where $\sigma = \{\lambda_n : n \in \mathbb{N}\}$. Since $\omega < \frac{2\pi}{k+1+\epsilon}$, the sum (3.45) is either zero or has exactly one term. The determination of the residue then follows exactly the same lines as in Section 3.2, so we obtain

$$u_1 = \begin{cases} u_0, & \omega < \frac{\pi}{1+k+\epsilon} \\ u_0 - \frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1}\sin(\lambda_1\varphi) f(x)\,dx\right) r^{\lambda_1}\sin(\lambda_1\phi), & \omega > \frac{\pi}{1+k+\epsilon} \end{cases},$$

where $u_0 \in H^{k+2+\epsilon}(\mathcal{C}_R)$ by Proposition 3.32. □

Proposition 3.35. *For $k \in \mathbb{N}_0$ and $\epsilon \in (0,1)$, let $\omega < \frac{2\pi}{k+1+\epsilon}$, $\omega \neq \frac{\pi}{k+1+\epsilon}$, and $f \in H^{k+\epsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$. Further assume $D^j f(0) = 0$ for $j = 0, \dots, k-1$. Then every function $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$ solving*

$$\begin{aligned} -\Delta u_1 &= f \in H^{k+\epsilon}(\mathcal{C}) \\ \partial_n u_1 &= 0, \quad \phi \in \{0, \omega\} \end{aligned} \tag{3.46}$$

satisfies

$$u_1 = \begin{cases} u_0, & k+\epsilon+1 < \frac{\pi}{\omega} \\ u_0 - \frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1}\cos(\lambda_1\varphi) f(x)\,dx\right) r^{\lambda_1}\cos(\lambda_1\phi), & k+\epsilon+1 > \frac{\pi}{\omega} \end{cases} \tag{3.47}$$

for a $u_0 \in H^{k+2+\epsilon}(\mathcal{C} \cap B_R(0))$ for $R > 0$, with the estimate

$$\|u_0\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}.$$

Proof. We follow the lines of Proposition 3.34 together with the evaluation of the residue as in Section 3.3. We point out that the main differences are the application of Proposition 3.27 instead of Theorem 3.22, that $\mathcal{M}u_1$ is holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > \delta\}$ for $\delta > 0$, and that we get

$$u_0 - u_1 = \sum_{\substack{\zeta \in -i\sigma \\ \operatorname{Im}\zeta \in (-1-k-\epsilon, \delta)}} \frac{2\pi i}{\sqrt{2\pi}} \operatorname*{Res}_{\zeta}\left(r^{i\zeta}\left(\mathcal{L}(\zeta)\right)^{-1}\mathcal{M}g(\zeta)\right)$$

instead of line (3.45). Taking the additional pole at zero into account, (3.47) follows from the calculations in Section 3.3 if we hide the constant $C(f)$ in the function u_0 - which does not violate the regularity and norm estimate of u_0, cf. (3.33). □

The next lemma helps us to get rid of the necessary decay properties at zero that appeared in Propositions 3.34 and 3.35.

Lemma 3.36. *Let $i, j, k \in \mathbb{N}_0$ with $i + j = k$. Further assume that $\frac{(k+2)\omega}{\pi} \notin \mathbb{N}$. Then there exists a solution $p_{i,j}$ of the equation*

$$\begin{aligned} -\Delta u &= x^i y^j \quad \text{on } \mathcal{C} \\ u|_\Gamma &= 0 \end{aligned} \tag{3.48}$$

that is a polynomial of degree $k+2$. Additionally, there exists a solution $p_{i,j,N}$ of the equation

$$\begin{aligned} -\Delta u &= x^i y^j \quad \text{on } \mathcal{C} \\ \partial_n u|_\Gamma &= 0 \end{aligned} \tag{3.49}$$

that is a polynomial of degree $k+2$.

Proof. As a first step we use induction to prove the existence of the desired polynomial without taking the boundary conditions into account. We use induction in the y-variable, the rest follows by symmetry arguments. For $i \in \mathbb{N}$ and $j=0$, we get $u = -\frac{1}{(i+2)(i+1)}x^{i+2}$ as a solution of the equation $-\Delta u = x^i y^0$. As induction hypothesis we assume that we already have polynomials $u_{i,j}$ of degrees $i+j+2$ such that $-\Delta u_{i,j} = x^i y^j$ for all $i \in \mathbb{N}_0$ and all $j \leq N$, and proceed with the induction step where we must verify existence of a polynomial for the equation $-\Delta u = x^i y^{N+1}$. We get

$$-\Delta(x^{i+2}y^{N+1}) = -(i+2)(i+1)x^i y^{N+1} - (N+1)N x^{i+2} y^{N-1}.$$

By the induction hypothesis we already know the existence of a polynomial $\widetilde{u}$ of degree $i+N+1$ such that $-\Delta\widetilde{u} = (N+1)N x^{i+2}y^{N-1}$. Thus the polynomial $u := -x^{i+2}y^{N+1} - \widetilde{u}$ is of the desired degree and solves $-\Delta u = (i+2)(i+1)x^i y^{N+1}$ which concludes the proof by induction. The existence of a polynomial additionally satisfying boundary conditions (both Dirichlet and Neumann) then follows with [47, Lemma 6.1.1]. □

Remark 3.37. A possible solution $p_{0,0}$ of (3.48) for $i=j=0$ is

$$p_{0,0}(x,y) = -\frac{1}{2}y^2 + \frac{\sin^2\omega}{\sin 2\omega}xy$$

in Cartesian coordinates. Thus, a solution of the problem

$$\begin{aligned} -\Delta\widetilde{p} &= f(0) \quad \text{in } \mathcal{C}, \\ \widetilde{p}|_\Gamma &= 0 \end{aligned}$$

is $\widetilde{p} = p_{0,0}f(0)$.

Corollary 3.38. *For $k \in \mathbb{N}_0$ and $\epsilon \in (0,1)$, let $\omega < \frac{2\pi}{k+1+\epsilon}$, $\omega \neq \frac{\pi}{k+1+\epsilon}$, $\frac{n\omega}{\pi} \notin \mathbb{N}$ for $n = 2,\dots,k+1$, and $f \in H^{k+\epsilon}(\mathcal{C})$ with $\operatorname{supp} f \subseteq B_1(0)$. Further let χ be a smooth cut-off-function with support in $B_1(0)$ satisfying $\chi \equiv 1$ near the origin. Then every function $u_1 \in H^1(\mathcal{C})$ with $\operatorname{supp} u_1 \subseteq B_1(0)$ solving*

$$\begin{aligned} -\Delta u_1 &= f \in H^{k+\epsilon}(\mathcal{C}) \\ u_1 &= 0, \quad \phi \in \{0,\omega\} \end{aligned} \tag{3.50}$$

satisfies

$$u_1 = \begin{cases} u_0, & k+\epsilon+1 < \frac{\pi}{\omega} \\ u_0 - \frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1}\sin(\lambda_1\varphi)(f(\mathbf{x}) + \Delta(\chi(\mathbf{x})P_{k-1}(\mathbf{x})))\,d\mathbf{x}\right) r^{\lambda_1}\sin(\lambda_1\phi), & k+\epsilon+1 > \frac{\pi}{\omega} \end{cases} \tag{3.51}$$

with $P_{k-1}(\mathbf{x}) := \sum_{i+j\le k-1} \frac{1}{i!j!} p_{i,j}(\mathbf{x})(\partial_x^i \partial_y^j f)(0)$, $u_0 \in H^{k+2+\epsilon}(\mathcal{C} \cap B_R(0))$ *for* $R > 0$, *where* $p_{i,j}$ *are fixed polynomial functions from Lemma 3.36. Furthermore, the estimate*

$$\|u_0\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C})} \tag{3.52}$$

holds, where the constant is dependent on the angle ω*, the choice of the polynomials* $p_{i,j}$ *and the choice of the cut-off function* χ*.*

Proof. We only consider $k \ge 1$, since the claim for $k = 0$ is a restatement of Proposition 3.34.
By Lemma 3.36 we obtain possible (fixed) polynomial solutions $p_{i,j}$ such that $P_{k-1} = \sum_{i+j\le k-1} \frac{1}{i!j!} p_{i,j} (\partial_x^i \partial_y^j f)(0)$ solves the problem

$$\begin{aligned} -\Delta P_{k-1} &= \sum_{i+j\le k-1} \frac{1}{i!j!} x^i y^j (\partial_x^i \partial_y^j f)(0) \\ P_{k-1} &= 0, \quad \phi \in \{0, \omega\}. \end{aligned} \tag{3.53}$$

We now define $\widetilde{u_1} := u_1 - \chi P_{k-1}$ which also has support in $B_1(0)$. Since u_1 solves (3.50), the function $\widetilde{u_1} \in H^1(\mathcal{C})$ solves the problem

$$\begin{aligned} -\Delta \widetilde{u_1} &= \widetilde{f} := f + \Delta(\chi P_{k-1}) \in H^{k+\epsilon}(\mathcal{C}) \\ \widetilde{u_1} &= 0, \quad \phi \in \{0, \omega\}. \end{aligned} \tag{3.54}$$

Note that the right-hand side $\widetilde{f}$ satisfies for $l = 0, \dots, k-1$ and $i + j = l$

$$\begin{aligned} D^l \widetilde{f}(0) &= (\partial_x^i \partial_y^j f)(0) + (\partial_x^i \partial_y^j \Delta(\chi P_{k-1}))(0) \\ &= (\partial_x^i \partial_y^j f)(0) + \underbrace{(\partial_x^i \partial_y^j \Delta\chi P_{k-1})(0)}_{=0} + \underbrace{2(\partial_x^i \partial_y^j (\nabla\chi \cdot \nabla P_{k-1}))(0)}_{=0} + \underbrace{(\partial_x^i \partial_y^j \chi \Delta P_{k-1})(0)}_{=(\partial_x^i \partial_y^j \Delta P_{k-1})(0)} \\ &= (\partial_x^i \partial_y^j f)(0) - \left(\partial_x^i \partial_y^j \sum_{i'+j'\le k-1} \frac{1}{i'!j'!} x^{i'} y^{j'} (\partial_x^{i'} \partial_y^{j'} f)(0) \right)(0) \\ &= (\partial_x^i \partial_y^j f)(0) - (\partial_x^i \partial_y^j f)(0) = 0. \end{aligned} \tag{3.55}$$

Thus Proposition 3.34 can be applied to problem (3.54) and we obtain

$$\widetilde{u_1} = \begin{cases} \widetilde{u_0}, & k+1+\epsilon < \frac{\pi}{\omega} \\ \widetilde{u_0} - \frac{1}{\pi} \left(\int_{\mathcal{C}} r^{-\lambda_1} \sin(\lambda_1 \varphi) \widetilde{f}(x)\, dx \right) r^{\lambda_1} \sin(\lambda_1 \phi), & k+1+\epsilon > \frac{\pi}{\omega} \end{cases}$$

with $\widetilde{u_0} \in H^{k+2+\epsilon}(\mathcal{C} \cap B_R(0))$ for $R > 0$. The proof of (3.51) is complete by defining $u_0 := \widetilde{u_0} + \chi P_{k-1}$, which obviously gives the desired regularity for u_0.

We now show the estimate (3.52). We have

$$\begin{aligned}\|u_0\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} &= \|\widetilde{u_0} + \chi P_{k-1}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \\ &\le \|\widetilde{u_0}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} + \|\chi P_{k-1}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \\ &\lesssim \|\widetilde{f}\|_{H^{k+\epsilon}(\mathcal{C})} + \|\chi P_{k-1}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \\ &\le \|f\|_{H^{k+\epsilon}(\mathcal{C})} + \|\Delta(\chi P_{k-1})\|_{H^{k+\epsilon}(\mathcal{C})} + \|\chi P_{k-1}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \\ &\lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C})} + \|\chi P_{k-1}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_1(0))}.\end{aligned}$$

The polynomial functions $p_{i,j}$ are independent of f, thus on the bounded domain $\mathcal{C}_1$, they can be bounded by a constant only dependent on the angle ω and on the polynomials. Thus, we get

$$\begin{aligned}\|\chi P_{k-1}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_1(0))} &= \left\| \sum_{i+j\le k-1} \frac{1}{i!j!} p_{i,j} (\partial_x^i \partial_y^j f)(0) \right\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_1(0))} \\ &\le \sum_{i+j\le k-1} \frac{1}{i!j!} \|\chi p_{i,j}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_1(0))} |(\partial_x^i \partial_y^j f)(0)| \\ &\lesssim \sum_{i+j\le k-1} \frac{1}{i!j!} C(\omega,i,j) |(\partial_x^i \partial_y^j f)(0)| \\ &\lesssim \sum_{i+j\le k-1} \frac{1}{i!j!} \max_{i+j\le k-1} C(\omega,i,j) |(\partial_x^i \partial_y^j f)(0)|.\end{aligned}$$

By a Sobolev embedding and the 1D and 2D trace theorems we get

$$|(\partial_x^i \partial_y^j f)(0)| \lesssim \|f\|_{H^{\epsilon+i+j+1}(\mathcal{C})} \lesssim \|f\|_{H^{k+\epsilon}(\mathcal{C})}.$$

Since $\sum_{i+j\le k-1} \frac{1}{i!j!} \le \exp(2)$, we have

$$\|\chi P_{k-1}\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_1(0))} \lesssim \max_{i+j\le k-1} C(\omega,i,j) \|f\|_{H^{k+\epsilon}(\mathcal{C})} \tag{3.56}$$

and hence

$$\|u_0\|_{H^{k+2+\epsilon}(\mathcal{C}\cap B_R(0))} \lesssim \max_{i+j\le k-1} C(\omega,i,j) \|f\|_{H^{k+\epsilon}(\mathcal{C})}.$$

□

We still have to deal with the singularity functions which is now accomplished through Lemma 2.15 together with the following result which is motivated by Lemma 2.16.

Lemma 3.39. *For $\alpha > 1$, $\alpha \notin \mathbb{N}$, set $k := \lfloor \alpha \rfloor - 1$ and let P_{k-1} be the polynomial function and χ be the cut-off function from Corollary 3.38. Then the mapping*

$$f \mapsto S(f) := \int_{\mathcal{C}_1} r^{-\alpha} \sin(\alpha\phi)(f + \Delta(\chi P_{k-1}))\, dx$$

is bounded and linear on $B_{2,1}^{\alpha-1}(\mathcal{C}_1)$.

Proof. Choose $0 < \epsilon \ll 1$ such that $k + 1 - \alpha + \epsilon < 0$. We have

$$\begin{aligned}
B^{\alpha-1}_{2,1}(\mathcal{C}_1) &= (H^{k+\epsilon}(\mathcal{C}_1), H^{k+1}(\mathcal{C}_1))_{\frac{\alpha-k-1-\epsilon}{1-\epsilon},1} \\
&= (H^{k+\epsilon}(\mathcal{C}_1), (H^{k+\epsilon}(\mathcal{C}_1), H^{k+2+\epsilon}(\mathcal{C}_1))_{\frac{1-\epsilon}{2},2})_{\frac{\alpha-k-1-\epsilon}{1-\epsilon},1} \\
&= (H^{k+\epsilon}(\mathcal{C}_1), (H^{k+\epsilon}(\mathcal{C}_1), H^{k+2+\epsilon}(\mathcal{C}_1))_{\frac{1-\epsilon}{2},1})_{\frac{\alpha-k-1-\epsilon}{1-\epsilon},1} \\
&= (H^{k+\epsilon}(\mathcal{C}_1), B^{k+1}_{2,1}(\mathcal{C}_1))_{\frac{\alpha-k-1-\epsilon}{1-\epsilon},1}
\end{aligned}$$

by the Reiteration Theorem 2.11. Now assume $f \in C^\infty(\overline{\mathcal{C}_1})$, $f \neq 0$, the general statement will then follow by density arguments. For

$$\delta := \min\left\{ \|f\|^{\frac{1}{1-\epsilon}}_{H^{k+\epsilon}(\mathcal{C}_1)} \|f\|^{-\frac{1}{1-\epsilon}}_{B^{k+1}_{2,1}(\mathcal{C}_1)}, \frac{1}{2}\operatorname{diam}\{x \in \mathcal{C} : \chi(x) = 1\} \right\},$$

denote by χ_δ a smooth cut-off-function that equals zero for $|x| < \delta$ and is one for $|x| > 2\delta$. We can then write

$$\begin{aligned}
S(f) &= \int_{\mathcal{C}_1} r^{-\alpha} \sin(\alpha\phi)(f + \Delta(\chi P_{k-1}))\,dx \\
&= \int_{\mathcal{C}_1} r^{-\alpha} \sin(\alpha\phi)\chi_\delta(f + \Delta(\chi P_{k-1}))\,dx + \int_{\mathcal{C}_1} r^{-\alpha} \sin(\alpha\phi)(1-\chi_\delta)(f + \Delta(\chi P_{k-1}))\,dx.
\end{aligned} \tag{3.57}$$

The first integral is estimated by

$$\begin{aligned}
&\left| \int_{\mathcal{C}_1} r^{-\alpha} \sin(\alpha\phi)\chi_\delta(f + \Delta(\chi P_{k-1}))\,dx \right| = \left| \int_{\mathcal{C}_1} r^{-\alpha+k+\epsilon} \sin(\alpha\phi)\chi_\delta \frac{f + \Delta(\chi P_{k-1})}{r^{k+\epsilon}}\,dx \right| \\
&\lesssim \left\| \frac{f + \Delta(\chi P_{k-1})}{r^{k+\epsilon}} \right\|_{L^2(\mathcal{C}_1)} \left| \int_\delta^1 r^{-2\alpha+2k+2\epsilon+1}\chi_\delta^2\,dr \right|^{\frac{1}{2}} \lesssim \left\| \frac{f + \Delta(\chi P_{k-1})}{r^{k+\epsilon}} \right\|_{L^2(\mathcal{C}_1)} \delta^{k+1-\alpha+\epsilon}.
\end{aligned}$$

The remaining L^2-norm can be handled with Lemma 3.31: Since $D^j(f + \Delta(\chi P_{k-1}))(0) = 0$ for $j = 0, \dots, k-1$, cf. (3.55), we have

$$\begin{aligned}
\left\| \frac{f + \Delta(\chi P_{k-1})}{r^{k+\epsilon}} \right\|^2_{L^2(\mathcal{C}_1)} &\leq \|f + \Delta(\chi P_{k-1})\|^2_{K^k_{-\epsilon}(\mathcal{C}_1)} \\
&\lesssim \|f + \Delta(\chi P_{k-1})\|^2_{H^{k+\epsilon}(\mathcal{C}_1)} \overset{(3.56)}{\lesssim} \|f\|^2_{H^{k+\epsilon}(\mathcal{C}_1)}.
\end{aligned}$$

For the delta term, we analyze two cases. In the case where $\|f\|^{\frac{1}{1-\epsilon}}_{H^{k+\epsilon}(\mathcal{C}_1)} \|f\|^{-\frac{1}{1-\epsilon}}_{B^{k+1}_{2,1}(\mathcal{C}_1)} \leq \frac{1}{2}\operatorname{diam}\{x \in \mathcal{C} : \chi(x) = 1\}$, we get directly

$$\delta^{k+1-\alpha+\epsilon} = \left(\|f\|^{\frac{1}{1-\epsilon}}_{H^{k+\epsilon}(\mathcal{C}_1)} \|f\|^{-\frac{1}{1-\epsilon}}_{B^{k+1}_{2,1}(\mathcal{C}_1)} \right)^{k+1-\alpha+\epsilon},$$

since the exponent $k+1-\alpha+\epsilon<0$, and if $\|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}^{\frac{1}{1-\epsilon}}\|f\|_{B^{k+1}_{2,1}(\mathcal{C}_1)}^{-\frac{1}{1-\epsilon}} > \frac{1}{2}\operatorname{diam}\{x\in\mathcal{C}:\chi(x)=1\}$, we note that $\|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}^{\frac{1}{1-\epsilon}}\|f\|_{B^{k+1}_{2,1}(\mathcal{C}_1)}^{-\frac{1}{1-\epsilon}}\lesssim 1$, since $B^{k+1}_{2,1}(\mathcal{C}_1)$ is imbedded in $H^{k+\epsilon}(\mathcal{C}_1)$. Thus we have

$$\delta^{k+1-\alpha+\epsilon}\lesssim\left(\frac{1}{2}\operatorname{diam}\{x\in\mathcal{C}:\chi(x)=1\}\right)^{k+1-\alpha+\epsilon}\lesssim\left(\|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}^{\frac{1}{1-\epsilon}}\|f\|_{B^{k+1}_{2,1}(\mathcal{C}_1)}^{-\frac{1}{1-\epsilon}}\right)^{k+1-\alpha+\epsilon},$$

since the cut-off function χ has support only in $B_1(0)$. For the second integral of (3.57) we obtain

$$\begin{aligned}\left|\int_{\mathcal{C}_1} r^{-\alpha}\sin(\alpha\phi)(1-\chi_\delta)(f+\Delta(\chi P_{k-1}))\,dx\right| &= \left|\int_{\mathcal{C}_1} r^{-\alpha+k}\sin(\alpha\phi)(1-\chi_\delta)\frac{f+\Delta(\chi P_{k-1})}{r^k}\,dx\right|\\ &\lesssim \left\|\frac{f+\Delta(\chi P_{k-1})}{r^k}\right\|_{L^\infty(\mathcal{C}_1\cap B_{2\delta}(0))}\int_0^{2\delta} r^{-\alpha+k+1}\,dr.\end{aligned}$$

Since $\Delta(\chi P_{k-1})=\Delta P_{k-1}=-\sum_{i+j\le k-1}\frac{1}{i!j!}x^iy^j(\partial_x^i\partial_y^j f)(0)$ on the region where $\chi\equiv 1$, it follows with the embedding $B^1_{2,1}(\mathcal{C}_1)\subseteq C(\mathcal{C}_1)$, cf. [60, Sec. 4.6.1],

$$\begin{aligned}&\left\|\frac{f+\Delta(\chi P_{k-1})}{r^k}\right\|_{L^\infty(\mathcal{C}_1\cap B_{2\delta}(0))}\int_0^{2\delta} r^{-\alpha+k+1}\,dr\\ &\quad\lesssim\left\|\frac{f-\sum_{i+j\le k-1}\frac{1}{i!j!}x^iy^j(\partial_x^i\partial_y^j f)(0)}{r^k}\right\|_{L^\infty(\mathcal{C}_1)}\delta^{k+2-\alpha}\\ &\quad\lesssim\|D^k f\|_{L^\infty(\mathcal{C}_1)}\delta^{k+2-\alpha}\lesssim\|f\|_{B^{k+1}_{2,1}(\mathcal{C}_1)}\left(\|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}^{\frac{1}{1-\epsilon}}\|f\|_{B^{k+1}_{2,1}(\mathcal{C}_1)}^{-\frac{1}{1-\epsilon}}\right)^{k+2-\alpha}.\end{aligned}$$

In total, we have arrived at

$$|S(f)|\lesssim\|f\|_{H^{k+\epsilon}(\mathcal{C}_1)}^{\frac{k+2-\alpha}{1-\epsilon}}\|f\|_{B^{k+1}_{2,1}(\mathcal{C}_1)}^{\frac{\alpha-k-1-\epsilon}{1-\epsilon}}.$$

By [59, Lemma 25.2], it follows $S\in\left((H^{k+\epsilon}(\mathcal{C}_1),B^{k+1}_{2,1}(\mathcal{C}_1))_{\frac{\alpha-k-1-\epsilon}{1-\epsilon},1}\right)^\star=(B^{\alpha-1}_{2,1}(\mathcal{C}_1))^\star$. □

Remark 3.40. It is obvious that Corollary 3.38 and Lemma 3.39 hold analogously for Neumann boundary conditions after replacing the sine functions by cosine functions and using a suitable polynomial function which solves (3.49).

Lemma 3.41. *Let Ω be a polygonal domain with vertices A_i and corresponding interior angles ω_i, $i=1,\dots,J$, and define $\omega_{max}:=\max_{i=1,\dots,J}\omega_i$. Then the following holds:*

(i) If $\frac{\pi}{\omega_{max}}\notin\mathbb{N}$, then $\frac{k\omega_i}{\pi}\notin\mathbb{N}$ for all $k\in\{1,\dots,\lfloor\frac{\pi}{\omega_{max}}\rfloor\}$.

(ii) If $\frac{\pi}{\omega_{max}}\in\mathbb{N}$, then $\frac{k\omega_i}{\pi}\notin\mathbb{N}$ for all $k\in\{1,\dots,\frac{\pi}{\omega_{max}}-1\}$.

Proof. A simple calculation gives $\frac{\pi}{\omega_{max}} \leq \frac{\pi}{\omega_i}$ and thus $\lfloor\frac{\pi}{\omega_{max}}\rfloor < \frac{\pi}{\omega_i}$ for all i if $\frac{\pi}{\omega_{max}}$ is not an integer. It follows

$$\frac{k\omega_i}{\pi} < k\frac{1}{\lfloor\frac{\pi}{\omega_{max}}\rfloor} \leq \lfloor\pi/\omega_{max}\rfloor\frac{1}{\lfloor\frac{\pi}{\omega_{max}}\rfloor} = 1,$$

i.e. $\frac{k\omega_i}{\pi} \notin \mathbb{N}$. If $\frac{\pi}{\omega_{max}} \in \mathbb{N}$ the result is an immediate consequence of

$$\frac{k\omega_i}{\pi} \leq k\frac{\omega_{max}}{\pi} \leq \left(\frac{\pi}{\omega_{max}} - 1\right)\frac{\omega_{max}}{\pi} = 1 - \frac{\omega_{max}}{\pi} < 1.$$

□

3.5 The shift theorem in Sobolev and Besov spaces

We can now formulate the shift theorem, first for Dirichlet boundary conditions and afterwards in the Neumann setting.

3.5.1 The Dirichlet case

For Dirichlet boundary conditions, we get the following result.

Theorem 3.42. *Let Ω be a polygonal domain with vertices A_i and corresponding interior angles ω_i, $i = 1, \dots, J$. Fix the vertex A_i, and fix $R > 0$ sufficiently small such that $A_j \notin \overline{B_R(A_i)}$ for $j \neq i$. Let $f \in H^{-1}(\Omega)$. Then for the solution $u \in H_0^1(\Omega)$ of the problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.58}$$

the following statements hold:

(i) For $f \in H^s(\Omega)$ with $-1 \leq s < \frac{\pi}{\omega_i} - 1$ it holds $u \in H^{s+2}(\Omega \cap B_R(A_i))$ with the estimate

$$\|u\|_{H^{s+2}(\Omega\cap B_R(A_i))} \lesssim \|f\|_{H^s(\Omega)}.$$

(ii) If $\frac{\pi}{\omega_i} \notin \mathbb{N}$ and $\omega_i < \pi$, then for $f \in B_{2,1}^{\pi/\omega_i - 1}(\Omega)$ it holds $u \in B_{2,\infty}^{\pi/\omega_i+1}(\Omega \cap B_R(A_i))$ with the estimate

$$\|u\|_{B_{2,\infty}^{\pi/\omega_i+1}(\Omega\cap B_R(A_i))} \lesssim \|f\|_{B_{2,1}^{\pi/\omega_i-1}(\Omega)}.$$

(iii) If $\omega_i > \pi$, then for $f \in (H^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_i},1}$ it holds $u \in B_{2,\infty}^{\pi/\omega_i+1}(\Omega \cap B_R(A_i))$ with the estimate

$$\|u\|_{B_{2,\infty}^{\pi/\omega_i+1}(\Omega\cap B_R(A_i))} \lesssim \|f\|_{B_{2,1}^{\pi/\omega_i-1}(\Omega)}.$$

Proof. We prove the result in several steps. We start with (ii) and proceed with (iii) from which (i) follows easily.

Step 1: Assume that the balls $B_1(A_i)$ are pairwise disjoint, otherwise scale the domain Ω and problem (3.58) appropriately. We then define a smooth cut-off function χ_i with $\operatorname{supp}\chi_i \subseteq B_1(A_i)$ and $\chi_i \equiv 1$ near A_i. Furthermore, we denote by T and T_i the solution operators

$$T : \begin{cases} H^{-1}(\Omega) & \to & H_0^1(\Omega) \\ f & \mapsto & u \end{cases}$$

and for $i = 1, \dots, J$

$$T_i : \begin{cases} H^{-1}(\Omega) & \to & H_0^1(\Omega) \\ f & \mapsto & \chi_i T f \end{cases} .$$

Step 2: We now show (ii). We assume $\frac{\pi}{\omega_i} \notin \mathbb{N}$ and $\omega_i < \pi$ and start with $f \in B_{2,1}^{\pi/\omega_i - 1}(\Omega)$. This Besov space can be seen as interpolation space between $H^{-1}(\Omega)$ and $H^{s^\star}(\Omega)$, where $s^\star > \frac{\pi}{\omega_i} - 1$ is chosen such that $\lfloor s^\star \rfloor = \lfloor \pi/\omega_i \rfloor - 1$ and $s^\star \notin \mathbb{N}$, cf. Lemma 2.13. Since by interpolation,

$$B_{2,1}^{\pi/\omega_i - 1}(\Omega) = (H^{-1}(\Omega), H^{s^\star}(\Omega))_{\frac{\pi}{\omega_i(1+s^\star)},1},$$

we can write $f = f_0 + f_1$ with $f_0 \in H^{-1}(\Omega)$ and $f_1 \in H^{s^\star}(\Omega)$ such that

$$\|f_0\|_{H^{-1}(\Omega)} + t\|f_1\|_{H^{s^\star}(\Omega)} \leq 2K(t,f) \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}} \|f\|_{\frac{\pi}{\omega_i(1+s^\star)},\infty} \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}} \|f\|_{\frac{\pi}{\omega_i(1+s^\star)},1}.$$

The next step is solving (3.58) for the right hand sides f_0 and f_1. We immediately get $T_i f_0 \in H_0^1(\Omega)$ with the norm estimate $\|T_i f_0\|_{H^1(\Omega)} \lesssim \|f_0\|_{H^{-1}(\Omega)}$.

Step 3: The function $T_i f_1$ satisfies

$$\begin{aligned} -\Delta(T_i f_1) &= -\Delta(\chi_i T f_1) = -\chi_i \Delta(T f_1) - 2\nabla\chi_i \cdot \nabla(T f_1) - (T f_1)\Delta\chi_i \\ &= \chi_i f_1 - 2\nabla\chi_i \cdot \nabla(T f_1) - (T f_1)\Delta\chi_i =: \widehat{f}_i. \end{aligned}$$

Note that $\nabla\chi_i \cdot \nabla(T f_1) \in H^{s^\star+1}(\Omega)$ and $(T f_1)\Delta\chi_i \in H^{s^\star+2}(\Omega)$, since these expressions are only non-zero away from the vertices where we get full regularity for $T f_1$. Thus it follows $\widehat{f}_i \in H^{s^\star}(\Omega)$. We also mention in passing that the mapping $f_1 \mapsto \widehat{f}_i$ is linear and bounded from $H^k(\Omega)$ to $H^k(\Omega)$ for all $k \in \mathbb{N}_0$ and thus from $B_{2,1}^{\pi/\omega_i - 1}(\Omega)$ to $B_{2,1}^{\pi/\omega_i - 1}(\Omega)$ by interpolation, and it holds $\operatorname{supp}\widehat{f}_i \subseteq \operatorname{supp}\chi_i \subseteq B_1(A_i)$ and $\operatorname{supp} T_i f_1 \subseteq \operatorname{supp}\chi_i \subseteq B_1(A_i)$. We now consider a coordinate transformation $\mathcal{B}_i : \Omega \to \mathcal{C}$ such that $A_i \mapsto 0$ and $\partial\Omega \cap B_1(A_i) \mapsto \partial\mathcal{C} \cap B_1(0)$ with the usual cone $\mathcal{C}$. Due to the support properties of $\mathcal{B}_i T_i f_i$ and $\mathcal{B}_i \widehat{f}_i$ we can extend them both by zero to functions on $\mathcal{C}$ without changing the regularity. Since the Laplacian is invariant under orthogonal transformations, they still satisfy the equation $-\Delta \mathcal{B}_i T_i f_1 = \mathcal{B}_i \widehat{f}_i$.

The functions $\mathcal{B}_i\widehat{f}_i$ and $\mathcal{B}_iT_if_1$ now satisfy the assumptions of Corollary 3.38. Lemma 3.41 gives $\frac{k\omega_i}{\pi} \notin \mathbb{N}$ for $k = 2, \ldots, \lfloor \frac{\pi}{\omega_i} \rfloor$. Since $s^\star > \frac{\pi}{\omega_i} - 1$, we get

$$\mathcal{B}_iT_if_1 = u_0(\widehat{f}_i) - \frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1}\sin(\lambda_1\varphi)(\mathcal{B}_i\widehat{f}_i + \Delta(\chi P_{\lfloor s^\star\rfloor -1}))\right) r^{\lambda_1}\sin(\lambda_1\phi), \tag{3.59}$$

with $u_0(\widehat{f}_i) \in H^{s^\star+2}(\mathcal{C}_1)$. Additionally, Corollary 3.38 gives the estimate

$$\|u_0(\widehat{f}_i)\|_{H^{s^\star+2}(\mathcal{C}_1)} \lesssim \|\mathcal{B}_i\widehat{f}_i\|_{H^{s^\star}(\mathcal{C})}.$$

We now decompose $s^+ := r^{\lambda_1}\sin(\lambda_1\phi)|_{\mathcal{C}_1} \in B^{\pi/\omega_i+1}_{2,\infty}(\mathcal{C}_1)$ as $s^+ = s_0 + s_1$ with $s_0 \in H^1(\mathcal{C}_1)$ and $s_1 \in H^{s^\star+2}(\mathcal{C}_1)$ such that

$$\|s_0\|_{H^1(\mathcal{C}_1)} + t\|s_1\|_{H^{s^\star+2}(\mathcal{C}_1)} \lesssim 2K(t,s^+) \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}}\|s^+\|_{B^{\pi/\omega_i+1}_{2,\infty}(\mathcal{C}_1)} \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}},$$

cf. Lemma 2.15. Altogether, we have

$$T_if = T_if_0 + T_if_1 = T_if_0 + \mathcal{B}_i^{-1}\left(u_0(\widehat{f}_i)|_{B_1(0)} + S(\mathcal{B}_i\widehat{f}_i)s_0|_{B_1(0)} + S(\mathcal{B}_i\widehat{f}_i)s_1|_{B_1(0)}\right),$$

where $S(\mathcal{B}_i\widehat{f}_i) := -\frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1}\sin(\lambda_1\varphi)(\mathcal{B}_i\widehat{f}_i + \Delta(\chi P_{\lfloor s^\star\rfloor-1}))\right)$. Since

$$\begin{aligned}|S(\mathcal{B}_i\widehat{f}_i)| &\overset{Lemma\ 3.39}{\lesssim} \|\mathcal{B}_i\widehat{f}_i\|_{B^{\pi/\omega_i-1}_{2,1}(\mathcal{C}_1)} \lesssim \|\widehat{f}_i\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} \\ &\lesssim \|f_1\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} \overset{Lemma\ 2.14}{\lesssim} \|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)},\end{aligned} \tag{3.60}$$

we obtain

$$\begin{aligned}\|T_if_0 + \mathcal{B}_i^{-1}S(\mathcal{B}_i\widehat{f}_i)s_0|_{B_1(0)}\|_{H^1(\Omega)} &\lesssim \|f_0\|_{H^{-1}(\Omega)} + |S(\mathcal{B}_i\widehat{f}_i)|\|s_0\|_{H^1(\mathcal{C}_1)} \\ &\lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}}\|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)}\end{aligned} \tag{3.61}$$

and

$$\begin{aligned}\|\mathcal{B}_i^{-1}u_0(\widehat{f}_i)|_{B_1(0)} + \mathcal{B}_i^{-1}S(\mathcal{B}_i\widehat{f}_i)s_1|_{B_1(0)}\|_{H^{s^\star+2}(\Omega)} &\lesssim \|\mathcal{B}_i\widehat{f}_i\|_{H^{s^\star}(\mathcal{C})} + |S(\mathcal{B}_i\widehat{f}_i)|\|s_1\|_{H^{s^\star+2}(\mathcal{C}_1)} \\ &\lesssim \|f_1\|_{H^{s^\star}(\Omega)} + t^{\frac{\pi}{\omega_i(1+s^\star)}-1}\|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}-1}\|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)}.\end{aligned} \tag{3.62}$$

Thus it follows

$$\inf_{v\in H^{s^\star+2}(\Omega)}\left(\|T_if - v\|_{H^1(\Omega)} + t\|v\|_{H^{s^\star+2}(\Omega)}\right) \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}}\|f\|_{\frac{\pi}{\omega_i(1+s^\star)},1},$$

which implies $T_if \in (H^1(\Omega), H^{s^\star+2}(\Omega))_{\frac{\pi}{\omega_i(1+s^\star)},\infty} = B^{\pi/\omega_i+1}_{2,\infty}(\Omega)$ together with the estimate $\|T_if\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega)} \lesssim \|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)}$, which proves (ii).

Step 4: We now prove (iii). Let $\omega_i > \pi$ and $f \in (H^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_i},1}$. We follow the lines of step 2 and 3. We can write $f = f_0 + f_1$ with $f_0 \in H^{-1}(\Omega)$ and $f_1 \in L^2(\Omega)$ such that

$$\|f_0\|_{H^{-1}(\Omega)} + t\|f_1\|_{L^2(\Omega)} \leq 2K(t,f) \lesssim t^{\frac{\pi}{\omega_i}} \|f\|_{\frac{\pi}{\omega_i},\infty} \lesssim t^{\frac{\pi}{\omega_i}} \|f\|_{\frac{\pi}{\omega_i},1}.$$

We immediately get $T_i f_0 \in H_0^1(\Omega)$ with the norm estimate $\|T_i f_0\|_{H^1(\Omega)} \lesssim \|f_0\|_{H^{-1}(\Omega)}$. The function $T_i f_1$ satisfies

$$-\Delta(T_i f_1) = \chi_i f_1 - 2\nabla\chi_i \cdot \nabla(T f_1) - (T f_1)\Delta\chi_i =: \widehat{f}_i \in L^2(\Omega).$$

Note that it holds $\operatorname{supp} \widehat{f}_i \subseteq \operatorname{supp}\chi_i \subseteq B_1(A_i)$ and $\operatorname{supp} T_i f_1 \subseteq \operatorname{supp}\chi_i \subseteq B_1(A_i)$. After the coordinate transformation we have the equation $-\Delta \mathcal{B}_i T_i f_1 = \mathcal{B}_i \widehat{f}_i$.
Since $\omega_i > \pi$, (3.29) (cf. also Corollary 3.25) yields

$$\mathcal{B}_i T_i f_1 = u_0(\widehat{f}_i) - \frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1}\sin(\lambda_1\varphi)\mathcal{B}_i\widehat{f}_i\right) r^{\lambda_1}\sin(\lambda_1\phi), \tag{3.63}$$

with $u_0(\widehat{f}_i) \in H^2(\mathcal{C}_1)$ and the estimate

$$\|u_0(\widehat{f}_i)\|_{H^2(\mathcal{C}_1)} \lesssim \|\mathcal{B}_i\widehat{f}_i\|_{L^2(\mathcal{C})}.$$

We now decompose $s^+ := r^{\lambda_1}\sin(\lambda_1\phi)|_{\mathcal{C}_1} \in B_{2,\infty}^{\pi/\omega_i+1}(\mathcal{C}_1)$ as $s^+ = s_0 + s_1$ with $s_0 \in H^1(\mathcal{C}_1)$ and $s_1 \in H^2(\mathcal{C}_1)$ such that

$$\|s_0\|_{H^1(\mathcal{C}_1)} + t\|s_1\|_{H^2(\mathcal{C}_1)} \lesssim 2K(t,s^+) \lesssim t^{\frac{\pi}{\omega_i}}\|s^+\|_{B_{2,\infty}^{\pi/\omega_i+1}(\mathcal{C}_1)} \lesssim t^{\frac{\pi}{\omega_i}},$$

cf. Lemma 2.15. Altogether, we have

$$T_i f = T_i f_0 + T_i f_1 = T_i f_0 + \mathcal{B}_i^{-1}\left(u_0(\widehat{f}_i)|_{B_1(0)} + S(\mathcal{B}_i\widehat{f}_i)s_0|_{B_1(0)} + S(\mathcal{B}_i\widehat{f}_i)s_1|_{B_1(0)}\right),$$

where $S(\mathcal{B}_i\widehat{f}_i) := -\frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1}\sin(\lambda_1\varphi)\mathcal{B}_i\widehat{f}_i\right)$. Since

$$|S(\mathcal{B}_i\widehat{f}_i)| \overset{Lemma\ 2.16,(i)}{\lesssim} \|\widehat{f}_i\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}} \overset{Lemma\ 2.14}{\lesssim} \|f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}},$$

we obtain

$$\begin{aligned}\|T_i f_0 + \mathcal{B}_i^{-1}S(\mathcal{B}_i\widehat{f}_i)s_0|_{B_1(0)}\|_{H^1(\Omega)} &\lesssim \|f_0\|_{H^{-1}(\Omega)} + |S(\mathcal{B}_i\widehat{f}_i)|\|s_0\|_{H^1(\mathcal{C}_1)} \\ &\lesssim t^{\frac{\pi}{\omega_i}}\|f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{B}_i^{-1}u_0(\widehat{f}_i)|_{B_1(0)} + \mathcal{B}_i^{-1}S(\mathcal{B}_i\widehat{f}_i)s_1|_{B_1(0)}\|_{H^2(\Omega)} &\lesssim \|\mathcal{B}_i\widehat{f}_i\|_{L^2(\mathcal{C})} + |S(\mathcal{B}_i\widehat{f}_i)|\|s_1\|_{H^2(\mathcal{C}_1)} \\ &\lesssim \|f_1\|_{L^2(\Omega)} + t^{\frac{\pi}{\omega_i}-1}\|f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}} \lesssim t^{\frac{\pi}{\omega_i}-1}\|f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}}.\end{aligned}$$

Thus it follows

$$\inf_{v\in H^2(\Omega)} \left(\|T_i f - v\|_{H^1(\Omega)} + t\|v\|_{H^2(\Omega)}\right) \lesssim t^{\frac{\pi}{\omega_i}} \|f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}},$$

which implies $T_i f \in (H^1(\Omega), H^2(\Omega))_{\frac{\pi}{\omega_i},\infty} = B^{\pi/\omega_i+1}_{2,\infty}(\Omega)$ together with the norm estimate $\|T_i f\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega)} \lesssim \|f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}}$, which completes the proof.

Step 5: We show (i). In this step, we assume $\frac{\pi}{\omega_i} \notin \mathbb{N}$. Let $f \in H^s(\Omega)$ for $-1 \le s < \frac{\pi}{\omega_i} - 1$. For $f \in H^{-1}(\Omega)$, we get $T_i f \in H^1_0(\Omega)$ directly by the regularity of the solution operator T_i, and for $s > -1$ we use the Reiteration Theorem to obtain

$$H^s(\Omega) = \left(H^{-1}(\Omega), H^{s^\star}(\Omega)\right)_{\frac{s+1}{s^\star+1},2} = \left(H^{-1}(\Omega), \left(H^{-1}(\Omega), H^{s^\star}(\Omega)\right)_{\frac{\pi}{\omega_i(1+s^\star)},1}\right)_{\frac{\omega_i(s+1)}{\pi},2}$$

and

$$H^{s+2}(\Omega) = \left(H^1(\Omega), H^{s^\star+2}(\Omega)\right)_{\frac{s+1}{s^\star+1},2} = \left(H^1(\Omega), \left(H^1(\Omega), H^{s^\star+2}(\Omega)\right)_{\frac{\pi}{\omega_i(1+s^\star)},\infty}\right)_{\frac{\omega_i(s+1)}{\pi},2}.$$

Since the operator T_i maps the space $H^{-1}(\Omega)$ to $H^1(\Omega)$ and the interpolation space $\left(H^{-1}(\Omega), H^{s^\star}(\Omega)\right)_{\frac{\pi}{\omega_i(1+s^\star)},1}$ to $\left(H^1(\Omega), H^{s^\star+2}(\Omega)\right)_{\frac{\pi}{\omega_i(1+s^\star)},\infty}$ (choose $s^\star = 0$ if $\omega_i > \pi$), Lemma 2.8 shows the statement.

Step 6: We now assume $\frac{\pi}{\omega_i} \in \mathbb{N}$ and $f \in H^s(\Omega)$ for $-1 \le s < \frac{\pi}{\omega_i} - 1$, $s \notin \mathbb{N}$. The argumentation of step 3 can now be repeated in the current setting with right-hand side f instead of f_1 (and $s^\star$ replaced by s), i.e. the function $T_i f$ solves the problem

$$-\Delta(T_i f) = \chi_i f - 2\nabla\chi_i \cdot \nabla(Tf) - (Tf)\Delta\chi_i =: \widehat{f}_i$$

with zero Dirichlet boundary conditions. Since now $s + 1 < \frac{\pi}{\omega_i}$ by assumption, it follows from Corollary 3.38 that the singular parts are not present, thus we arrive at

$$\mathcal{B}_i T_i f = u_0(\widehat{f}_i) \in H^{s+2}(\mathcal{C}).$$

Note that the conditions of Corollary 3.38 are satisfied since $\lfloor s \rfloor + 1 = \frac{\pi}{\omega_i} - 1$ implies $\frac{k\omega_i}{\pi} \notin \mathbb{N}$ for $k = 1, \ldots, \lfloor s \rfloor + 1$ by Lemma 3.41. This implies $T_i f \in H^{s+2}(\Omega)$.

Step 7: We again assume $\frac{\pi}{\omega_i} \in \mathbb{N}$ and $f \in H^s(\Omega)$ for $0 < s < \frac{\pi}{\omega_i} - 1$, but now $s \in \mathbb{N}$. We can then write $H^s(\Omega)$ as the interpolation space $(H^{s-\epsilon}(\Omega), H^{s+\epsilon}(\Omega))_{1/2,2}$ for a small $\epsilon > 0$. The result now follows from step 6, and (i) is proved. □

The following corollary deals with global regularity on the whole domain Ω.

Corollary 3.43. *With the assumptions of Theorem 3.42, we define the largest interior angle* $\omega_{max} := \max_{i=1,\ldots,J} \omega_i$. *Then for the solution of the problem* (3.58)*, the following statements hold:*

(i) *For* $f \in H^s(\Omega)$ *with* $-1 \leq s < \frac{\pi}{\omega_{max}} - 1$ *it holds* $u \in H^{s+2}(\Omega)$ *with the estimate*

$$\|u\|_{H^{s+2}(\Omega)} \lesssim \|f\|_{H^s(\Omega)}.$$

(ii) *If* $\frac{\pi}{\omega_{max}} \notin \mathbb{N}$ *and* $\omega_{max} < \pi$, *then for* $f \in B^{\pi/\omega_{max}-1}_{2,1}(\Omega)$ *it holds* $u \in B^{\pi/\omega_{max}+1}_{2,\infty}(\Omega)$ *with the estimate*

$$\|u\|_{B^{\pi/\omega_{max}+1}_{2,\infty}(\Omega)} \lesssim \|f\|_{B^{\pi/\omega_{max}-1}_{2,1}(\Omega)}.$$

(iii) *If* $\omega_{max} > \pi$, *then for* $f \in (H^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_{max}},1}$ *it holds* $u \in B^{\pi/\omega_{max}+1}_{2,\infty}(\Omega)$ *with the estimate*

$$\|u\|_{B^{\pi/\omega_{max}+1}_{2,\infty}(\Omega)} \lesssim \|f\|_{B^{\pi/\omega_{max}-1}_{2,1}(\Omega)}.$$

Proof. The definition $\chi_0 := 1 - \sum_{i=1}^J \chi_i$ gives $Tf = \chi_0 Tf + \sum_{i=1}^J T_i f$. Note that $\chi_0 Tf$ is non-zero away from the vertices and thus attains full regularity only restricted by the right-hand side f. This infers together with the regularity results from Theorem 3.42 that we obtain the desired regularity. □

Better regularity near a corner of the polygon already follows if the right-hand side f is only of higher regularity in a neighborhood of the corner.

Theorem 3.44 (Shift theorem, Dirichlet)**.** *With the assumptions of Theorem 3.42, let* $\chi_{i,r}$ *denote a smooth cut-off function with* $\operatorname{supp}\chi_{i,r} \subseteq B_r(A_i)$ *and* $\chi_{i,r} \equiv 1$ *on* $B_{r/2}(A_i)$. *Then for the solution of the problem* (3.58)*, the following statements hold:*

(i) *For* $\chi_{i,2R} f \in H^s(\Omega)$ *with* $0 \leq s < \frac{\pi}{\omega_i} - 1$ *it holds* $u \in H^{s+2}(\Omega \cap B_{R/4}(A_i))$ *with the estimate*

$$\|u\|_{H^{s+2}(\Omega\cap B_{R/4}(A_i))} \lesssim \|f\|_{H^s(\Omega\cap B_R(A_i))} + \|u\|_{H^1(\Omega)}.$$

(ii) *For* $\chi_{i,R/2} f \in H^s(\Omega)$ *with* $-1 < s < \frac{\pi}{\omega_i} - 1$ *it holds* $u \in H^{s+2}(\Omega \cap B_{R/4}(A_i))$ *with the estimate*

$$\|u\|_{H^{s+2}(\Omega\cap B_{R/4}(A_i))} \lesssim \|\chi_{i,R/2} f\|_{H^s(\Omega\cap B_R(A_i))} + \|u\|_{H^1(\Omega)}.$$

(iii) *If* $\frac{\pi}{\omega_i} \notin \mathbb{N}$ *and* $\omega_i < \pi$, *then for* $\chi_{i,2R} f \in B^{\pi/\omega_i-1}_{2,1}(\Omega)$ *it holds* $u \in B^{\pi/\omega_i+1}_{2,\infty}(\Omega \cap B_{R/4}(A_i))$ *with the estimate*

$$\|u\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega\cap B_{R/4}(A_i))} \lesssim \|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega\cap B_R(A_i))} + \|u\|_{H^1(\Omega)}.$$

(iv) *If* $\omega_i > \pi$, *then for* $\chi_{i,R/2} f \in (H^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_i},1}$ *it follows* $u \in B^{\pi/\omega_i+1}_{2,\infty}(\Omega \cap B_{R/4}(A_i))$ *with the estimate*

$$\|u\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega\cap B_{R/4}(A_i))} \lesssim \|\chi_{i,R/2} f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}} + \|u\|_{H^1(\Omega)}.$$

Proof. We prove (i). The function $u_i := \chi_{i,R/2}u$ solves

$$\begin{aligned} -\Delta u_i &= -\chi_{i,R/2}\Delta u - 2\nabla\chi_{i,R/2}\cdot\nabla u - \Delta\chi_{i,R/2}u \\ &= \chi_{i,R/2}f - 2\nabla\chi_{i,R/2}\cdot\nabla u - \Delta\chi_{i,R/2}u =: f_i. \end{aligned} \tag{3.64}$$

On $A(R/4,R/2) := \Omega\cap(B_{R/2}(A_i)\backslash\overline{B_{R/4}(A_i)})$ (note that the notation coincides with the notation introduced in (3.8) after a suitable coordinate transformation) we have the regularity estimate

$$\|u\|_{H^{s+2}(A(R/4,R/2))} \lesssim \|f\|_{H^s(A(R/8,R))} + \|u\|_{H^1(A(R/8,R))},$$

cf. Lemma 3.10, (i). We have

$$\begin{aligned} \|f_i\|_{H^s(\Omega)} &\lesssim \|\chi_{i,R/2}f\|_{H^s(\Omega)} + \|\nabla\chi_{i,R/2}\cdot\nabla u\|_{H^s(\Omega)} + \|\Delta\chi_{i,R/2}u\|_{H^s(\Omega)} \\ &\lesssim \|f\|_{H^s(\Omega\cap B_R(A_i))} + \|u\|_{H^{s+2}(A(R/4,R/2))} \\ &\lesssim \|f\|_{H^s(\Omega\cap B_R(A_i))} + \|u\|_{H^1(\Omega)}. \end{aligned}$$

Thus it follows $f_i \in H^s(\Omega)$, and we can apply Theorem 3.42 to obtain $u_i \in H^{s+2}(\Omega\cap B_{R/2}(A_i))$ with the estimate

$$\|u_i\|_{H^{s+2}(\Omega\cap B_{R/2}(A_i))} \lesssim \|f_i\|_{H^s(\Omega)},$$

from which

$$\|u\|_{H^{s+2}(\Omega\cap B_{R/4}(A_i))} = \|u_i\|_{H^{s+2}(\Omega\cap B_{R/4}(A_i))} \lesssim \|f\|_{H^s(\Omega\cap B_R(A_i))} + \|u\|_{H^1(\Omega)}$$

follows.
We now prove (ii). The function u_i again solves (3.64), thus we have

$$\begin{aligned} \|f_i\|_{H^s(\Omega)} &\lesssim \|\chi_{i,R/2}f\|_{H^s(\Omega)} + \|\nabla\chi_{i,R/2}\cdot\nabla u\|_{H^s(\Omega)} + \|\Delta\chi_{i,R/2}u\|_{H^s(\Omega)} \\ &\lesssim \|\chi_{i,R/2}f\|_{H^s(\Omega\cap B_R(A_i))} + \|\nabla\chi_{i,R/2}\cdot\nabla u\|_{L^2(\Omega)} + \|\Delta\chi_{i,R/2}u\|_{L^2(\Omega)} \\ &\lesssim \|\chi_{i,R/2}f\|_{H^s(\Omega\cap B_R(A_i))} + \|u\|_{H^1(\Omega)}. \end{aligned}$$

We now show (iii). Here we estimate with $\epsilon > 0$, such that $2 - \frac{\pi}{\omega_i} < \epsilon < 1$,

$$\begin{aligned} \|f_i\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} &\lesssim \|\chi_{i,R/2}f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} + \|\nabla\chi_{i,R/2}\cdot\nabla u\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} + \|\Delta\chi_{i,R/2}u\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} \\ &\lesssim \|\chi_{i,R/2}f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} + \|u\|_{H^{\pi/\omega_i+\epsilon}(A(R/4,R/2))} \\ &\overset{\text{Lemma 3.10}}{\lesssim} \|\chi_{i,R/2}f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)} + \|f\|_{H^{\pi/\omega_i-2+\epsilon}(A(R/8,R))} + \|u\|_{H^1(\Omega)} \\ &\lesssim \|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega\cap B_R(A_i))} + \|u\|_{H^1(\Omega)}, \end{aligned}$$

cf. Lemma 2.6, (iv) for the changes in norms between the Sobolev and Besov spaces. Hence it follows $f_i \in B^{\pi/\omega_i-1}_{2,1}(\Omega)$, and we can apply Theorem 3.42 to obtain $u_i \in B^{\pi/\omega+1}_{2,\infty}(\Omega\cap B_{R/2}(A_i))$ with the estimate

$$\|u_i\|_{B^{\pi/\omega+1}_{2,\infty}(\Omega\cap B_{R/2}(A_i))} \lesssim \|f_i\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)},$$

from which

$$\|u\|_{B^{\pi/\omega+1}_{2,\infty}(\Omega\cap B_{R/4}(A_i))} = \|u_i\|_{B^{\pi/\omega+1}_{2,\infty}(\Omega\cap B_{R/4}(A_i))} \lesssim \|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega\cap B_R(A_i))} + \|u\|_{H^1(\Omega)}$$

follows.
For the proof of (iv), we mention the inequality

$$\begin{aligned}\|f_i\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}} &\lesssim \|\chi_{i,R/2}f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}} + \|\nabla\chi_{i,R/2}\cdot\nabla u\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}}\\ &\quad + \|\Delta\chi_{i,R/2}u\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}}\\ &\lesssim \|\chi_{i,R/2}f\|_{(H^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}} + \|u\|_{H^1(\Omega)}.\end{aligned}$$

Theorem 3.42 again implies the desired estimate. □

3.5.2 The Neumann case

A similar shift theorem holds for Neumann boundary conditions.

Theorem 3.45. *Let Ω be a polygonal domain with vertices A_i and corresponding interior angles $\omega_i \neq \pi$, $i = 1,\dots,J$. Fix the vertex A_i, and fix $R > 0$ sufficiently small such that $A_j \notin \overline{B_R(A_i)}$ for $j \neq i$. Let $f \in \widetilde{H}^{-1}(\Omega) = \left(H^1(\Omega)\right)^\star$ with the compatibility condition $\langle f, 1\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)} = 0$. Then for the solution $u \in H^1(\Omega)$ with $\int_\Omega u = 0$ of the problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega,\\ \partial_n u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.65}$$

the following statements hold:

(i) For $f \in \widetilde{H}^s(\Omega)$ with $-1 \le s < \min(\frac{\pi}{\omega_i}-1, 0)$ or for $f \in H^s(\Omega)$ with $0 \le s < \frac{\pi}{\omega_i}-1$ it holds $u \in H^{s+2}(\Omega\cap B_R(A_i))$ with the estimate

$$\|u\|_{H^{s+2}(\Omega\cap B_R(A_i))} \lesssim \begin{cases} \|f\|_{\widetilde{H}^s(\Omega)}, & -1 \le s < \min(\frac{\pi}{\omega_i}-1,0)\\ \|f\|_{H^s(\Omega)}, & 0 \le s < \frac{\pi}{\omega_i}-1 \end{cases}.$$

(ii) If $\frac{\pi}{\omega_i} \notin \mathbb{N}$ and $\omega_i < \pi$, then for $f \in B^{\pi/\omega_i-1}_{2,1}(\Omega)$ it holds $u \in B^{\pi/\omega_i+1}_{2,\infty}(\Omega\cap B_R(A_i))$ with the estimate

$$\|u\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega\cap B_R(A_i))} \lesssim \|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)}.$$

(iii) If $\omega_i > \pi$, then for $f \in (\widetilde{H}^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_i},1}$ it holds $u \in B^{\pi/\omega_i+1}_{2,\infty}(\Omega\cap B_R(A_i))$ with the estimate

$$\|u\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega\cap B_R(A_i))} \lesssim \|f\|_{(\widetilde{H}^{-1}(\Omega),L^2(\Omega))_{\frac{\pi}{\omega_i},1}}.$$

Proof. We follow the lines of Theorem 3.42.
In the Neumann case, we take the solution operator

$$T : \begin{cases} \widetilde{H}^{-1}(\Omega) & \to & H^1(\Omega) \\ f & \mapsto & u \end{cases},$$

where u with $\int_\Omega u = 0$ is the solution of the problem

$$\begin{aligned} -\Delta u &= f - |\Omega|^{-1} \langle f, 1\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)} \\ \partial_n u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{3.66}$$

Note that the right-hand side of (3.66) satisfies the compatibility condition for the Neumann problem and thus admits a solution. The solution operators T_i for $i = 1, \dots, J$ are defined analogously by

$$T_i : \begin{cases} \widetilde{H}^{-1}(\Omega) & \to & H^1(\Omega) \\ f & \mapsto & \chi_i T f \end{cases}.$$

We now assume $\frac{\pi}{\omega_i} \notin \mathbb{N}$ and $\omega_i < \pi$ and start with $f \in B^{\pi/\omega_i - 1}_{2,1}(\Omega)$. This Besov space can be seen as interpolation space between $\widetilde{H}^{-1}(\Omega)$ and $H^{s^\star}(\Omega)$, where $s^\star > \frac{\pi}{\omega_i} - 1$ is chosen such that $\lfloor s^\star \rfloor = \lfloor \pi/\omega_i \rfloor - 1$ and $s^\star \notin \mathbb{N}$, cf. Lemma 2.13. Since by interpolation,

$$B^{\pi/\omega_i - 1}_{2,1}(\Omega) = (\widetilde{H}^{-1}(\Omega), H^{s^\star}(\Omega))_{\frac{\pi}{\omega_i(1+s^\star)},1},$$

we can write $f = f_0 + f_1$ with $f_0 \in \widetilde{H}^{-1}(\Omega)$ and $f_1 \in H^{s^\star}(\Omega)$ such that

$$\|f_0\|_{\widetilde{H}^{-1}(\Omega)} + t\|f_1\|_{H^{s^\star}(\Omega)} \leq 2K(t,f) \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}} \|f\|_{\frac{\pi}{\omega_i(1+s^\star)},\infty} \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}} \|f\|_{\frac{\pi}{\omega_i(1+s^\star)},1}.$$

We then get $T_i f_0 \in H^1(\Omega)$ with the norm estimate

$$\begin{aligned} \|T_i f_0\|_{H^1(\Omega)} &\lesssim \|f_0 - |\Omega|^{-1} \langle f_0, 1\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)} \|_{\widetilde{H}^{-1}(\Omega)} \\ &\leq \|f_0\|_{\widetilde{H}^{-1}(\Omega)} + |\Omega|^{-1} \left| \langle f_0, 1\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)} \right| \|1\|_{\widetilde{H}^{-1}(\Omega)} \\ &\lesssim \|f_0\|_{\widetilde{H}^{-1}(\Omega)} + |\Omega|^{-1} \|f_0\|_{\widetilde{H}^{-1}(\Omega)} \|1\|_{H^1(\Omega)} \|1\|_{\widetilde{H}^{-1}(\Omega)} \\ &\lesssim \|f_0\|_{\widetilde{H}^{-1}(\Omega)} + |\Omega|^{-1} \|f_0\|_{\widetilde{H}^{-1}(\Omega)} \|1\|_{H^1(\Omega)} \|1\|_{\widetilde{H}^{-1}(\Omega)} \\ &\lesssim \|f_0\|_{\widetilde{H}^{-1}(\Omega)}. \end{aligned}$$

The function $T_i f_1$ satisfies

$$\begin{aligned} -\Delta(T_i f_1) &= -\Delta(\chi_i T f_1) = -\chi_i \Delta(T f_1) - 2\nabla\chi_i \cdot \nabla(T f_1) - (T f_1)\Delta\chi_i \\ &= \chi_i (f_1 - |\Omega|^{-1} \int_\Omega f_1) - 2\nabla\chi_i \cdot \nabla(T f_1) - (T f_1)\Delta\chi_i =: \widehat{f}_i. \end{aligned}$$

As in the proof of Theorem 3.42, step 3, we obtain $\widehat{f}_i \in H^{s^\star}(\Omega)$ with the same support properties. We mention that the mapping $f_1 \mapsto \widehat{f}_i$ is linear and bounded from $H^k(\Omega)$ to $H^k(\Omega)$ for $k \in \mathbb{N}_0$ since we obtain with Jensen's inequality

$$\left\| |\Omega|^{-1} \int_\Omega f_1 \right\|^2_{H^k(\Omega)} = \int_\Omega \left| |\Omega|^{-1} \int_\Omega f_1 \right|^2 \lesssim \int_\Omega |\Omega|^{-1} \int_\Omega |f_1|^2$$
$$= \int_\Omega |\Omega|^{-1} \|f_1\|^2_{L^2(\Omega)} \leq \|f_1\|^2_{H^k(\Omega)}.$$

We now get after a coordinate transformation

$$\mathcal{B}_i T_i f_1 = u_0(\widehat{f}_i) - \frac{1}{\pi} \left(\int_{\mathcal{C}} r^{-\lambda_1} \cos(\lambda_1 \varphi)(\mathcal{B}_i \widehat{f}_i + \Delta(\chi P_{\lfloor s^\star \rfloor -1})) \right) r^{\lambda_1} \cos(\lambda_1 \phi), \tag{3.67}$$

with $u_0(\widehat{f}_i) \in H^{s^\star+2}(\mathcal{C}_1)$, together with the estimate

$$\|u_0(\widehat{f}_i)\|_{H^{s^\star+2}(\mathcal{C}_1)} \lesssim \|\mathcal{B}_i \widehat{f}_i\|_{H^{s^\star}(\mathcal{C})},$$

cf. Remark 3.40. We now decompose $s^+ := r^{\lambda_1} \cos(\lambda_1 \phi)|_{\mathcal{C}_1} \in B^{\pi/\omega_i+1}_{2,\infty}(\mathcal{C}_1)$ as $s^+ = s_0 + s_1$ with $s_0 \in H^1(\mathcal{C}_1)$ and $s_1 \in H^{s^\star+2}(\mathcal{C}_1)$ such that

$$\|s_0\|_{H^1(\mathcal{C}_1)} + t\|s_1\|_{H^{s^\star+2}(\mathcal{C}_1)} \lesssim 2K(t, s^+) \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}} \|s^+\|_{B^{\pi/\omega_i+1}_{2,\infty}(\mathcal{C}_1)} \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}},$$

cf. Lemma 2.15. Altogether, we have

$$T_i f = T_i f_0 + T_i f_1 = T_i f_0 + \mathcal{B}_i^{-1} \left(u_0(\widehat{f}_i)|_{B_1(0)} + S(\mathcal{B}_i \widehat{f}_i) s_0|_{B_1(0)} + S(\mathcal{B}_i \widehat{f}_i) s_1|_{B_1(0)} \right),$$

where $S(\mathcal{B}_i \widehat{f}_i) := -\frac{1}{\pi} \left(\int_{\mathcal{C}} r^{-\lambda_1} \cos(\lambda_1 \varphi)(\mathcal{B}_i \widehat{f}_i + \Delta(\chi P_{\lfloor s^\star \rfloor -1})) \right)$. Imitating the equations (3.60), (3.61) and (3.62), it follows

$$\inf_{v \in H^{s^\star+2}(\Omega)} \left(\|T_i f - v\|_{H^1(\Omega)} + t\|v\|_{H^{s^\star+2}(\Omega)} \right) \lesssim t^{\frac{\pi}{\omega_i(1+s^\star)}} \|f\|_{\frac{\pi}{\omega_i(1+s^\star)},1},$$

which implies $T_i f \in (H^1(\Omega), H^{s^\star+2}(\Omega))_{\frac{\pi}{\omega_i(1+s^\star)},\infty} = B^{\pi/\omega_i+1}_{2,\infty}(\Omega)$ together with the estimate $\|T_i f\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega)} \lesssim \|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega)}$, which proves (ii).

We now prove (iii). Let $\omega_i > \pi$ and $f \in (\widetilde{H}^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_i},1}$. We can write $f = f_0 + f_1$ with $f_0 \in \widetilde{H}^{-1}(\Omega)$ and $f_1 \in L^2(\Omega)$ such that

$$\|f_0\|_{\widetilde{H}^{-1}(\Omega)} + t\|f_1\|_{L^2(\Omega)} \leq 2K(t, f) \lesssim t^{\frac{\pi}{\omega_i}} \|f\|_{\frac{\pi}{\omega_i},\infty} \lesssim t^{\frac{\pi}{\omega_i}} \|f\|_{\frac{\pi}{\omega_i},1}.$$

We immediately get $T_i f_0 \in H^1(\Omega)$ with the norm estimate $\|T_i f_0\|_{\widetilde{H}^1(\Omega)} \lesssim \|f_0\|_{H^{-1}(\Omega)}$. The function $T_i f_1$ satisfies again

$$-\Delta(T_i f_1) = \chi_i(f_1 - |\Omega|^{-1} \int_\Omega f_1) - 2\nabla\chi_i \cdot \nabla(T f_1) - (T f_1)\Delta\chi_i =: \widehat{f}_i \in L^2(\Omega).$$

Since $\omega_i > \pi$, (3.34) (cf. also Corollary 3.28) yields

$$\mathcal{B}_i T_i f_1 = u_0(\widehat{f}_i) - \frac{1}{\pi}\left(\int_{\mathcal{C}} r^{-\lambda_1}\cos(\lambda_1\varphi)\mathcal{B}_i\widehat{f}_i\right) r^{\lambda_1}\cos(\lambda_1\phi), \tag{3.68}$$

with $u_0(\widehat{f}_i) \in H^2(\mathcal{C}_1)$ and the estimate

$$\|u_0(\widehat{f}_i)\|_{H^2(\mathcal{C}_1)} \lesssim \|\mathcal{B}_i\widehat{f}_i\|_{L^2(\mathcal{C})}.$$

The rest of the proof follows as in step 4 of the proof of Theorem 3.42, however, use Lemma 2.16, (ii) instead of (i).
Statement (i) follows with the same arguments as in Theorem 3.42, considering Lemma 2.13. □

An analogous proof to Corollary 3.43 gives the following result which deals with global regularity on the domain Ω.

Corollary 3.46. *With the assumptions of Theorem 3.45, we define the largest interior angle $\omega_{max} := \max_{i=1,\ldots,J}\omega_i$. Then for the solution of the problem (3.58), the following statements hold:*

(i) For $f \in \widetilde{H}^s(\Omega)$ with $-1 \leq s < \min(\frac{\pi}{\omega_i} - 1, 0)$ or for $f \in H^s(\Omega)$ with $0 \leq s < \frac{\pi}{\omega_{max}} - 1$ it holds $u \in H^{s+2}(\Omega)$ with the estimate

$$\|u\|_{H^{s+2}(\Omega)} \lesssim \begin{cases} \|f\|_{\widetilde{H}^s(\Omega)}, & -1 \leq s < \min(\frac{\pi}{\omega_i} - 1, 0) \\ \|f\|_{H^s(\Omega)}, & 0 \leq s < \frac{\pi}{\omega_{max}} - 1 \end{cases}.$$

(ii) If $\frac{\pi}{\omega_{max}} \notin \mathbb{N}$ and $\omega_{max} < \pi$, then for $f \in B^{\pi/\omega_{max}-1}_{2,1}(\Omega)$ it holds $u \in B^{\pi/\omega_{max}+1}_{2,\infty}(\Omega)$ with the estimate

$$\|u\|_{B^{\pi/\omega_{max}+1}_{2,\infty}(\Omega)} \lesssim \|f\|_{B^{\pi/\omega_{max}-1}_{2,1}(\Omega)}.$$

(iii) If $\omega_{max} > \pi$, then for $f \in (\widetilde{H}^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_{max}},1}$ it holds $u \in B^{\pi/\omega_{max}+1}_{2,\infty}(\Omega)$ with the estimate

$$\|u\|_{B^{\pi/\omega_{max}+1}_{2,\infty}(\Omega)} \lesssim \|f\|_{(\widetilde{H}^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_{max}},1}}.$$

Also in the Neumann setting, regularity near a corner only depends on local regularity of the right-hand side in a neighborhood of the corner. The proof fully imitates the methods of Theorem 3.44, but uses Lemma 3.10, (ii) and Theorem 3.45 instead.

Theorem 3.47 (Shift theorem, Neumann)**.** *With the assumptions of Theorem 3.45, let $\chi_{i,r}$ denote a smooth cut-off function with $\operatorname{supp}\chi_{i,r} \subseteq B_r(A_i)$ and $\chi_{i,r} \equiv 1$ on $B_{r/2}(A_i)$. Then for the solution of the problem (3.65), the following statements hold:*

(i) *For* $\chi_{i,2R} f \in H^s(\Omega)$ *with* $0 \le s < \frac{\pi}{\omega_i} - 1$ *it holds* $u \in H^{s+2}(\Omega \cap B_{R/4}(A_i))$ *with the estimate*

$$\|u\|_{H^{s+2}(\Omega \cap B_{R/4}(A_i))} \lesssim \|f\|_{H^s(\Omega \cap B_R(A_i))} + \|u\|_{H^1(\Omega)}.$$

(ii) *For* $\chi_{i,R/2} f \in \widetilde{H}^s(\Omega)$ *with* $-1 < s < \min(\frac{\pi}{\omega_i} - 1, 0)$ *or* $\chi_{i,R/2} f \in H^s(\Omega)$ *with* $0 \le s < \frac{\pi}{\omega_i} - 1$, *it holds* $u \in H^{s+2}(\Omega \cap B_{R/4}(A_i))$ *with the estimate*

$$\|u\|_{H^{s+2}(\Omega \cap B_{R/4}(A_i))} \lesssim \begin{cases} \|\chi_{i,R/2} f\|_{\widetilde{H}^s(\Omega \cap B_R(A_i))} + \|u\|_{H^1(\Omega)}, & -1 < s < \min(\frac{\pi}{\omega_i} - 1, 0) \\ \|\chi_{i,R/2} f\|_{H^s(\Omega \cap B_R(A_i))} + \|u\|_{H^1(\Omega)}, & 0 \le s < \frac{\pi}{\omega_i} - 1 \end{cases}.$$

(iii) *If* $\frac{\pi}{\omega_i} \notin \mathbb{N}$ *and* $\omega_i < \pi$, *then for* $\chi_{i,2R} f \in B^{\pi/\omega_i - 1}_{2,1}(\Omega)$ *it holds* $u \in B^{\pi/\omega_i+1}_{2,\infty}(\Omega \cap B_{R/4}(A_i))$ *with the estimate*

$$\|u\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega \cap B_{R/4}(A_i))} \lesssim \|f\|_{B^{\pi/\omega_i-1}_{2,1}(\Omega \cap B_R(A_i))} + \|u\|_{H^1(\Omega)}.$$

(iv) *If* $\omega_i > \pi$, *then for* $\chi_{i,R/2} f \in (\widetilde{H}^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_i},1}$ *it follows* $u \in B^{\pi/\omega_i+1}_{2,\infty}(\Omega \cap B_{R/4}(A_i))$ *with the estimate*

$$\|u\|_{B^{\pi/\omega_i+1}_{2,\infty}(\Omega \cap B_{R/4}(A_i))} \lesssim \|\chi_{i,R/2} f\|_{(\widetilde{H}^{-1}(\Omega), L^2(\Omega))_{\frac{\pi}{\omega_i},1}} + \|u\|_{H^1(\Omega)}.$$

The last result of this chapter deals with the shift theorem for Neumann problems with inhomogeneous boundary conditions.

Proposition 3.48. *Let* Ω *be a polygonal domain with vertices* A_i *and corresponding interior angles* $\omega_i < \pi$, $i = 1, \ldots, J$, *and define the largest interior angle* $\omega_{max} := \max_{i=1,\ldots,J} \omega_i$. *Then for every* $s \in [0, \frac{\pi}{\omega_{max}} - 1)$ *there is* $C_s > 0$ *such that the following shift theorem is true:*
For every $v \in H^s(\Omega)$ *and* $g \in L^2(\partial\Omega)$ *with* $g|_e \in H^{s+1/2}(e)$ *for each* $e \in \mathcal{E}(\Omega)$ *that satisfies additionally the compatibility condition* $\int_\Omega v + \int_{\partial\Omega} g = 0$, *the solution* z *of the problem*

$$-\Delta z = v \text{ in } \Omega, \quad \partial_n z = g \text{ on } \partial\Omega, \qquad \int_{\widehat{f}} z = 0,$$

satisfies $z \in H^{s+2}(\Omega)$ *with the estimate*

$$\|z\|_{H^{s+2}(\Omega)} \le C_s \left[\|v\|_{H^s(\Omega)} + \sum_{e \in \mathcal{E}(\Omega)} \|g\|_{H^{s+1/2}(e)} \right].$$

Proof. Let $\boldsymbol{\sigma} \in \mathbf{H}^{s+1}(\Omega)$ be a vector field with the condition $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ on $\partial\Omega$. Such a vector field exists, since constructing such a vector field away from the vertices is easy, and near the vertices, the construction is reduced to one in a quarter plane by an affine coordinate change together with a Piola transformation for $\boldsymbol{\sigma}$. Each component of $\boldsymbol{\sigma}$ can there be

constructed separately by lifting from one of the coordinate axes, since one component of $\mathbf{n}$ is always zero.
We now solve the two problems

$$-\Delta z_0 = v + \operatorname{div} \boldsymbol{\sigma} \quad \text{in } \Omega, \qquad \partial_n z_0 = 0 \quad \text{on } \partial\Omega,$$

and

$$-\Delta \widetilde{z}_0 = \operatorname{curl} \boldsymbol{\sigma} \quad \text{in } \Omega, \qquad \widetilde{z}_0 = 0 \quad \text{on } \partial\Omega.$$

Both problems have homogeneous boundary conditions, thus we obtain z_0, $\widetilde{z}_0 \in H^{s+2}(\Omega)$ with Corollaries 3.43 and 3.46. It remains to see that

$$\nabla z = \boldsymbol{\sigma} - \mathbf{curl}\, \widetilde{z}_0 + \nabla z_0.$$

For this purpose, we define the difference

$$\boldsymbol{\delta} := \nabla z - (\boldsymbol{\sigma} - \mathbf{curl}\, \widetilde{z}_0 + \nabla z_0),$$

which satisfies $\operatorname{div} \boldsymbol{\delta} = 0$, $\operatorname{curl} \boldsymbol{\delta} = 0$ and $\boldsymbol{\delta} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Since $\operatorname{curl} \boldsymbol{\delta} = 0$, there exists a function φ such that $\boldsymbol{\delta} = \nabla\varphi$, and $\operatorname{div} \boldsymbol{\delta} = 0$ gives $-\Delta\varphi = 0$. Together with $\partial_n \varphi = \boldsymbol{\delta} \cdot \mathbf{n} = 0$ we can conclude that $\boldsymbol{\delta} = \nabla\varphi = 0$. □

4 The projection-based interpolation operators

The goal of this chapter is the definition of the desired projection-based interpolation operators that have the optimal polynomial approximation properties assumed the given functions are sufficiently regular. This chapter is organized as follows: In Section 4.1 we explain the ideas of the proofs, using the example of the 2D-operator $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$. In Section 4.2 the interpolation operators are defined, first for 3D and then for 2D. The following results in Section 4.3 then deal with the questions of well-definedness of the interpolation operators. In Section 4.4, the commuting diagram property is shown. The next Sections 4.5, 4.6 and 4.7 are about the interpolation error estimates, from 1D up to 3D. The 1D-result is rather straightforward, whereas there are more technical difficulties to deal with in higher dimensions, especially for the interpolation operators for the $\mathbf{H}(\mathrm{curl})$ (or $\mathbf{H}(\mathbf{curl})$). In order to have a clear structure, these sections are subdivided in few subsections: First we deal with the operators mapping in the H^1-conforming spaces, then with those mapping in the $\mathbf{H}(\mathrm{curl})$-conforming spaces (in 3D the same also for $\mathbf{H}(\mathrm{div})$), and each section finishes with a collection of the main results for each interpolation operator. The main results for the two-dimensional operators are stated in Theorem 4.24, those for the 3D-case in Theorem 4.42. The concluding Section 4.8 then deals with finite elements of the second kind.

4.1 Outline about the concepts on the example of a 2D-operator

In order to clarify the structure of this chapter, we want to present the ideas of the proofs of the interpolation errors based on the example of the 2D-operator $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$ which maps into $\mathcal{P}_{p+1}(\widehat{f})$, the space of polynomials of degree $p+1$ on the reference triangle $\widehat{f}$, see Lemma 4.18:
The 2D-operator $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d} : H^{3/2}(\widehat{f}) \to \mathcal{P}_{p+1}(\widehat{f})$ is defined by the conditions

$$
\begin{aligned}
u(V) - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d} u(V) &= 0 \quad \forall V \in \mathcal{V}(\widehat{f}), \\
(\nabla_e (u - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d} u), \nabla_e v)_{L^2(e)} &= 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall \text{ edges } e \text{ of } \widehat{f}, \\
(\nabla (u - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d} u), \nabla v)_{L^2(\widehat{f})} &= 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{f}),
\end{aligned}
$$

cf. Definition 4.5. The 1D-operator $\widehat{\Pi}_{p}^{\mathrm{grad},1d} : H^1(\widehat{e}) \to \mathcal{P}_p(\widehat{e})$ is defined by

$$
\begin{aligned}
((u - \widehat{\Pi}_{p}^{\mathrm{grad},1d} u)', v')_{L^2(\widehat{e})} = 0 \qquad \forall v \in \mathcal{P}_p(\widehat{e}) \cap H_0^1(\widehat{e}), \\
u(\pm 1) = (\widehat{\Pi}_{p}^{\mathrm{grad},1d} u)(\pm 1),
\end{aligned}
$$

where $\widehat{e}$ denotes the reference interval $(-1,1)$. We mention that the 1D-operator $\widehat{\Pi}_p^{\mathrm{grad},1d}$ coincides with the restriction of the 2D-operator $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$ to an edge e if we identify e with $\widehat{e}$.
For an estimate of the interpolation error $\|u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u\|_{H^{1-s}(\widehat{f})}$, $s\in[0,1]$, we need to bound the stronger norm $\|u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u\|_{H^1(\widehat{f})}$ and the weaker norm $\|u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u\|_{L^2(\widehat{f})}$, from which an interpolation argument gives the desired result. For the stronger norm, we introduce the best approximation operator $P^{\mathrm{grad},2d}$, defined by

$$\begin{aligned}
(\nabla(u-P^{\mathrm{grad},2d}u),\nabla v)_{L^2(\widehat{f})} &= 0 \qquad \forall v\in W_{p+1}(\widehat{f}),\\
(u-P^{\mathrm{grad},2d}u,1)_{L^2(\widehat{f})} &= 0,
\end{aligned}$$

cf. Lemma 4.16, and apply the triangle inequality to obtain

$$\|u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u\|_{H^1(\widehat{f})} \le \|u-P^{\mathrm{grad},2d}u\|_{H^1(\widehat{f})} + \|P^{\mathrm{grad},2d}u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u\|_{H^1(\widehat{f})}.$$

Now the first term is handled with the best approximation result Lemma 4.16,

$$\|u-P^{\mathrm{grad},2d}u\|_{H^1(\widehat{f})} \lesssim p^{-1/2}\|u\|_{H^{3/2}(\widehat{f})}.$$

For the second term, we use a continuous polynomial preserving lifting $\mathcal{L}: H^{1/2}(\widehat{f}) \to H^1(\widehat{f})$ from the boundary. The conditions imposed on $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$ and $P^{\mathrm{grad},2d}$ show

$$\Big(\nabla(P^{\mathrm{grad},2d}u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u),\nabla(P^{\mathrm{grad},2d}u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u-\mathcal{L}(P^{\mathrm{grad},2d}u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u))\Big)_{L^2(\widehat{f})} = 0$$

and as a consequence

$$\begin{aligned}
|P^{\mathrm{grad},2d}u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u|_{H^1(\widehat{f})} &\le |\mathcal{L}(P^{\mathrm{grad},2d}u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u)|_{H^1(\widehat{f})}\\
&\lesssim \|P^{\mathrm{grad},2d}u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u\|_{H^{1/2}(\partial\widehat{f})}.
\end{aligned}$$

Together with the triangle inequality, this yields

$$|P^{\mathrm{grad},2d}u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u|_{H^1(\widehat{f})} \lesssim \|u-P^{\mathrm{grad},2d}u\|_{H^{1/2}(\partial\widehat{f})} + \|u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u\|_{H^{1/2}(\partial\widehat{f})}. \tag{4.1}$$

A trace theorem and the best approximation result now give an estimate for the first term of (4.1), whereas we use the 1D-estimate

$$\|u-\widehat{\Pi}_p^{\mathrm{grad},1d}u\|_{H^{1/2}(\widehat{e})} \lesssim p^{-1/2}\|u\|_{H^1(\widehat{e})},$$

cf. Lemma 4.15, for the second expression.
The weaker norm is treated by a duality argument and integration by parts which yields two expressions, i.e.

$$\|u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u\|_{L^2(\widehat{f})}^2 = \int_{\widehat{f}}\nabla(u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u)\cdot\nabla z - \int_{\partial\widehat{f}}\partial_n z(u-\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}u),$$

where z solves the problem $-\Delta z = u - \widehat{\Pi}_{p+1}^{\text{grad},2d} u$ with homogeneous Dirichlet boundary conditions. Regularity theory then gives norm estimates for z. The integral on $\widehat{f}$ is again handled by best approximation properties: The orthogonality properties satisfied by $\widehat{\Pi}_{p+1}^{\text{grad},2d}$ allow us to insert an arbitrary polynomial function which gives

$$\left| \int_{\widehat{f}} \nabla(u - \widehat{\Pi}_{p+1}^{\text{grad},2d} u) \cdot \nabla z \right| \leq \inf_{\pi \in \mathcal{P}_p \cap H_0^1(\widehat{f})} \|z - \pi\|_{H^1(\widehat{f})} \|\nabla(u - \widehat{\Pi}_{p+1}^{\text{grad},2d} u)\|_{L^2(\widehat{f})}.$$

Hence, the integral on $\widehat{f}$ is estimated by standard best approximation (Lemma 2.23) and the already established H^1-result. The boundary integral on $\partial\widehat{f}$ is then written as sum over the edge contributions, and we estimate on each edge

$$\int_e \partial_n z (u - \widehat{\Pi}_{p+1}^{\text{grad},2d} u) \lesssim \|u - \widehat{\Pi}_{p+1}^{\text{grad},2d} u\|_{\widetilde{H}^{-1/2}(e)} \|\partial_n z\|_{H^{1/2}(e)}.$$

The proof is finished by the following error estimate for 1D from Lemma 4.15,

$$\|u - \widehat{\Pi}_p^{\text{grad},1d} u\|_{\widetilde{H}^{-1/2}(\widehat{e})} \lesssim p^{-3/2} \|u\|_{H^1(\widehat{e})},$$

together with the norm estimates for z.

The results for the remaining interpolation operators follow with a similar structure, but more technical effort. Note that the proofs are built up by spatial dimension, thus we can use the 2D-results for the boundary expressions appearing at the error estimates in 3D (just as we used the 1D-results for the boundary terms in 2D).

4.2 Definition of the interpolation operators

In this section, we define the projection-based interpolation operators. As introduced in Section 2.4, we denote by $\widehat{K}$ a fixed reference tetrahedron in 3D, and by $\widehat{f}$ a fixed reference triangle in 2D. In 3D, the H^1-conforming operator is then denoted by $\widehat{\Pi}_{p+1}^{\text{grad},3d}$, the $\mathbf{H}(\mathbf{curl})$-conforming one by $\widehat{\Pi}_p^{\text{curl},3d}$ and the $\mathbf{H}(\text{div})$-conforming operator by $\widehat{\Pi}_p^{\text{div},3d}$. In 2D, the notation is similar, i.e. we define the operators $\widehat{\Pi}_{p+1}^{\text{grad},2d}$ and $\widehat{\Pi}_p^{\text{curl},2d}$.

The operators are built up by spatial dimension. For $\widehat{\Pi}_{p+1}^{\text{grad},3d}$, we start with fixing the values in the vertices first, and then follow up on the edges, the faces and finally in the interior. For $\widehat{\Pi}_p^{\text{curl},3d}$, we just start on the edges, and for $\widehat{\Pi}_p^{\text{div},3d}$, fixing only on the faces and the interior remains. We note that the procedure of building the interpolation operators can also be seen as a sequence of constrained optimizations in which the value on a subsimplex of $\widehat{K}$ is determined as the solution of a minimization problem, where the values on the boundary subsimplices have already been fixed. In 2D, analogous statements hold.

We start with the operators in 3D.

Definition 4.1 ($\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$)**.** $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d} : H^2(\widehat{K}) \to W_{p+1}(\widehat{K})$ *is given by*

$$u(V) - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} u(V) = 0 \quad \forall V \in \mathcal{V}(\widehat{K}), \tag{4.2a}$$
$$(\nabla_e(u - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} u), \nabla_e v)_{L^2(e)} = 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall e \in \mathcal{E}(\widehat{K}), \tag{4.2b}$$
$$(\nabla_f(u - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} u), \nabla_f v)_{L^2(f)} = 0 \quad \forall v \in \mathring{W}_{p+1}(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{4.2c}$$
$$(\nabla(u - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} u), \nabla v)_{L^2(\widehat{K})} = 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{K}). \tag{4.2d}$$

Definition 4.2 ($\widehat{\Pi}_{p}^{\mathrm{curl},3d}$)**.** $\widehat{\Pi}_{p}^{\mathrm{curl},3d} : \mathbf{H}^1(\widehat{K}, \mathbf{curl}) \to \mathbf{Q}_p(\widehat{K})$ *is given by*

$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{curl},3d} \mathbf{u}), 1)_{L^2(e)} = 0 \quad \forall e \in \mathcal{E}(\widehat{K}), \tag{4.3a}$$
$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{curl},3d} \mathbf{u}), \nabla_e v)_{L^2(e)} = 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall e \in \mathcal{E}(\widehat{K}), \tag{4.3b}$$
$$(\Pi_\tau(\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{curl},3d} \mathbf{u}), \nabla_f v)_{L^2(f)} = 0 \quad \forall v \in \mathring{W}_{p+1}(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{4.3c}$$
$$(\mathrm{curl}_f \, \Pi_\tau(\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{curl},3d} \mathbf{u}), \mathrm{curl}_f \, \mathbf{v})_{L^2(f)} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{4.3d}$$
$$((\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{curl},3d} \mathbf{u}), \nabla v)_{L^2(\widehat{K})} = 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{K}), \tag{4.3e}$$
$$(\mathbf{curl}(\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{curl},3d} \mathbf{u}), \mathbf{curl}\, \mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}). \tag{4.3f}$$

Definition 4.3 ($\widehat{\Pi}_{p}^{\mathrm{div},3d}$)**.** $\widehat{\Pi}_{p}^{\mathrm{div},3d} : \mathbf{H}^{1/2}(\widehat{K}, \mathrm{div}) \to \mathbf{V}_p(\widehat{K})$ *is given by*

$$(\mathbf{n}_f \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{div},3d} \mathbf{u}), 1)_{L^2(f)} = 0 \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{4.4a}$$
$$(\mathbf{n}_f \cdot (\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{div},3d} \mathbf{u}), v)_{L^2(f)} = 0 \quad \forall v \in \mathring{V}_p(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{4.4b}$$
$$((\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{div},3d} \mathbf{u}), \mathbf{curl}\, \mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}), \tag{4.4c}$$
$$(\mathrm{div}(\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{div},3d} \mathbf{u}), \mathrm{div}\, \mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}). \tag{4.4d}$$

In order to obtain a complete commuting diagram property, we also need the following interpolation operator on the space $L^2(\widehat{K})$ which is defined as the L^2-projection.

Definition 4.4 ($\widehat{\Pi}_{p}^{L^2}$)**.** $\widehat{\Pi}_{p}^{L^2} : L^2(\widehat{K}) \to W_p(\widehat{K})$ *is given by*

$$(u - \widehat{\Pi}_{p}^{L^2} u, v)_{L^2(\widehat{K})} = 0 \qquad \forall v \in W_p(\widehat{K}). \tag{4.5}$$

In 2D, the definitions are similar.

Definition 4.5 ($\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$)**.** $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d} : H^{3/2}(\widehat{f}) \to W_{p+1}(\widehat{f})$ *is given by*

$$u(V) - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d} u(V) = 0 \quad \forall V \in \mathcal{V}(\widehat{f}), \tag{4.6a}$$
$$(\nabla_e(u - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d} u), \nabla_e v)_{L^2(e)} = 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall e \in \mathcal{E}(\widehat{f}), \tag{4.6b}$$
$$(\nabla(u - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d} u), \nabla v)_{L^2(\widehat{f})} = 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{f}). \tag{4.6c}$$

Definition 4.6 ($\widehat{\Pi}_p^{\mathrm{curl},2d}$)**.** $\widehat{\Pi}_p^{\mathrm{curl},2d} : \mathbf{H}^{1/2}(\widehat{f}, \mathrm{curl}) \to \mathbf{Q}_p(\widehat{f})$ *is given by*

$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}), 1)_{L^2(e)} = 0 \quad \forall e \in \mathcal{E}(\widehat{f}), \tag{4.7a}$$

$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}), \nabla_e v)_{L^2(e)} = 0 \quad \forall v \in \mathring{W}_{p+1}(e) \quad \forall e \in \mathcal{E}(\widehat{f}), \tag{4.7b}$$

$$((\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}), \nabla v)_{L^2(\widehat{f})} = 0 \quad \forall v \in \mathring{W}_{p+1}(\widehat{f}), \tag{4.7c}$$

$$(\mathrm{curl}(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}), \mathrm{curl}\, \mathbf{v})_{L^2(\widehat{f})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{f}). \tag{4.7d}$$

Definition 4.7 ($\widehat{\Pi}_p^{L^2}$)**.** *The operator* $\widehat{\Pi}_p^{L^2} : L^2(\widehat{f}) \to W_p(\widehat{f})$ *is defined by*

$$(u - \widehat{\Pi}_p^{L^2} u, v)_{L^2(\widehat{f})} = 0 \qquad \forall v \in W_p(\widehat{f}). \tag{4.8}$$

Remark 4.8. A closer look at the definitions of the operators reveals that the operators in 2D and 3D are closely related. In fact, if we identify a face $f \in \mathcal{F}(\widehat{K})$ with $\widehat{f}$ via an affine congruence map, the restrictions of the operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$ (and $\widehat{\Pi}_p^{\mathrm{curl},3d}$) to the face f coincide with the operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$ (and $\widehat{\Pi}_p^{\mathrm{curl},2d}$).

4.3 Well-definedness of the projection-based operators

We now show that the interpolation operators defined in Section 4.2 are well-defined and projections.

Lemma 4.9. *For* $u \in H^2(\widehat{K})$ *there holds* $u|_e \in H^1(e)$ *for each* $e \in \mathcal{E}(\widehat{K})$ *and*

$$\|u\|_{H^1(e)} \lesssim \|u\|_{H^2(\widehat{K})}.$$

Moreover, the operator $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$ *is well-defined.*

Proof. We prove the first claim by applying the trace theorem twice. For $\epsilon > 0$ sufficiently small, the first application shows that the trace operator maps $H^{2+\epsilon}(\widehat{K}) \to H^{3/2+\epsilon}(f)$ and $H^{2-\epsilon}(\widehat{K}) \to H^{3/2-\epsilon}(f)$, the second application gives $H^{3/2+\epsilon}(f) \to H^{1+\epsilon}(e)$ and $H^{3/2-\epsilon}(f) \to H^{1-\epsilon}(e)$. Interpolation then asserts $H^2(\widehat{K}) \to H^1(e)$.
We now show the well-definedness of $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$ by dimension arguments. Since we have

$$\dim W_{p+1}(\widehat{K}) = \frac{1}{6}(p+4)(p+3)(p+2),$$

and the number N_{cond} of posed conditions in 4.2 is

$$N_{cond} = 4 + 6p + 4\frac{(p-1)p}{2} + \frac{1}{6}(p-2)(p-1)p = \frac{1}{6}(p+4)(p+3)(p+2),$$

we observe $N_{cond} = \dim W_{p+1}(\widehat{K})$. Thus, (4.2) represents a square linear system. What is still left to show, is uniqueness of the system. Let $u = 0$. Then (4.2a) shows $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}u(V) = 0$ for all vertices $V \in \mathcal{V}(\widehat{K})$. The conditions (4.2b) then imply that $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}u = 0$ on all edges of $\widehat{K}$. Equations (4.2c) then lead to $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}u = 0$ vanishing on all faces of $\widehat{K}$ and finally (4.2d) shows $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}u = 0$. Thus, $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$ is well-defined. □

Lemma 4.10. *For* $\mathbf{u} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$ *there holds* $\mathbf{u} \cdot \mathbf{t}_e \in L^2(e)$ *for each edge* $e \in \mathcal{E}(\widehat{K})$ *and*

$$\|\mathbf{u} \cdot \mathbf{t}_e\|_{L^2(e)} \lesssim \|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K}, \mathbf{curl})}.$$

Moreover, the operator $\widehat{\Pi}_p^{\text{curl},3d}$ *is well-defined.*

Proof. For $\mathbf{u} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$ the trace theorem gives, for each face f, $\Pi_\tau \mathbf{u} \in \mathbf{H}^{1/2}(f, \text{curl}_f)$. Lemma 2.25 now shows that we can decompose this functions as $\Pi_\tau \mathbf{u} = \nabla\varphi + \mathbf{z}$ with $\varphi \in H^{3/2}(f)$ and $\mathbf{z} \in \mathbf{H}^{3/2}(f)$, which implies $\Pi_\tau \mathbf{u} \cdot \mathbf{t}_e = \nabla_e \varphi|_e + \mathbf{z}|_e \cdot \mathbf{t}_e \in L^2(e)$.
The well-definedness of $\widehat{\Pi}_p^{\text{curl},3d}$ is seen in a similar way to Lemma 4.9. Here we introduce the notation

$$\ker \mathbf{curl} = \{\mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K}) : \mathbf{curl}\, \mathbf{q} = \mathbf{0}\}.$$

Considering the exactness of the sequence (2.33), we obtain

$$\dim \mathring{\mathbf{Q}}_p(\widehat{K}) = \dim \mathbf{curl}\, \mathring{\mathbf{Q}}_p(\widehat{K}) + \dim \ker \mathbf{curl} = \dim \mathbf{curl}\, \mathring{\mathbf{Q}}_p(\widehat{K}) + \dim \nabla \mathring{W}_{p+1}(\widehat{K}).$$

Thus, the number of conditions in (4.3e), (4.3f) equals $\dim \mathring{\mathbf{Q}}_p(\widehat{K})$. Analogously, by the exactness of the second sequence in (2.33) we obtain, for each face $f \in \mathcal{F}(\widehat{K})$, the number $\dim \mathring{\mathbf{Q}}_p(f)$ as the number of conditions in (4.3c), (4.3d). Finally, for each edge $e \in \mathcal{E}(\widehat{K})$, the number of conditions in (4.3b) is p and the number of conditions in (4.3a) is 6.
We now calculate the various dimensions. Using the formulas from [51, Sec. 5.5], we get

$$\dim \mathbf{Q}_p(\widehat{K}) = \frac{1}{2}(p+4)(p+3)(p+1)$$

and

$$\begin{aligned}\dim \mathring{\mathbf{Q}}_p(\widehat{K}) &= \dim \mathbf{Q}_p(\widehat{K}) - 4 \cdot 2 \dim(\mathcal{P}_{p-1}(\mathbb{R}^2)) - 6(p+1)\\ &= \frac{1}{2}(p+1)p(p-1),\end{aligned}$$

cf. [51, Def. 5.30, Lemma 5.35]. Analogous considerations for 2D give for each face $f \in \mathcal{F}(\widehat{K})$

$$\begin{aligned}\dim \mathbf{Q}_p(f) &= 2 \dim \mathcal{P}_p(f) + 2 \dim \widetilde{\mathcal{P}}_{p+1}(f) - \dim \widetilde{\mathcal{P}}_{p+2}(f)\\ &= (p+1)(p+2) + 2(p+2) - (p+3) = (p+1)(p+3)\end{aligned}$$

and

$$\dim \mathring{\mathbf{Q}}_p(f) = (p+1)(p+3) - 3(p+1) = p(p+1).$$

Hence, an elementary calculation yields

$$\dim \mathbf{Q}_p(\widehat{K}) = \dim \mathring{\mathbf{Q}}_p(\widehat{K}) + 4 \dim \mathring{\mathbf{Q}}_p(f) + 6p + 6,$$

which implies that the number of conditions in (4.3) coincides with $\dim \mathbf{Q}_p(\widehat{K})$. Thus, (4.3) represents a square system of equations.

As in Lemma 4.9, we now show that $\mathbf{u} = 0$ implies $\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} = 0$. Conditions (4.3a) and (4.3b) imply that the tangential component of $\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}$ vanishes on all edges of $\widehat{K}$. This observation together with the orthogonality conditions (4.3c) and (4.3d) and the exact sequence property (2.34) gives that the tangential component $\Pi_\tau\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}$ vanishes on all faces of $\widehat{K}$. Finally, it follows from (4.3e) and (4.3f) together with the exact sequence property (2.33) that $\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} = 0$. □

Lemma 4.11. *For $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$ there holds $\mathbf{u}\cdot\mathbf{n}_f \in L^2(f)$ for each face $f \in \mathcal{F}(\widehat{K})$ and*

$$\|\mathbf{u}\cdot\mathbf{n}_f\|_{L^2(f)} \lesssim \|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}.$$

Moreover, the operator $\widehat{\Pi}_p^{\mathrm{div},3d}$ is well-defined.

Proof. We decompose $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$ as $\mathbf{u} = \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z}$ with $\boldsymbol{\varphi}, \mathbf{z} \in \mathbf{H}^{3/2}(\widehat{K})$, cf. Lemma 2.33. It follows $\mathbf{n}_f\cdot\mathbf{z} \in \mathbf{H}^1(f)$ and $\boldsymbol{\varphi}|_f \in \mathbf{H}^1(f)$. Since $(\mathbf{n}_f\cdot\mathbf{curl}\,\boldsymbol{\varphi})|_f = \mathrm{curl}_f(\Pi_\tau\boldsymbol{\varphi})|_f$, we obtain $(\mathbf{n}_f\cdot\mathbf{curl}\,\boldsymbol{\varphi})|_f \in L^2(f)$, and thus $\mathbf{n}_f\cdot\mathbf{u} \in L^2(f)$.
The well-definedness of the operator is again seen by dimension arguments. We introduce the notation

$$\ker\mathrm{div} = \{\mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}) : \mathrm{div}\,\mathbf{v} = 0\}.$$

Considering the exactness of the sequence in (2.33), we get the equality

$$\dim\mathring{\mathbf{V}}_p(\widehat{K}) = \dim\mathrm{div}\,\mathring{\mathbf{V}}_p(\widehat{K}) + \dim\ker\mathrm{div} = \dim\mathrm{div}\,\mathring{\mathbf{V}}_p(\widehat{K}) + \dim\mathbf{curl}\,\mathring{\mathbf{Q}}_p(\widehat{K}).$$

Thus, the number of conditions in (4.4c), (4.4d) equals $\dim\mathring{\mathbf{V}}_p(\widehat{K})$. Furthermore, the number of conditions in (4.4a), (4.4b) is

$$4 + 4\dim\mathring{V}_p(f) = 4 + 4\left(\frac{(p+1)(p+2)}{2} - 1\right) = 4\dim W_p(f)$$

such that

$$\dim\mathring{\mathbf{V}}_p(\widehat{K}) + 4\dim W_p(f) = \frac{1}{2}(p+2)(p+1)p + 4\frac{(p+1)(p+2)}{2} = \dim\mathbf{V}_p(\widehat{K}).$$

Hence, (4.4) represents a square linear system. In order to show uniqueness, we observe that $\mathbf{u} = 0$ implies $\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} = 0$, since the conditions (4.4a) and (4.4b) give $\mathbf{n}_f\cdot\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} = 0$ for all faces $f \in \mathcal{F}(\widehat{K})$, and the exact sequence property (2.33) and conditions (4.4c), (4.4d) then imply $\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} = 0$. □

Since the operators are defined in a similar way in 2D, it seems obvious that they are also well-defined.

Lemma 4.12. *The following statements hold:*

(i) For $u \in H^{3/2}(\widehat{f})$ there holds $\nabla_e u \in L^2(e)$ for each edge $e \in \mathcal{E}(\widehat{f})$ and

$$\|\nabla_e u\|_{L^2(e)} \lesssim \|u\|_{H^{3/2}(\widehat{f})}.$$

Moreover, the operator $\widehat{\Pi}^{\text{grad},2d}_{p+1}$ is well-defined.

(ii) For $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \text{curl})$ there holds $\mathbf{u} \cdot \mathbf{t}_e \in L^2(e)$ for each edge $e \in \mathcal{E}(\widehat{f})$ and

$$\|\mathbf{u} \cdot \mathbf{t}_e\|_{L^2(e)} \lesssim \|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{f},\text{curl})}.$$

Moreover, the operator $\widehat{\Pi}^{\text{curl},2d}_{p}$ is well-defined.

Proof. For $\varepsilon > 0$ sufficiently small, the trace theorem implies $u|_e \in H^{1+\varepsilon}(e)$ for $u \in H^{3/2+\varepsilon}(\widehat{f})$ and $u|_e \in H^{1-\varepsilon}(e)$ for $u \in H^{3/2-\varepsilon}(\widehat{f})$. It follows $u|_e \in H^1(e)$ for $u \in H^{3/2}(\widehat{f})$ by interpolation.
By Lemma 2.25, we can write $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \text{curl})$ as $\mathbf{u} = \nabla\varphi + \mathbf{z}$ with $\varphi \in H^{3/2}(\widehat{f})$ and $\mathbf{z} \in \mathbf{H}^{3/2}(\widehat{f})$. Hence, we have $\mathbf{u} \cdot \mathbf{t}_e = \nabla_e \varphi|_e + \mathbf{z}|_e \cdot \mathbf{t}_e \in L^2(e)$.
The well-definedness of the operators is seen completely analogously to the 3D case by dimension arguments. □

Lemma 4.13. *The operators defined in Definitions 4.1-4.7 are projections.*

Proof. We start with the operator $\widehat{\Pi}^{\text{grad},3d}_{p+1}$. Let $u \in W_{p+1}(\widehat{K})$. Then property (4.2a) yields

$$u(V) = \widehat{\Pi}^{\text{grad},3d}_{p+1} u(V) \tag{4.9}$$

for all $V \in \mathcal{V}(\widehat{K})$ and therefore $(u - \widehat{\Pi}^{\text{grad},3d}_{p+1} u)|_e \in \mathring{W}_{p+1}(e)$ for all $e \in \mathcal{E}(\widehat{K})$. Thus, we can use $v = (u - \widehat{\Pi}^{\text{grad},3d}_{p+1} u)|_e$ as test function for (4.2b) and obtain

$$\|\nabla_e (u - \widehat{\Pi}^{\text{grad},3d}_{p+1} u)|_e\|_{L^2(e)} = 0,$$

which implies $\nabla_e (u - \widehat{\Pi}^{\text{grad},3d}_{p+1} u)|_e = 0$. This is equivalent to $(u - \widehat{\Pi}^{\text{grad},3d}_{p+1} u)|_e$ being constant on e, and together with (4.9) it follows $(u - \widehat{\Pi}^{\text{grad},3d}_{p+1} u)|_e = 0$ on e, hence $(u - \widehat{\Pi}^{\text{grad},3d}_{p+1} u)|_f \in \mathring{W}_{p+1}(f)$ for all $f \in \mathcal{F}(\widehat{K})$. Repeating the argumentation above for the faces f in view of (4.2c) and for the volume $\widehat{K}$ in view of (4.2d) shows that $\widehat{\Pi}^{\text{grad},3d}_{p+1}$ is a projection.
In order to show the projection property for $\widehat{\Pi}^{\text{curl},3d}_{p}$, we take $\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$. Let $\widetilde{e} := \left(\mathbf{u} - \widehat{\Pi}^{\text{curl},3d}_{p} \mathbf{u}\right) \cdot \mathbf{t}_e$ be the error. We now identify an edge $e \in \mathcal{E}(\widehat{f})$ with the interval $(0, L)$ of length $L = \operatorname{diam} e$. Note that a function $w \in \mathcal{P}_p(e)$ can then be decomposed into

$$w(x) = \overline{w} + \left(\int_0^x w(t)\,dt - x\overline{w}\right)',$$

where $\overline{w}$ denotes the average of w on e. Since we have

$$\int_0^x w(t)\,dt - x\overline{w} \in \mathring{W}_{p+1}(e),$$

it follows $(\widetilde{e}, w)_{L^2(e)} = 0$ by (4.3a) and (4.3b). Hence, we have shown that the operator $\widehat{\Pi}_p^{\mathrm{curl},3d}$ is the L^2-projection on edges $e \in \mathcal{E}(\widehat{f})$, i.e.

$$(\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), w)_{L^2(e)} = 0 \qquad \forall w \in Q_p(e). \tag{4.10}$$

Thus, we can use $w = \mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}) \in Q_p(e)$ as test function in (4.10) and obtain $\mathbf{t}_e \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}) = 0$ on edges $e \in \mathcal{E}(\widehat{K})$. This implies $\Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}) \in \mathring{\mathbf{Q}}_p(f)$ on all faces $f \in \mathcal{F}(\widehat{K})$, thus

$$\mathrm{curl}_f\, \Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}) = 0$$

on f by (4.3d). By the exact sequence property, there exists $\varphi \in \mathring{W}_{p+1}(f)$ such that $\Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}) = \nabla\varphi$, thus (4.3c) shows

$$\Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}) = 0$$

on each face f, which implies $\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} \in \mathring{\mathbf{Q}}_p(\widehat{K})$. The exact sequence property and equations (4.3e) and (4.3f) now give the desired result.
For the projection property of $\widehat{\Pi}_p^{\mathrm{div},3d}$, let $\mathbf{u} \in \mathbf{V}_p(\widehat{K})$. We define $\widetilde{e} := \left(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\right) \cdot \mathbf{n}_f$. Since every function $w \in \mathcal{P}_p(f)$ can be written as $w = \overline{w} + (w - \overline{w})$, where $\overline{w}$ denotes the average of w on f, the fact $w - \overline{w} \in \mathring{V}_p(f)$ gives

$$(\mathbf{n}_f \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}), w)_{L^2(f)} = 0 \qquad \forall w \in V_p(f). \tag{4.11}$$

in view of (4.4a) and (4.4b). Hence, we have shown that the operator $\widehat{\Pi}_p^{\mathrm{div},3d}$ is the L^2-projection on faces $\mathcal{F}(\widehat{K})$. This observation justifies the use of $w = \mathbf{n}_f \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}) \in V_p(f)$ as test function in (4.11), from which

$$\mathbf{n}_f \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}) = 0$$

on each face $f \in \mathcal{F}(\widehat{K})$ follows. Now by definition, $\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} \in \mathring{\mathbf{V}}_p(\widehat{K})$, thus we get

$$\mathrm{div}(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}) = 0$$

by (4.4d). From the exact sequences, there follows the existence of a function $\boldsymbol{\varphi} \in \mathring{\mathbf{Q}}_p(\widehat{K})$ such that $\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} = \mathbf{curl}\,\boldsymbol{\varphi}$. Equation (4.4c) then shows $\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} = \mathbf{u}$ on $\widehat{K}$.
The projection property of $\widehat{\Pi}_p^{L^2}$ is clear by definition.
Since the operators in 2D are only the restrictions of the versions in 3D (cf. Remark 4.8), the projection properties of the 2D-operators follow from the argumentation above. □

4.4 The commuting diagram property

We use this short section to show that the interpolation operators defined in Section 4.2 satisfy the commuting diagram properties

$$\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\mathrm{id}} & H^2(\widehat{K}) & \xrightarrow{\nabla} & \mathbf{H}^1(\widehat{K},\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \mathbf{H}^1(\widehat{K},\mathrm{div}) & \xrightarrow{\mathrm{div}} & H^1(\widehat{K}) & \xrightarrow{0} & \{0\} \\
 & & \downarrow \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} & & \downarrow \widehat{\Pi}_p^{\mathrm{curl},3d} & & \downarrow \widehat{\Pi}_p^{\mathrm{div},3d} & & \downarrow \widehat{\Pi}_p^{L^2} & & \\
\mathbb{R} & \xrightarrow{\mathrm{id}} & W_{p+1}(\widehat{K}) & \xrightarrow{\nabla} & \mathbf{Q}_p(\widehat{K}) & \xrightarrow{\mathbf{curl}} & \mathbf{V}_p(\widehat{K}) & \xrightarrow{\mathrm{div}} & W_p(\widehat{K}) & \xrightarrow{0} & \{0\}
\end{array} \tag{4.12}$$

in 3D and

$$\begin{array}{ccccccccc}
\mathbb{R} & \xrightarrow{\text{id}} & H^{3/2}(\widehat{f}) & \xrightarrow{\nabla} & \mathbf{H}^{1/2}(\widehat{f},\text{curl}) & \xrightarrow{\text{curl}} & H^{1/2}(\widehat{f}) & \xrightarrow{0} & \{0\} \\
 & & \big\downarrow \widehat{\Pi}^{\text{grad},2d}_{p+1} & & \big\downarrow \widehat{\Pi}^{\text{curl},2d}_{p} & & \big\downarrow \widehat{\Pi}^{L^2}_{p} & & \\
\mathbb{R} & \xrightarrow{\text{id}} & W_{p+1}(\widehat{f}) & \xrightarrow{\nabla} & \mathbf{Q}_p(\widehat{f}) & \xrightarrow{\text{curl}} & W_p(\widehat{f}) & \xrightarrow{0} & \{0\}
\end{array} \tag{4.13}$$

in 2D.

Theorem 4.14. *The diagrams* (4.12) *and* (4.13) *commute.*

Proof. We show the 3D case (4.12). Note that the proof follows with similar arguments as in [26, Thm. 5.1].

Step 1: We show $\widehat{\Pi}^{\text{curl},3d}_p \nabla\varphi = \nabla\varphi_p$ for some $\varphi \in H^2(\widehat{K})$ and some $\varphi_p \in W_{p+1}(\widehat{K})$.
Let $\mathbf{u} = \nabla\varphi$ for some $\varphi \in H^2(\widehat{K})$. For each face $f \in \mathcal{F}(\widehat{K})$ and each edge $e \in \mathcal{E}(f)$ with endpoints $V_{e,1}$, $V_{e,2}$, we obtain $\int_e \mathbf{u}\cdot\mathbf{t}_e = \varphi(V_1) - \varphi(V_2)$, where $\mathbf{t}_e$ denotes the tangential vector of e in the mathematical positive direction with regard to f. Thus, (4.3a) implies

$$\int_{\partial f} \Pi_{\tau,f}\widehat{\Pi}^{\text{curl},3d}_p \mathbf{u} = \sum_{e\subset\partial f}\int_e \mathbf{u}\cdot\mathbf{t}_e = 0. \tag{4.14}$$

Note that $\text{curl}_f\, \Pi_\tau \mathbf{u} = \text{curl}_f\, \Pi_\tau \nabla\varphi = 0$, thus we can conclude with integration by parts

$$\int_f \text{curl}_f\, \Pi_\tau \widehat{\Pi}^{\text{curl},3d}_p \mathbf{u} = \int_{\partial f} \Pi_{\tau,f}\widehat{\Pi}^{\text{curl},3d}_p \mathbf{u} \overset{(4.14)}{=} 0. \tag{4.15}$$

Furthermore, the exact sequence property (2.33) gives us $\text{curl}_f\, \mathring{\mathbf{Q}}_p(f) = \mathring{V}_p(f)$. Hence, (4.3d) with test function $\text{curl}_f\, \Pi_\tau \widehat{\Pi}^{\text{curl},3d}_p \mathbf{u} \in \mathring{V}_p(f)$ (cf. (4.15)) leads to

$$\text{curl}_f\, \Pi_\tau \widehat{\Pi}^{\text{curl},3d}_p \mathbf{u} = 0, \tag{4.16}$$

which implies that on each face $(\Pi_\tau \widehat{\Pi}^{\text{curl},3d}_p \mathbf{u})|_f$ is a gradient of a polynomial, i.e.

$$(\Pi_\tau \widehat{\Pi}^{\text{curl},3d}_p \mathbf{u})|_f = \nabla\varphi_{p,f}$$

for some $\varphi_{p,f} \in W_{p+1}(f)$ for each face $f \in \mathcal{F}(\widehat{K})$.
We now claim that this piecewise polynomial can be chosen to be continuous on $\partial\widehat{K}$. Fix a vertex $V \in \mathcal{V}(\widehat{K})$. By fixing the constant of the polynomials $\varphi_{p,f}$ we may assume that $\varphi_{p,f}(V) = 0$ for each face f with V as a vertex. We now take an edge e that has V as vertex, and denote by f_1 and f_2 both faces sharing the edge e. The conditions in (4.3a) and (4.3b) read for $i = 1, 2$ as

$$(\mathbf{t}_e \cdot (\nabla\varphi - \nabla\varphi_{p,f_i}), \nabla_e v)_{L^2(e)} = 0$$

for $v \in W_{p+1}(e)$, by definition of the functions φ_{p,f_i}. By taking $v = \varphi_{p,f_1} - \varphi_{p,f_2}$ as test function and subtracting both scalar products, we can conclude that $\varphi_{p,f}$ is continuous across all edges e that have V as an endpoint. Hence, the piecewise polynomial φ_p defined by $\varphi_p|_f = \varphi_{p,f}$ is continuous in all vertices of $\widehat{K}$ and thus also on $\partial\widehat{K}$. This continuous, piecewise polynomial φ_p has a polynomial lifting to $\widehat{K}$ that we again denote by $\varphi_p \in W_{p+1}(\widehat{K})$, cf. [52, 30]. Since

$$\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} - \nabla\varphi_p \in \mathring{\mathbf{Q}}_p(\widehat{K}),$$

(4.3f) with test function $\mathbf{v} = \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} - \nabla\varphi_p \in \mathring{\mathbf{Q}}_p(\widehat{K})$ implies

$$\mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} = 0. \tag{4.17}$$

Note that the second line of (4.12) expresses an exact sequence property, hence step 1 is complete.

Step 2: We show $\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla = \nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$.
The first step gives us $\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi = \nabla\varphi_p$ for some $\varphi_p \in W_{p+1}(\widehat{K})$. We fix the constant in the function φ_p by fixing a vertex $V \in \mathcal{V}(\widehat{K})$ and then setting $\varphi_p(V) = \varphi(V)$. Now let $V' \in \mathcal{V}(\widehat{K})$ another arbitrary vertex, and let e be the edge between V and V'. From (4.3a), we then get

$$0 = (\mathbf{t}_e \cdot (\nabla\varphi - \nabla\varphi_p), 1)_{L^2(e)} = (\varphi - \varphi_p)(V) - (\varphi - \varphi_p)(V') = -(\varphi - \varphi_p)(V'),$$

and it follows $\varphi_p(V') = \varphi(V')$ for all vertices $V' \in \mathcal{V}(\widehat{K})$ by repeating the argument.
Next, (4.2b) and (4.3b) imply

$$(\mathbf{t}_e \cdot (\nabla\varphi_p - \nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi), \nabla_e v)_{L^2(e)} = 0 \tag{4.18}$$

for all $v \in \mathring{W}_{p+1}(e)$. Since $\varphi \in H^2(\widehat{K})$, (4.2a) shows

$$\varphi_p(V) = \varphi(V) = \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi(V)$$

for all vertices $V \in \mathcal{V}(\widehat{K})$ and thus

$$\varphi_p|_e - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi|_e \in \mathring{W}_{p+1}(e).$$

Choosing $v = \varphi_p|_e - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi|_e$ as test function in (4.18) then shows $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi = \varphi_p$ on all edges $e \in \mathcal{E}(\widehat{K})$.
The argument on the faces is similar. Here, (4.2c) and (4.3c) reveal

$$(\nabla_f(\varphi_p - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi), \nabla_f v)_{L^2(f)} = 0 \tag{4.19}$$

for $v \in \mathring{W}_{p+1}(f)$ on each face $f \in \mathcal{F}(\widehat{K})$. Since we have recently seen $\varphi_p - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi = 0$ on each edge, it follows

$$\varphi_p - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi \in \mathring{W}_{p+1}(f)$$

which makes it a suitable test function for (4.19). Hence, we obtain $\nabla_f \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi = \Pi_\tau \widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi$ on each face $f \in \mathcal{F}(\widehat{K})$.
Finally on $\widehat{K}$, we compare (4.2d) with (4.3e), which yields

$$(\nabla\varphi_p - \nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi, \nabla v)_{L^2(\widehat{K})} = 0 \tag{4.20}$$

for all $v \in \mathring{W}_{p+1}(\widehat{K})$. The observation $\nabla_f(\varphi_p - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi) = 0$ on each face f together with the fact that $\varphi_p - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi = 0$ on each edge e implies

$$\varphi_p - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi \in \mathring{W}_{p+1}(\widehat{K}),$$

thus it can be used as test function for (4.20). This yields $\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi = \nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi$ on $\widehat{K}$, which finishes step 2.

Step 3: We prove $\mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d} = \widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}$.
First, we show

$$\operatorname{div}\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = 0. \tag{4.21}$$

To see this, we note that the second line of (4.12) implies $\operatorname{div}\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} \in W_p(\widehat{K})$. Integration by parts together with (4.4a) then shows

$$\int_{\widehat{K}} \operatorname{div}\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = \int_{\partial\widehat{K}} \mathbf{n}\cdot\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = \int_{\partial\widehat{K}} \mathbf{n}\cdot\mathbf{curl}\,\mathbf{u} = \int_{\widehat{K}} \operatorname{div}\mathbf{curl}\,\mathbf{u} = 0, \tag{4.22}$$

thus $\operatorname{div}\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} \in W_p^{aver}(\widehat{K})$. By the exact sequence property of the first line of diagram (2.33) we obtain that $\operatorname{div}: \mathring{\mathbf{V}}_p(\widehat{K}) \to W_p^{aver}(\widehat{K})$ is surjective. Hence, we get from (4.4d) that $\operatorname{div}\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = 0$, i.e. the claim (4.21) holds.
Next, by (4.21) and the exact sequence property of (4.12), there exists a vector field $\mathbf{u}_p \in \mathbf{Q}_p(\widehat{K})$ such that

$$\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = \mathbf{curl}\,\mathbf{u}_p. \tag{4.23}$$

In order to show $\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u} = \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}$, we prove that $\mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ satisfies the equations (4.4) for $\widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}\,\mathbf{u}$. In this case, they are reformulated as

$$(\mathbf{n}_f\cdot(\mathbf{curl}\,\mathbf{u} - \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), 1)_{L^2(f)} = 0 \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{4.24a}$$

$$(\mathbf{n}_f\cdot(\mathbf{curl}\,\mathbf{u} - \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), v)_{L^2(f)} = 0 \quad \forall v \in \mathring{V}_p(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{4.24b}$$

$$(\mathbf{curl}\,\mathbf{u} - \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}), \tag{4.24c}$$

$$(\operatorname{div}(\mathbf{curl}\,\mathbf{u} - \mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), \operatorname{div}\mathbf{v})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}). \tag{4.24d}$$

(4.24d) is obviously true since the whole line is zero, and (4.24c) is exactly (4.3f). Using the observation

$$\mathbf{n}_f\cdot\mathbf{curl} = \operatorname{curl}_f \Pi_\tau,$$

we can write (4.24b) as

$$(\operatorname{curl}_f \Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), v)_{L^2(f)} = 0 \quad \forall v \in \mathring{V}_p(f) \quad \forall f \in \mathcal{F}(\widehat{K}), \tag{4.25}$$

which is (4.3d) considering $\mathring{V}_p(f) = \operatorname{curl}_f \mathring{\mathbf{Q}}_p(f)$ by the exact sequence property of (2.33). Finally, (4.24a) follows by integration by parts, which gives

$$(\operatorname{curl}_f \Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), 1)_{L^2(f)} = \sum_{e\subset\partial f} (\Pi_\tau(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}), \mathbf{t}_e)_{L^2(e)} \overset{(4.3a)}{=} 0.$$

This completes step 3.

Step 4: We prove $\operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d} = \widehat{\Pi}_p^{L^2} \operatorname{div}$.
First, we have

$$(\operatorname{div} \mathbf{u} - \operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}, 1)_{L^2(\widehat{K})} = \int_{\partial\widehat{K}} \mathbf{n} \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}) = 0 \tag{4.26}$$

by (4.4a). The exact sequence property (2.33) implies that every $w \in W_p^{aver}(\widehat{K})$ can be represented as $w = \operatorname{div} \mathbf{w}$ for some $\mathbf{w} \in \mathring{\mathbf{V}}_p(\widehat{K})$. Hence, for any $w \in W_p^{aver}(\widehat{K})$, we get

$$(\operatorname{div} \mathbf{u} - \operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}, w)_{L^2(\widehat{K})} = (\operatorname{div} \mathbf{u} - \operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}, \operatorname{div} \mathbf{w})_{L^2(\widehat{K})} \overset{(4.4d)}{=} 0. \tag{4.27}$$

Both equations (4.26) and (4.27) now show that $\operatorname{div} \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}$ satisfies (4.5), which completes step 4.
Thus, (4.12) is proved in the three-dimensional setting. The proof of (4.13) in the 2D case follows analogously. Since the operators in 2D are simply restrictions of the operators in 3D (cf. Remark 4.8), the biggest change in the proof is just to stop the argumentation at the level of the faces. □

4.5 Stability estimates in 1D

As we mentioned in the introductory chapter, we need to analyze two different norms for proving stability estimates for our interpolation operators. Here, the estimates in the weaker norms are obtained by duality arguments, and the proofs are, in a similar way to the definitions of the operators themselves, built up by spatial dimension. Without going into excessive details here, we start with the norm in 3D and use integration by parts to obtain two expressions: The 3D-term is handled by best approximation results, and for the boundary term, we use suitable 2D results, which are themselves based on an 1D result after an integration by parts argument.
This section will deal with a stability result in the one-dimensional case, which is needed for the proofs in higher dimensions. The 1D-case is rather simple and not really exciting in terms of exact sequences or commuting diagrams, thus we haven't included the interpolation operator $\widehat{\Pi}_p^{\mathrm{grad},1d}$ introduced here, which is just the restriction of the 2D-operator to one edge, in the definitions of Section 4.2.

Lemma 4.15. *Let $\widehat{e} = (-1,1)$ the reference element in 1D, and let $\widehat{\Pi}_p^{\mathrm{grad},1d} : H^1(\widehat{e}) \to \mathcal{P}_p(\widehat{e})$ be defined by*

$$((u - \widehat{\Pi}_p^{\mathrm{grad},1d} u)', v')_{L^2(\widehat{e})} = 0 \qquad \forall v \in \mathcal{P}_p(\widehat{e}) \cap H_0^1(\widehat{e}), \tag{4.28a}$$

$$u(\pm 1) = (\widehat{\Pi}_p^{\mathrm{grad},1d} u)(\pm 1). \tag{4.28b}$$

Then, for every $s \geq 0$ there is a constant $C_s > 0$ such that

$$\|u - \widehat{\Pi}_p^{\mathrm{grad},1d} u\|_{H^{1-s}(\widehat{e})} \leq C_s p^{-s} \inf_{v \in \mathcal{P}_p(\widehat{e})} \|u - v\|_{H^1(\widehat{e})}, \qquad \text{if } s \in [0,1], \tag{4.29a}$$

$$\|u - \widehat{\Pi}_p^{\mathrm{grad},1d} u\|_{\widetilde{H}^{1-s}(\widehat{e})} \leq C_s p^{-s} \inf_{v \in \mathcal{P}_p(\widehat{e})} \|u - v\|_{H^1(\widehat{e})}, \qquad \text{if } s \geq 1, \tag{4.29b}$$

$$\|(u - \widehat{\Pi}_p^{\mathrm{grad},1d} u)'\|_{\widetilde{H}^{-s}(\widehat{e})} \leq C_s p^{-s} \inf_{v \in \mathcal{P}_p(\widehat{e})} \|u - v\|_{H^1(\widehat{e})}, \qquad \text{if } s \geq 0. \tag{4.29c}$$

Proof. We start with the case $s = 0$. For $u \in H^1(\widehat{e})$, let $\mathcal{L}u \in \mathcal{P}_1(\widehat{e})$ interpolate u in the endpoints ± 1. Sobolev's embedding theorem then gives $\|\mathcal{L}u\|_{H^1(\widehat{e})} \lesssim \|u\|_{H^1(\widehat{e})}$. Since $u - \mathcal{L}u \in H_0^1(\widehat{e})$, there holds $\widehat{\Pi}_p^{\mathrm{grad},1d}(u - \mathcal{L}u) \in H_0^1(\widehat{e})$ by (4.28b), and (4.28a) yields

$$|\widehat{\Pi}_p^{\mathrm{grad},1d}(u - \mathcal{L}u)|_{H^1(\widehat{e})}^2 = ((u - \mathcal{L}u)', (\widehat{\Pi}_p^{\mathrm{grad},1d}(u - \mathcal{L}u))')_{L^2(\widehat{e})},$$

which implies

$$|\widehat{\Pi}_p^{\mathrm{grad},1d}(u - \mathcal{L}u)|_{H^1(\widehat{e})} \leq |u - \mathcal{L}u|_{H^1(\widehat{e})}.$$

We then have

$$\|\widehat{\Pi}_p^{\mathrm{grad},1d}(u - \mathcal{L}u)\|_{H^1(\widehat{e})} \lesssim \|u - \mathcal{L}u\|_{H^1(\widehat{e})}$$

with Poincaré's inequality. It follows

$$\begin{aligned} \|\widehat{\Pi}_p^{\mathrm{grad},1d} u\|_{H^1(\widehat{e})} &= \|\widehat{\Pi}_p^{\mathrm{grad},1d}(u - \mathcal{L}u) + \widehat{\Pi}_p^{\mathrm{grad},1d} \mathcal{L}u\|_{H^1(\widehat{e})} \\ &\leq \|\widehat{\Pi}_p^{\mathrm{grad},1d}(u - \mathcal{L}u)\|_{H^1(\widehat{e})} + \|\mathcal{L}u\|_{H^1(\widehat{e})} \lesssim \|u\|_{H^1(\widehat{e})}. \end{aligned} \tag{4.30}$$

The projection property of $\widehat{\Pi}_p^{\mathrm{grad},1d}$, which follows immediately from the definition, shows for any $v \in \mathcal{P}_p(\widehat{e})$ that $u - \widehat{\Pi}_p^{\mathrm{grad},1d} u = u - v - \widehat{\Pi}_p^{\mathrm{grad},1d}(u - v)$. Together with (4.30) follows the estimate (4.29a) for $s = 0$.
We now prove (4.29b) for $s \geq 1$ by a duality argument. With the notation $\widetilde{e} := u - \widehat{\Pi}_p^{\mathrm{grad},1d} u$ and $t = -(1-s) \geq 0$, we want to estimate

$$\|\widetilde{e}\|_{\widetilde{H}^{-t}(\widehat{e})} = \sup_{v \in H^t(\widehat{e})} \frac{(\widetilde{e}, v)_{L^2(\widehat{e})}}{\|v\|_{H^t(\widehat{e})}}.$$

For every $v \in H^t(\widehat{e})$, there exists a unique solution $z \in H^{t+2}(\widehat{e}) \cap H_0^1(\widehat{e})$ of the problem

$$\begin{aligned} -z'' &= v \text{ in } \widehat{e} \\ z &= 0 \text{ on } \partial\widehat{e}, \end{aligned} \tag{4.31}$$

which satisfies the regularity estimate $\|z\|_{H^{t+2}(\widehat{e})} \lesssim \|v\|_{H^t(\widehat{e})}$. Using integration by parts, condition (4.28a) and the estimate (4.29a) for $s = 0$ already shown above, we obtain

$$\begin{aligned}
|(\widetilde{e}, v)_{L^2(\widehat{e})}| &= |(\widetilde{e}', z')_{L^2(\widehat{e})}| \overset{(4.28a)}{\leq} \|\widetilde{e}'\|_{L^2(\widehat{e})} \inf_{\pi \in \mathcal{P}_p(\widehat{e}) \cap H_0^1(\widehat{e})} \|z' - \pi'\|_{L^2(\widehat{e})} \\
&\lesssim \|\widetilde{e}'\|_{L^2(\widehat{e})} p^{-(t+1)} \|z\|_{H^{t+2}(\widehat{e})} \overset{(4.29a) \text{ with } s=0}{\lesssim} p^{-(t+1)} \inf_{v \in \mathcal{P}_p(\widehat{e})} \|u - v\|_{H^1(\widehat{e})} \|v\|_{H^t(\widehat{e})},
\end{aligned}$$

cf. [57, Thm. 3.17]. This implies (4.29b) for $s \geq 1$. We mention that $\|\cdot\|_{\widetilde{H}^0(\widehat{e})} = \|\cdot\|_{L^2(\widehat{e})} = \|\cdot\|_{H^0(\widehat{e})}$, thus the intermediate cases $s \in (0,1)$ follow by interpolation.
Finally, (4.29c) is shown by similar duality arguments. Here, we have to estimate the norm

$$\|\widetilde{e}'\|_{\widetilde{H}^{-s}(\widehat{e})} = \sup_{v \in H^s(\widehat{e})} \frac{(\widetilde{e}', v)_{L^2(\widehat{e})}}{\|v\|_{H^s(\widehat{e})}}.$$

We write the scalar product as

$$(\widetilde{e}', v)_{L^2(\widehat{e})} = (\widetilde{e}', v - \overline{v})_{L^2(\widehat{e})} + \underbrace{(\widetilde{e}', \overline{v})_{L^2(\widehat{e})}}_{=\overline{v}(\widetilde{e}(1) - \widetilde{e}(-1)) = 0},$$

where $\overline{v} = (\int_{\widehat{e}} v)/|\widehat{e}|$ denotes the average. Posing the dual problem

$$\begin{aligned}
-z'' &= v - \overline{v} \text{ in } \widehat{e} \\
z' &= 0 \text{ on } \partial\widehat{e}
\end{aligned}$$

for $v \in H^s(\widehat{e})$, we obtain

$$\begin{aligned}
|(\widetilde{e}', v)_{L^2(\widehat{e})}| &= |(\widetilde{e}', (z')')_{L^2(\widehat{e})}| \leq \|\widetilde{e}'\|_{L^2(\widehat{e})} \inf_{\pi \in \mathcal{P}_p(\widehat{e}) \cap H_0^1(\widehat{e})} \|(z')' - \pi'\|_{L^2(\widehat{e})} \\
&\lesssim \|\widetilde{e}'\|_{L^2(\widehat{e})} p^{-s} \|z\|_{H^{s+2}(\widehat{e})} \lesssim p^{-s} \inf_{v \in \mathcal{P}_p(\widehat{e})} \|u - v\|_{H^1(\widehat{e})} \|v\|_{H^s(\widehat{e})}.
\end{aligned}$$

□

4.6 Stability estimates in 2D

After the definition of additional projection operators from [26, Thm. 4.2], we proceed with the stability estimates for the operators $\widehat{\Pi}_{p+1}^{\text{grad},2d}$ and $\widehat{\Pi}_p^{\text{curl},2d}$. In this section, we also make frequent use of the right inverses and the Helmholtz-like decompositions introduced in Section 2.5.

Lemma 4.16 ([26, Thm. 4.2]). *Let $P^{\text{grad},2d} u \in W_{p+1}(\widehat{f})$ be defined by*

$$(\nabla(u - P^{\text{grad},2d} u), \nabla v)_{L^2(\widehat{f})} = 0 \qquad \forall v \in W_{p+1}(\widehat{f}), \tag{4.32a}$$

$$(u - P^{\text{grad},2d} u, 1)_{L^2(\widehat{f})} = 0. \tag{4.32b}$$

Then, for $r > 1$, there holds

$$\|u - P^{\text{grad},2d} u\|_{H^1(\widehat{f})} \leq C_r p^{-(r-1)} \|u\|_{H^r(\widehat{f})}.$$

Lemma 4.17 ([26, Thm. 4.2]). *Let* $P^{\mathrm{curl},2d}\mathbf{u} \in \mathbf{Q}_p(\widehat{f})$ *be defined by*

$$(\operatorname{curl}(\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}), \operatorname{curl}\mathbf{v})_{L^2(\widehat{f})} = 0 \qquad \forall \mathbf{v} \in \mathbf{Q}_p(\widehat{f}), \tag{4.33a}$$

$$(\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}, \nabla v)_{L^2(\widehat{f})} = 0 \qquad \forall v \in W_{p+1}(\widehat{f}). \tag{4.33b}$$

Then, for $r > 0$*, there holds*

$$\|\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} \leq C_r p^{-r} \|\mathbf{u}\|_{\mathbf{H}^r(\widehat{f},\mathrm{curl})}.$$

4.6.1 Stability of $\widehat{\Pi}^{\mathrm{grad},2d}_{p+1}$

The following theorem deals with the stability of the gradient interpolation operator in two dimensions. In parts, the proof uses the similar 1D-result Lemma 4.15.
For $s = 0$, a similar result can be found in [10, Thm. 4.1].

Lemma 4.18. *Let* $s \in [0, \pi/\omega_{max})$*, where* ω_{max} *denotes the largest interior angle of* $\widehat{f}$*. Then there exists* $C_s > 0$ *such that for* $u \in H^{3/2}(\widehat{f})$*, the following stability estimates hold.*

$$\|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{H^{1-s}(\widehat{f})} \leq C_s p^{-(1/2+s)} \inf_{v \in W_{p+1}(\widehat{f})} \|u - v\|_{H^{3/2}(\widehat{f})} \quad \textit{if } s \in [0,1], \tag{4.34a}$$

$$\|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{\widetilde{H}^{1-s}(\widehat{f})} \leq C_s p^{-(1/2+s)} \inf_{v \in W_{p+1}(\widehat{f})} \|u - v\|_{H^{3/2}(\widehat{f})} \quad \textit{if } s \in [1, \pi/\omega_{max}), \tag{4.34b}$$

$$\|\nabla(u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u)\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f})} \leq C_s p^{-(1/2+s)} \inf_{v \in W_{p+1}(\widehat{f})} \|u - v\|_{H^{3/2}(\widehat{f})} \quad \textit{if } s \in [0, \pi/\omega_{max}). \tag{4.34c}$$

Proof. By the projection property of $\widehat{\Pi}^{\mathrm{grad},2d}_{p+1}$ it is sufficient to prove the estimates (4.34a), (4.34b) and (4.34c) only for the special case $v = 0$ in the infimum, cf. the proof of Lemma 4.15, where this fact is demonstrated for the sake of completeness.

Step 1: We show (4.34a) for the case $s = 0$.
From $u \in H^{3/2}(\widehat{f})$, we obtain $u \in H^1(e)$ for each edge $e \in \mathcal{E}(\widehat{f})$ by the trace theorem, together with the bound $\|u\|_{H^1(e)} \lesssim \|u\|_{H^{3/2}(\widehat{f})}$, cf. Lemma 4.12. On every edge $e \in \mathcal{E}(\widehat{f})$, we can now use the 1D result, thus Lemma 4.15 yields

$$\|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{H^{1-s}(e)} \leq C p^{-s} \|u\|_{H^{3/2}(\widehat{f})}, \qquad s \in [0,1].$$

Note that by definition, $\widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u$ is piecewise polynomial and continuous on $\partial\widehat{f}$, hence we get in particular for $s = 0$ and $s = 1$ the estimates

$$\|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1} u\|_{H^{1-s}(\partial\widehat{f})} \leq C p^{-s} \|u\|_{H^{3/2}(\widehat{f})}. \tag{4.35}$$

The bounds for $s \in (0,1)$ follow as usual by interpolation.

We now observe that $P^{\mathrm{grad},2d}u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1}u$ is discrete harmonic, i.e.,

$$(\nabla \underbrace{(P^{\mathrm{grad},2d}u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1}u)}_{=:\delta_p \in W_{p+1}(\widehat{f})}, \nabla v)_{L^2(\widehat{f})} = 0 \qquad \forall v \in \mathring{W}_{p+1}(\widehat{f}), \tag{4.36}$$

cf. (4.6c) and (4.32a). We now use the lifting $\mathcal{L}^{\mathrm{grad},2d} : H^{1/2}(\partial\widehat{f}) \to H^1(\widehat{f})$ of [5, Thm. 7.4]. It bears the properties that for a piecewise polynomial function w on the boundary, $\mathcal{L}^{\mathrm{grad},2d}w$ is a polynomial of same degree, and moreover, it is a continuous lifting, i.e.

$$\|\mathcal{L}^{\mathrm{grad},2d}w\|_{H^1(\widehat{f})} \lesssim \|w\|_{H^{1/2}(\partial\widehat{f})}.$$

It obviously holds $\delta_p - \mathcal{L}^{\mathrm{grad},2d}\delta_p \in \mathring{W}_{p+1}(\widehat{f})$, thus

$$\left(\nabla\delta_p, \nabla(\delta_p - \mathcal{L}^{\mathrm{grad},2d}\delta_p)\right)_{L^2(\widehat{f})} = 0$$

by (4.36), which implies

$$|\delta_p|^2_{H^1(\widehat{f})} = (\nabla\delta_p, \nabla\delta_p)_{L^2(\widehat{f})} = (\nabla\delta_p, \nabla(\mathcal{L}^{\mathrm{grad},2d}\delta_p))_{L^2(\widehat{f})} \lesssim |\delta_p|_{H^1(\widehat{f})}\|\delta_p\|_{H^{1/2}(\partial\widehat{f})}. \tag{4.37}$$

Hence, we obtain with Lemma 4.16, (4.35) and (4.37)

$$\begin{aligned}
|u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1}u|_{H^1(\widehat{f})} &\leq |u - P^{\mathrm{grad},2d}u|_{H^1(\widehat{f})} + |P^{\mathrm{grad},2d}u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1}u|_{H^1(\widehat{f})} \\
&\overset{\text{Lemma 4.16, (4.37)}}{\lesssim} p^{-1/2}\|u\|_{H^{3/2}(\widehat{f})} + \|P^{\mathrm{grad},2d}u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1}u\|_{H^{1/2}(\partial\widehat{f})} \\
&\overset{(4.35)}{\lesssim} p^{-1/2}\|u\|_{H^{3/2}(\widehat{f})} + \|u - P^{\mathrm{grad},2d}u\|_{H^1(\widehat{f})} \\
&\overset{\text{Lemma 4.16}}{\lesssim} p^{-1/2}\|u\|_{H^{3/2}(\widehat{f})},
\end{aligned} \tag{4.38}$$

which proves (4.34a) for the case $s = 0$.

Step 2: We show (4.34b) for the case $s \in [1, \pi/\omega_{max})$ by a duality argument.
With the notation $\widetilde{e} = u - \widehat{\Pi}^{\mathrm{grad},2d}_{p+1}u$ and $t = -(1-s)$, we want to estimate the norm

$$\|\widetilde{e}\|_{\widetilde{H}^{-t}(\widehat{f})} = \sup_{v \in H^t(\widehat{f})} \frac{(\widetilde{e}, v)_{L^2(\widehat{f})}}{\|v\|_{H^t(\widehat{f})}}. \tag{4.39}$$

For every $v \in H^t(\widehat{f})$, there exists a solution $z \in H^{t+2}(\widehat{f}) \cap H^1_0(\widehat{f})$ of the problem

$$\begin{aligned}
-\Delta z &= v \quad \text{in } \widehat{f}, \\
z &= 0 \quad \text{on } \partial\widehat{f},
\end{aligned}$$

with the estimate $\|z\|_{H^{t+2}(\widehat{f})} \leq C\|v\|_{H^t(\widehat{f})}$, cf. Corollary 3.43. We now proceed by integration by parts to get

$$(\widetilde{e}, v)_{L^2(\widehat{f})} = \int_{\widehat{f}} \nabla\widetilde{e} \cdot \nabla z - \int_{\partial\widehat{f}} \partial_n z \widetilde{e}. \tag{4.40}$$

We now estimate the first term in (4.40), using the orthogonality properties satisfied by $\widetilde{e}$, Lemma 2.23 and (4.38), and get

$$\begin{aligned} \left|\int_{\widehat{f}} \nabla z \cdot \nabla\widetilde{e}\right| &\leq \inf_{\pi\in\mathcal{P}_p\cap H_0^1(\widehat{f})} \|z-\pi\|_{H^1(\widehat{f})}\|\nabla\widetilde{e}\|_{L^2(\widehat{f})} \overset{\text{Lemma 2.23}}{\lesssim} p^{-(t+1)}\|z\|_{H^{t+2}(\widehat{f})}\|\nabla\widetilde{e}\|_{L^2(\widehat{f})} \\ &\lesssim p^{-(t+1)}\|\nabla\widetilde{e}\|_{L^2(\widehat{f})}\|v\|_{H^t(\widehat{f})} \overset{(4.38)}{\lesssim} p^{-(1/2+s)}\|u\|_{H^{3/2}(\widehat{f})}\|v\|_{H^t(\widehat{f})}. \end{aligned} \tag{4.41}$$

Note that we have already established an 1D stability result in Lemma 4.15, which we now apply to estimate the second term in (4.40). Together with trace theorems, we obtain for each edge $e \in \mathcal{E}(\widehat{f})$

$$\begin{aligned} |(\partial_n z, \widetilde{e})_{L^2(e)}| &\lesssim \|\widetilde{e}\|_{\widetilde{H}^{-(t+1/2)}(e)}\|\partial_n z\|_{H^{t+1/2}(e)} \overset{\text{Lemma 4.15}}{\lesssim} p^{-(3/2+t)}\|u\|_{H^1(e)}\|z\|_{H^{t+2}(\widehat{f})} \\ &\lesssim p^{-(1/2+s)}\|u\|_{H^{3/2}(\widehat{f})}\|v\|_{H^t(\widehat{f})}. \end{aligned} \tag{4.42}$$

Inserting (4.41) and (4.42) in (4.40) and (4.39) implies the desired estimate (4.34b) in the case $s \in [1, \pi/\omega_{max})$.

Step 3: We show (4.34a) for $s \in (0, 1]$.
Note that (4.34a) and (4.34b) coincide for $s = 1$, thus the result follows by interpolation between $s = 0$ and $s = 1$, cf. steps 1 and 2.

Step 4: We show the estimate (4.34c) for $s \in [1, \pi/\omega_{max})$ by a duality argument.
With the notation $\widetilde{e} := u - \widehat{\Pi}^{\text{grad},2d}_{p+1} u$, we need an estimate for the norm

$$\|\nabla\widetilde{e}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f})} = \sup_{\mathbf{v}\in\mathbf{H}^s(\widehat{f})} \frac{(\nabla\widetilde{e}, \mathbf{v})_{L^2(\widehat{f})}}{\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})}}. \tag{4.43}$$

As shown in Lemma 2.26, any $\mathbf{v} \in \mathbf{H}^s(\widehat{f})$ satisfies the decomposition $\mathbf{v} = \nabla\varphi + \mathbf{curl}\, z$ with $\varphi \in H^{s+1}(\widehat{f}) \cap H_0^1(\widehat{f})$, $z \in H^{s+1}(\widehat{f})$ and the corresponding norm estimates. We then proceed by integration by parts (cf. (2.25)) to get

$$(\nabla\widetilde{e}, \mathbf{v})_{L^2(\widehat{f})} = (\nabla\widetilde{e}, \nabla\varphi)_{L^2(\widehat{f})} + (\nabla\widetilde{e}, \mathbf{curl}\, z)_{L^2(\widehat{f})} = (\nabla\widetilde{e}, \nabla\varphi)_{L^2(\widehat{f})} + (\mathbf{t} \cdot \nabla\widetilde{e}, z)_{L^2(\partial\widehat{f})}.$$

The rest of the proof follows the lines of step 2. For the first term, we obtain with

Lemma 2.23 and (4.38)

$$\begin{aligned}\left|(\nabla\widetilde{e},\nabla\varphi)_{L^2(\widehat{f})}\right| &\lesssim \|\nabla\widetilde{e}\|_{L^2(\widehat{f})} \inf_{\pi\in\mathring{W}_{p+1}(\widehat{f})}\|\varphi-\pi\|_{H^1(\widehat{f})}\\ &\lesssim p^{-1/2}\|u\|_{H^{3/2}(\widehat{f})}p^{-s}\|\varphi\|_{H^{s+1}(\widehat{f})}\\ &\lesssim p^{-(s+1/2)}\|u\|_{H^{3/2}(\widehat{f})}\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})},\end{aligned}$$

and the estimate of the second term follows with Lemma 4.15

$$\begin{aligned}\left|(\mathbf{t}\cdot\nabla\widetilde{e},z)_{L^2(e)}\right| &= \left|(\nabla_e\widetilde{e},z)_{L^2(e)}\right| \lesssim \|\nabla_e\widetilde{e}\|_{\widetilde{H}^{-(s+1/2)}(e)}\|z\|_{H^{s+1/2}(e)}\\ &\lesssim p^{-(s+1/2)}\|u\|_{H^1(e)}\|z\|_{H^{s+1/2}(e)} \lesssim p^{-(s+1/2)}\|u\|_{H^{3/2}(\widehat{f})}\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})},\end{aligned}$$

since $z\in H^{s+1}(\widehat{f})$ implies $z\in H^{s+1/2}(e)$ for each edge $e\in\mathcal{E}(\widehat{f})$. Inserting the last two estimates in (4.43) gives us (4.34c) for $s\in[1,\pi/\omega_{max})$.

Step 5: The estimate (4.34c) for $s\in(0,1)$ follows by interpolation between $s=0$ and $s=1$. □

4.6.2 Stability of $\widehat{\Pi}_p^{\mathrm{curl},2d}$

In this section, we deal with the stability of the curl-interpolation operator in two dimensions. The ideas are often similar to those used in the proof of Lemma 4.18, however, the proofs are technically more difficult. The concepts of the proofs here rely on [26] in parts. We start with two results about estimations of negative Sobolev norms.

Lemma 4.19. *Let $\mathbf{E}\in\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})$ satisfy the orthogonality conditions*

$$(\mathrm{curl}\,\mathbf{E},\mathrm{curl}\,\mathbf{v})_{L^2(\widehat{f})}=0 \qquad \forall\mathbf{v}\in\mathring{\mathbf{Q}}_p(\widehat{f}), \tag{4.44a}$$

$$(\mathbf{E},\nabla\varphi)_{L^2(\widehat{f})}=0 \qquad \forall\varphi\in\mathring{W}_{p+1}(\widehat{f}), \tag{4.44b}$$

$$(\mathbf{E}\cdot\mathbf{t}_e,\nabla_e\varphi)_{L^2(e)}=0 \qquad \forall\varphi\in\mathring{W}_{p+1}(e)\quad\forall e\in\mathcal{E}(\widehat{f}), \tag{4.44c}$$

$$(\mathbf{E}\cdot\mathbf{t}_e,1)_{L^2(e)}=0 \qquad \forall e\in\mathcal{E}(\widehat{f}). \tag{4.44d}$$

Then, for $s\in[0,\pi/\omega_{max})$, where ω_{max} denotes the largest interior angle of $\widehat{f}$, there holds the estimate

$$\|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f})}\le C_s p^{-s}\|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}.$$

Proof. Step 1: Since $\mathbf{E}\in\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})$, Lemma 4.12 shows that $\mathbf{E}\cdot\mathbf{t}_e\in L^2(e)$ for each $e\in\mathcal{E}(\widehat{f})$, such that the conditions (4.44c) and (4.44d) are meaningful. Additionally, we have the estimates

$$\|\mathbf{E}\cdot\mathbf{t}_e\|_{L^2(e)}\lesssim\|\mathbf{E}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}.$$

Step 2: We assume $s \geq 1$. By Lemma 2.26, any $\mathbf{v} \in \mathbf{H}^s(\widehat{f})$ can be decomposed as

$$\mathbf{v} = \nabla\varphi + \mathbf{curl}\, z, \tag{4.45}$$

where $\varphi \in H^{s+1}(\widehat{f}) \cap H^1_0(\widehat{f})$ and $z \in H^{s+1}(\widehat{f})$ with $(z,1)_{L^2(\widehat{f})} = 0$, together with the norm estimate $\|\varphi\|_{H^{s+1}(\widehat{f})} + \|z\|_{H^{s+1}(\widehat{f})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})}$. Integration by parts (cf. (2.25)) then leads to

$$\begin{aligned}(\mathbf{E},\mathbf{v})_{L^2(\widehat{f})} &= (\mathbf{E},\nabla\varphi)_{L^2(\widehat{f})} + (\mathbf{E},\mathbf{curl}\, z)_{L^2(\widehat{f})} \\ &= (\mathbf{E},\nabla\varphi)_{L^2(\widehat{f})} + (\operatorname{curl}\mathbf{E}, z)_{L^2(\widehat{f})} - \int_{\partial\widehat{f}} z\mathbf{E}\cdot\mathbf{t}.\end{aligned} \tag{4.46}$$

Step 3: We estimate the first term in (4.46).
With the orthogonality property (4.44b) and Lemma 2.23, we obtain

$$\begin{aligned}\left|(\mathbf{E},\nabla\varphi)_{L^2(\widehat{f})}\right| &= \left|\inf_{w\in\mathring{W}_{p+1}(\widehat{f})}(\mathbf{E},\nabla(\varphi-w))_{L^2(\widehat{f})}\right| \lesssim p^{-s}\|\varphi\|_{H^{s+1}(\widehat{f})}\|\mathbf{E}\|_{L^2(\widehat{f})} \\ &\lesssim p^{-s}\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})}\|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\operatorname{curl})}.\end{aligned}$$

Step 4: We estimate the second term in (4.46).
Here, we pose the Neumann problem

$$\begin{aligned}-\Delta\widetilde{z} &= z \quad \text{in } \widehat{f}, \\ \partial_n\widetilde{z} &= 0 \quad \text{on } \partial\widehat{f}.\end{aligned}$$

Since the function z has the property $\int_{\widehat{f}} z = 0$, the compatibility condition is satisfied, thus we get a solution $\widetilde{z}$ with the estimate $\|\widetilde{z}\|_{H^{s+1}(\widehat{f})} \lesssim \|z\|_{H^{s-1}(\widehat{f})}$, cf. Proposition 3.48. We then define the function $\mathbf{z} := \mathbf{curl}\,\widetilde{z}$, which has the properties $\operatorname{curl}\mathbf{z} = z$ and $\mathbf{z}\cdot\mathbf{t} = 0$. It follows $\|\mathbf{z}\|_{\mathbf{H}^s(\widehat{f},\operatorname{curl})} \lesssim \|z\|_{H^{s+1}(\widehat{f})}$. By the orthogonality property (4.44a), we have

$$(\operatorname{curl}\mathbf{E}, z)_{L^2(\widehat{f})} = (\operatorname{curl}\mathbf{E}, \operatorname{curl}\mathbf{z})_{L^2(\widehat{f})} \overset{(4.44a)}{=} \inf_{\mathbf{w}\in\mathring{\mathbf{Q}}_p(\widehat{f})}(\operatorname{curl}\mathbf{E}, \operatorname{curl}(\mathbf{z}-\mathbf{w}))_{L^2(\widehat{f})}. \tag{4.47}$$

Now note that there exists a continuous, polynomial-preserving lifting

$$\boldsymbol{\mathcal{L}}^{\operatorname{curl},2d} : H^{-1/2}(\partial\widehat{f}) \to \mathbf{H}(\widehat{f},\operatorname{curl})$$

that is in p uniformly bounded, cf. [2] and [26, eq. (164)]. Hence, it follows

$$\boldsymbol{\mathcal{L}}^{\operatorname{curl},2d}(P^{\operatorname{curl},2d}\mathbf{z}\cdot\mathbf{t}) - P^{\operatorname{curl},2d}\mathbf{z} \in \mathring{\mathbf{Q}}_p(\widehat{f}),$$

which implies that this function can be used as $\mathbf{w}$ in the infimum in (4.47). Since

$$\begin{aligned}\|\boldsymbol{\mathcal{L}}^{\operatorname{curl},2d}(P^{\operatorname{curl},2d}\mathbf{z}\cdot\mathbf{t})\|_{\mathbf{H}(\widehat{f},\operatorname{curl})} &\lesssim \|P^{\operatorname{curl},2d}\mathbf{z}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})} = \|(\mathbf{z}-P^{\operatorname{curl},2d}\mathbf{z})\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})} \\ &\lesssim \|\mathbf{z}-P^{\operatorname{curl},2d}\mathbf{z}\|_{\mathbf{H}(\widehat{f},\operatorname{curl})}\end{aligned}$$

by the properties of the lifting operator and a trace inequality, cf. [26, eq. (154)], we get with Lemma 4.17 and Lemma 2.26

$$(\operatorname{curl}\mathbf{E}, z)_{L^2(\widehat{f})} \leq \|\operatorname{curl}\mathbf{E}\|_{L^2(\widehat{f})}\|\mathbf{z} - (P^{\mathrm{curl},2d}\mathbf{z} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(P^{\mathrm{curl},2d}\mathbf{z}\cdot\mathbf{t}))\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} \tag{4.48}$$

$$\lesssim \|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}\left(\|\mathbf{z} - P^{\mathrm{curl},2d}\mathbf{z}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} + \|\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(P^{\mathrm{curl},2d}\mathbf{z}\cdot\mathbf{t})\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}\right) \tag{4.49}$$

$$\overset{\text{Lem. 4.17}}{\lesssim} \|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}p^{-s}\|\mathbf{z}\|_{\mathbf{H}^s(\widehat{f},\mathrm{curl})} \tag{4.50}$$

$$\overset{\text{Lem. 2.26}}{\lesssim} p^{-s}\|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})}. \tag{4.51}$$

Step 5: We estimate the third term in (4.46).
We use the orthogonalities (4.44c) and (4.44d). Since $z \in H^{s+1}(\widehat{f})$, we get $z \in C(\partial\widehat{f})$ and $z \in H^{s+1/2}(e)$ for each edge $e \in \mathcal{E}(\widehat{f})$, thus

$$\left|\int_{\partial\widehat{f}}\mathbf{E}\cdot\mathbf{t}z\right| = \inf_{w\in W_p(\widehat{f})}\left|\int_{\partial\widehat{f}}\mathbf{E}\cdot\mathbf{t}(z-w)\right| \lesssim \|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})}\inf_{w\in W_p(\widehat{f})}\|z-w\|_{H^{1/2}(\partial\widehat{f})}$$
$$\lesssim p^{-s}\|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})}\|z\|_{H^{s+1}(\widehat{f})} \lesssim p^{-s}\|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f})},$$

where we again used Lemma 2.23 and also the continuity of the tangential trace map,

$$\|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})},$$

cf. for example [26, eq. (154)].

Step 6: Since the case $s = 0$ is completely trivial and the case $s = 1$ has been shown in steps 2-5, the cases $s \in (0,1)$ now follow immediately by interpolation. □

Lemma 4.20. *Let* $\mathbf{E} \in \mathbf{H}^{1/2}(\widehat{f}, \mathrm{curl})$ *satisfy the orthogonality conditions* (4.44a) *and* (4.44d). *Then, for* $s \in [0, \pi/\omega_{max})$, *where* ω_{max} *denotes the largest interior angle of* $\widehat{f}$, *there holds*

$$\|\operatorname{curl}\mathbf{E}\|_{\widetilde{H}^{-s}(\widehat{f})} \leq C_s p^{-s}\|\operatorname{curl}\mathbf{E}\|_{L^2(\widehat{f})}.$$

Proof. Let $s \geq 1$, and let $v \in H^s(\widehat{f})$. With the notation $\overline{v} := (\int_{\widehat{f}} v)/|\widehat{f}| \in \mathbb{R}$ for its average, we obtain by integration by parts and the orthogonality (4.44d) that

$$(\operatorname{curl}\mathbf{E}, v)_{L^2(\widehat{f})} = (\operatorname{curl}\mathbf{E}, v - \overline{v})_{L^2(\widehat{f})} + \overline{v}(\mathbf{E}\cdot\mathbf{t}, 1)_{L^2(\partial\widehat{f})} = (\operatorname{curl}\mathbf{E}, v - \overline{v})_{L^2(\widehat{f})}.$$

Posing the Neumann problem

$$-\Delta\varphi = v - \overline{v} \quad \text{in } \widehat{f},$$
$$\partial_n\varphi = 0 \quad \text{on } \partial\widehat{f},$$

where the compatibility condition is obviously satisfied, Proposition 3.48 gives us the existence of a solution $\varphi \in H^{s+1}(\widehat{f})$ with the estimate

$$\|\varphi\|_{H^{s+1}(\widehat{f})} \lesssim \|v - \overline{v}\|_{H^{s-1}(\widehat{f})} \lesssim \|v\|_{H^s(\widehat{f})}.$$

Setting $\mathbf{v} := \mathbf{curl}\,\varphi$, we see that $\operatorname{curl}\mathbf{v} = -\Delta\varphi = v - \overline{v}$ in $\widehat{f}$ and $\mathbf{t}\cdot\mathbf{v} = -\partial_n\varphi = 0$ on $\partial\widehat{f}$. The orthogonality property (4.44a), Lemma 4.17 and again integration by parts now leads to

$$\begin{aligned}(\operatorname{curl}\mathbf{E}, v-\overline{v})_{L^2(\widehat{f})} &= (\operatorname{curl}\mathbf{E}, \operatorname{curl}\mathbf{v})_{L^2(\widehat{f})} = \inf_{\mathbf{w}\in\mathring{\mathbf{Q}}_p(\widehat{f})}(\operatorname{curl}\mathbf{E}, \operatorname{curl}(\mathbf{v}-\mathbf{w}))_{L^2(\widehat{f})} \\ &\lesssim p^{-s}\|\operatorname{curl}\mathbf{E}\|_{L^2(\widehat{f})}\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{f},\operatorname{curl})} \lesssim p^{-s}\|\operatorname{curl}\mathbf{E}\|_{L^2(\widehat{f})}\|v\|_{H^s(\widehat{f})},\end{aligned}$$

cf. the arguments with the lifting operator in (4.48). This concludes the proof for $s \geq 1$. The case $s = 0$ is again trivial, thus standard interpolation arguments give us the result for $s \in (0, 1)$. □

The stability result for the operator $\widehat{\Pi}_p^{\operatorname{curl},2d}$ is again built up by spatial dimension. The next lemma deals with stability on edges, before we can prove the desired stability result on $\widehat{f}$ (which is later on used for the analogous result on the three-dimensional tetrahedron $\widehat{K}$).

Lemma 4.21. *Let $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \operatorname{curl})$. Then there holds for each edge $e \in \mathcal{E}(\widehat{f})$ and $s \geq 0$*

$$\|(\mathbf{u} - \widehat{\Pi}_p^{\operatorname{curl},2d}\mathbf{u})\cdot\mathbf{t}_e\|_{\widetilde{H}^{-s}(e)} \leq C_s p^{-s} \inf_{v\in\mathcal{P}_p(e)}\|\mathbf{u}\cdot\mathbf{t}_e - v\|_{L^2(e)}. \tag{4.52}$$

Proof. Note that Lemma 4.12 gives us $\mathbf{u}\cdot\mathbf{t}_e \in L^2(e)$, thus all expressions in the statement are indeed meaningful.
Let $\widetilde{e} := \left(\mathbf{u} - \widehat{\Pi}_p^{\operatorname{curl},2d}\mathbf{u}\right)\cdot\mathbf{t}_e$ be the error. We have already shown that the operator $\widehat{\Pi}_p^{\operatorname{curl},2d}$ is the L^2-projection on edges $e \in \mathcal{E}(\widehat{f})$, i.e.

$$(\mathbf{t}_e\cdot(\mathbf{u} - \widehat{\Pi}_p^{\operatorname{curl},2d}\mathbf{u}), w)_{L^2(e)} = 0 \qquad \forall w \in Q_p(e), \tag{4.53}$$

cf. the proof of Lemma 4.13. Thus, we can use $w = (\widehat{\Pi}_p^{\operatorname{curl},2d}\mathbf{u})\cdot\mathbf{t}_e$ as test function in (4.53). Simple estimates then imply (4.52) for $s = 0$.
For $s > 0$, (4.52) is seen by a standard duality argument. We have to estimate the norm

$$\|\widetilde{e}\|_{\widetilde{H}^{-s}(e)} = \sup_{v\in H^s(e)}\frac{(\widetilde{e}, v)_{L^2(e)}}{\|v\|_{H^s(e)}}.$$

Since $(\widetilde{e}, w)_{L^2(e)} = 0$ for all $w \in \mathcal{P}_p(e)$ by (4.53), we obtain

$$\begin{aligned}\left|(\widetilde{e}, v)_{L^2(e)}\right| &= \left|\inf_{w\in\mathcal{P}_p(e)}(\widetilde{e}, v-w)_{L^2(e)}\right| \leq \|\widetilde{e}\|_{L^2(e)}\inf_{w\in\mathcal{P}_p(e)}\|v-w\|_{L^2(e)} \\ &\lesssim p^{-s}\|\widetilde{e}\|_{L^2(e)}\|v\|_{H^s(e)}\end{aligned}$$

by Lemma 2.23. □

We now show the stability result on $\widehat{f}$. We mention that the case $s = 0$ can also be found in [10, Thm. 4.2].

Lemma 4.22. *Let $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \mathrm{curl})$. Then, for $s \in [0, \pi/\omega_{max})$, where ω_{max} denotes the largest interior angle of $\widehat{f}$, there holds*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \leq C_s p^{-(1/2+s)} \inf_{\mathbf{v}\in\mathbf{Q}_p(\widehat{f})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}.$$

Proof. Note that by the projection property of $\widehat{\Pi}_p^{\mathrm{curl},2d}$, it again suffices to only show the bound with $\mathbf{v} = 0$ in the infimum.
Let $P^{\mathrm{curl},2d}\mathbf{u}$ defined as in Lemma 4.17. Similar to Lemma 4.18, we define

$$\mathbf{E} := P^{\mathrm{curl},2d}\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u} \in \mathbf{Q}_p(\widehat{f}).$$

With the continuous, polynomial-preserving lifting

$$\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d} : H^{-1/2}(\partial\widehat{f}) \to \mathbf{H}(\widehat{f}, \mathrm{curl})$$

already introduced in Lemma 4.19, cf. [2] and [26, eq. (164)], it follows $\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t}) \in \mathring{\mathbf{Q}}_p(\widehat{f})$, and the orthogonalities (4.3d) and (4.33) imply

$$(\mathrm{curl}(\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})), \mathrm{curl}\,\mathbf{E})_{L^2(\widehat{f})} = 0. \tag{4.54}$$

Thus, we can estimate

$$\begin{aligned}\|\mathrm{curl}\,\mathbf{E}\|^2_{L^2(\widehat{f})} &\overset{(4.54)}{=} \Big(\mathrm{curl}\,\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t}), \mathrm{curl}\,\mathbf{E}\Big)_{L^2(\widehat{f})}\\ &\leq \|\mathrm{curl}\,\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{L^2(\widehat{f})}\|\mathrm{curl}\,\mathbf{E}\|_{L^2(\widehat{f})},\end{aligned}$$

and the continuity of the lifting operator $\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}$ yields the bound

$$\|\mathrm{curl}\,\mathbf{E}\|_{L^2(\widehat{f})} \lesssim \|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})}. \tag{4.55}$$

We now use the discrete Friedrichs inequality (Lemma 2.34, (ii)) to obtain

$$\begin{aligned}\|\mathbf{E}\|_{L^2(\widehat{f})} &\leq \|\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{L^2(\widehat{f})} + \|\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{L^2(\widehat{f})}\\ &\lesssim \|\mathrm{curl}(\mathbf{E} - \boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t}))\|_{L^2(\widehat{f})} + \|\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{L^2(\widehat{f})}\\ &\lesssim \|\mathrm{curl}\,\mathbf{E}\|_{L^2(\widehat{f})} + \|\boldsymbol{\mathcal{L}}^{\mathrm{curl},2d}(\mathbf{E}\cdot\mathbf{t})\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} \overset{(4.55)}{\lesssim} \|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})}.\end{aligned} \tag{4.56}$$

Since $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \mathrm{curl})$ implies $\mathbf{u}\cdot\mathbf{t}_e \in L^2(e)$ for each edge $e \in \mathcal{E}(\widehat{f})$ with the estimate $\|\mathbf{u}\cdot\mathbf{t}_e\|_{L^2(e)} \lesssim \|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}$ according to Lemma 4.12, we get with the help of Lemma 4.17 and Lemma 4.21

$$\begin{aligned}\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} &\lesssim \|\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} + \|\mathbf{E}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})}\\ &\overset{(4.55),(4.56)}{\lesssim} \|\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} + \|\mathbf{E}\cdot\mathbf{t}\|_{H^{-1/2}(\partial\widehat{f})}\\ &\lesssim \|\mathbf{u} - P^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} + \|((\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u})\cdot\mathbf{t})\|_{H^{-1/2}(\partial\widehat{f})}\\ &\overset{\text{Lem. 4.17, 4.21}}{\lesssim} p^{-1/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})},\end{aligned} \tag{4.57}$$

cf. also [45, Thm. 3.34, Thm. 3.40], where $\widetilde{H}^t(\Omega) = H^t(\Omega)$ for a Lipschitz domain Ω and $t \in [0, 1/2)$ is shown. The proof is now complete considering

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \lesssim p^{-s}\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\mathbf{H}(\widehat{f},\mathrm{curl})} \overset{(4.57)}{\lesssim} p^{-(1/2+s)}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}$$

by Lemma 4.19 and Lemma 4.20. □

If $\mathbf{u}$ is even more regular, and if its curl turns out to be a polynomial function, we get the following result.

Lemma 4.23. *Let $k \geq 1$ and $\mathbf{u} \in \mathbf{H}^k(\widehat{f})$ with $\operatorname{curl}\mathbf{u} \in \mathcal{P}_p(\widehat{f})$. Then, for $s \in [0, \pi/\omega_{max})$, where ω_{max} denotes the largest interior angle of $\widehat{f}$, there holds*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \leq C_{s,k}p^{-(k+s)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}. \tag{4.58}$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{f})}$.

Proof. We follow the lines of [39, Lemma 5.8]. Using the right inverses introduced in Section 2.5, we can decompose $\mathbf{u}$ as

$$\mathbf{u} = \nabla R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}\operatorname{curl}\mathbf{u}) + \mathbf{R}^{\mathrm{curl}}\operatorname{curl}\mathbf{u} =: \nabla\varphi + \mathbf{v}$$

with $\varphi \in H^{k+1}(\widehat{f})$ and $\mathbf{v} \in \mathbf{H}^k(\widehat{f})$ together with the estimate

$$\|\varphi\|_{H^{k+1}(\widehat{f})} + \|\mathbf{v}\|_{\mathbf{H}^k(\widehat{f})} \leq C\left(\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})} + \|\operatorname{curl}\mathbf{u}\|_{H^{k-1}(\widehat{f})}\right) \leq C\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}, \tag{4.59}$$

cf. Lemma 2.25. Since $\operatorname{curl}\mathbf{u} \in \mathcal{P}_p(\widehat{f})$, Lemma 2.24, (iv) implies $\mathbf{v} = \mathbf{R}^{\mathrm{curl}}\operatorname{curl}\mathbf{u} \in \mathbf{Q}_p(\widehat{f})$. It immediately follows

$$\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{v} = 0 \tag{4.60}$$

due to the fact that the operator $\widehat{\Pi}_p^{\mathrm{curl},2d}$ is a projection. Hence, the commuting diagram property $\nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},2d} = \widehat{\Pi}_p^{\mathrm{curl},2d}\nabla$, cf. (4.13), and the estimate (4.34c) yields

$$\begin{aligned}\|(\mathrm{I} - \widehat{\Pi}_p^{\mathrm{curl},2d})\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} &= \|(\mathrm{I} - \widehat{\Pi}_p^{\mathrm{curl},2d})\nabla\varphi + (\mathrm{I} - \widehat{\Pi}_p^{\mathrm{curl},2d})\mathbf{v}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})}\\ &\overset{(4.60)}{=} \|\nabla(\mathrm{I} - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d})\varphi\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f})} \overset{(4.34c)}{\lesssim} p^{-(k+s)}\|\varphi\|_{H^{k+1}(\widehat{f})},\end{aligned}$$

from which (4.58) follows by (4.59).
The additional claim that we can replace the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}$ with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{f})}$ is clear since the operator $\widehat{\Pi}_p^{\mathrm{curl},2d}$ reproduces polynomials of degree p. □

4.6.3 The main results in 2D

We now collect the important results from the previous subsections. The following theorem will then be our main result concerning the interpolation operators in two dimensions.

Theorem 4.24. *For a reference triangle $\widehat{f} \subset \mathbb{R}^2$, define $\widehat{s} := \frac{\pi}{\omega_{max}}$, where ω_{max} denotes the largest interior angle of $\widehat{f}$. Then there are constants $C_{s,k}$ depending only on s, k, and the choice of $\widehat{f}$ such that the following assertions hold:*

(i) The operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$, $\widehat{\Pi}_p^{\mathrm{curl},2d}$, $\widehat{\Pi}_p^{L^2}$ are well-defined, projections, and the diagram (4.13) commutes.

(ii) For all $\varphi \in H^{3/2}(\widehat{f})$ there holds

$$\begin{aligned}
\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d}\varphi\|_{H^{1-s}(\widehat{f})} &\leq C_{s,k}p^{-(1/2+s)} \inf_{v\in W_{p+1}(\widehat{f})} \|\varphi - v\|_{H^{3/2}(\widehat{f})}, && s \in [0,1],\\
\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d}\varphi\|_{\widetilde{H}^{1-s}(\widehat{f})} &\leq C_{s,k}p^{-(1/2+s)} \inf_{v\in W_{p+1}(\widehat{f})} \|\varphi - v\|_{H^{3/2}(\widehat{f})}, && s \in [1,\widehat{s}),\\
\|\nabla(\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d}\varphi)\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f})} &\leq C_{s,k}p^{-(1/2+s)} \inf_{v\in W_{p+1}(\widehat{f})} \|\varphi - v\|_{H^{3/2}(\widehat{f})}, && s \in [0,\widehat{s}).
\end{aligned}$$

(iii) For all $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{f}, \mathrm{curl})$ there holds

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \leq C_{s,k}p^{-(1/2+s)} \inf_{\mathbf{v}\in\mathbf{Q}_p(\widehat{f})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}, \qquad s \in [0,\widehat{s}).$$

(iv) For all $k \geq 1$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{f})$ with $\mathrm{curl}\,\mathbf{u} \in \mathcal{P}_p(\widehat{f})$ there holds

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} \leq C_{s,k}p^{-(k+s)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}, \qquad s \in [0,\widehat{s}). \tag{4.61}$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{f})}$.

Proof. For the subjects stated in (i), see Lemma 4.12, Lemma 4.13 and Theorem 4.14. Item (ii) is exactly Lemma 4.18, and the results in (iii) have been shown in Lemma 4.22. Finally, (iv) coincides with Lemma 4.23. □

If our given function is more regular, we even get better approximation properties in p. This observation is now stated in the following simple corollary.

Corollary 4.25. *Using the notation of Theorem 4.24, the following statements hold for $k \geq 1/2$:*

$$\begin{aligned}
\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d}\varphi\|_{H^{1-s}(\widehat{f})} &\leq C_{s,k}p^{-(k+s)}\|\varphi\|_{H^{k+1}(\widehat{f})}, && s \in [0,1], && (4.62)\\
\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},2d}\varphi\|_{\widetilde{H}^{1-s}(\widehat{f})} &\leq C_{s,k}p^{-(k+s)}\|\varphi\|_{H^{k+1}(\widehat{f})}, && s \in [1,\widehat{s}), && (4.63)\\
\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},2d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} &\leq C_{s,k}p^{-(k+s)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f},\mathrm{curl})}, && s \in [0,\widehat{s}). && (4.64)
\end{aligned}$$

Proof. The estimates (4.62) and (4.63) follow directly from Theorem 4.24, (ii) together with the best approximation property of Lemma 2.23.
In order to show (4.64), we write

$$\mathbf{u} = \nabla\varphi + \mathbf{z}$$

with $\varphi \in H^{k+1}(\widehat{f})$, $\mathbf{z} \in \mathbf{H}^{k+1}(\widehat{f})$, together with the bounds $\|\varphi\|_{H^{k+1}(\widehat{f})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f},\mathbf{curl})}$ and $\|\mathbf{z}\|_{\mathbf{H}^{k+1}(\widehat{f})} \lesssim \|\operatorname{\mathbf{curl}}\mathbf{u}\|_{\mathbf{H}^k(\widehat{f})}$, cf. Lemma 2.25. Thus, Theorem 4.24, (iii) and Lemma 2.23 yield

$$\begin{aligned}\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{f},\mathrm{curl})} &\lesssim p^{-(1/2+s)} \inf_{\substack{v\in W_{p+1}(\widehat{f}),\\ \mathbf{q}\in\mathbf{Q}_p(\widehat{f})}} \|\nabla\varphi + \mathbf{z} - (\nabla v + \mathbf{q})\|_{\mathbf{H}^{1/2}(\widehat{f},\mathrm{curl})}\\ &\lesssim p^{-(1/2+s)}\left[\inf_{v\in W_{p+1}(\widehat{f})}\|\varphi - v\|_{H^{3/2}(\widehat{f})} + \inf_{\mathbf{q}\in\mathbf{Q}_p(\widehat{f})}\|\mathbf{z}-\mathbf{q}\|_{\mathbf{H}^{3/2}(\widehat{f})}\right]\\ &\overset{\text{Lem. 2.23}}{\lesssim} p^{-(1/2+s)-(k+1-3/2)}\left[\|\varphi\|_{H^{k+1}(\widehat{f})} + \|\mathbf{z}\|_{\mathbf{H}^{k+1}(\widehat{f})}\right] \lesssim p^{-(s+k)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{f},\mathrm{curl})}.\end{aligned}$$

□

4.7 Stability estimates in 3D

This section is about the stability estimates for the operators in 3D. The concepts are similar to the ideas in the 2D-case, which includes reducing the integrals to manifolds of lower dimensions by integration by parts arguments. Hence, the results of Theorem 4.24 are frequently applied.
Note that for functions $f \in H^s(\widehat{K})$ with s sufficiently large, it is well-known that solutions u of the Poisson problem satisfy $u \in H^2(\widehat{K})$ since $\widehat{K}$ is a convex domain. However, the exact value for the maximal regularity in a shift theorem for tetrahedra cannot be explicitly stated as easily as in the 2D-case in Chapter 3. For that reason, we will formulate the stability estimates only for negative norms $\widetilde{H}^{-s}(\widehat{K})$ with $s \in [0,1]$ on the left-hand side. We however mention that generalization to $s \in [0, \pi/\omega_{max} - 1)$ is easily possible with only slightly changed proofs, if the choice of the tetrahedron $\widehat{K}$ admits sufficient regularity for u.
We start with the introduction of best approximation results.

Lemma 4.26 ([26, Thm. 5.2])**.** *Let $P^{\mathrm{grad},3d}u \in W_{p+1}(\widehat{K})$ be defined by*

$$(\nabla(u - P^{\mathrm{grad},3d}u), \nabla v)_{L^2(\widehat{K})} = 0 \qquad \forall v \in W_{p+1}(\widehat{K}), \tag{4.65a}$$

$$(u - P^{\mathrm{grad},3d}u, 1)_{L^2(\widehat{K})} = 0. \tag{4.65b}$$

Then, for $r > 1$, there holds

$$\|u - P^{\mathrm{grad},3d}u\|_{H^1(\widehat{K})} \leq C_r p^{-(r-1)}\|u\|_{H^r(\widehat{K})}.$$

Lemma 4.27 ([26, Thm. 5.2], [29]). *Let $P^{\mathrm{curl},3d}\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$ be defined by*

$$(\mathbf{curl}(\mathbf{u} - P^{\mathrm{curl},3d}\mathbf{u}), \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathbf{Q}_p(\widehat{K}), \tag{4.66a}$$

$$(\mathbf{u} - P^{\mathrm{curl},3d}\mathbf{u}, \nabla v)_{L^2(\widehat{K})} = 0 \qquad \forall v \in W_{p+1}(\widehat{K}). \tag{4.66b}$$

Then, for $r > 0$, there holds

$$\|\mathbf{u} - P^{\mathrm{curl},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \leq C_r p^{-r} \|\mathbf{u}\|_{\mathbf{H}^r(\widehat{K},\mathbf{curl})}.$$

Lemma 4.28 ([26, Thm. 5.2]). *Let $P^{\mathrm{div},3d}\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ be defined by*

$$(\mathrm{div}(\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u}), \mathrm{div}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_p(\widehat{K}), \tag{4.67a}$$

$$(\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u}, \mathrm{div}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathbf{Q}_p(\widehat{K}). \tag{4.67b}$$

Then, for $r > 0$, there holds

$$\|\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \leq C_r p^{-r} \|\mathbf{u}\|_{\mathbf{H}^r(\widehat{K},\mathrm{div})}.$$

4.7.1 Stability of $\widehat{\Pi}^{\mathrm{grad},3d}_{p+1}$

This section is devoted to the analogous result of Lemma 4.18.

Proposition 4.29. *Let $\omega \in (0, \pi)$, and assume that all interior angles of all faces of $\widehat{K}$ are smaller than ω. Then, for every $s \in [0, \min(\pi/\omega - 1/2, 1)]$, there exists $C_s > 0$ such that for $u \in H^2(\widehat{K})$, the following stability estimates hold.*

$$\|u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u\|_{H^{1-s}(\widehat{K})} \leq C_s p^{-(1+s)} \inf_{v \in W_{p+1}(\widehat{K})} \|u - v\|_{H^2(\widehat{K})}, \tag{4.68a}$$

$$\|\nabla(u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u)\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K})} \leq C_s p^{-(1+s)} \inf_{v \in W_{p+1}(\widehat{K})} \|u - v\|_{H^2(\widehat{K})}. \tag{4.68b}$$

Proof. The proof follows the lines of Lemma 4.18. Again, by the projection property of $\widehat{\Pi}^{\mathrm{grad},3d}_{p+1}$, it is sufficient to show the estimates (4.68a) and (4.68b) with $v = 0$ in the infimum.

Step 1: We show (4.68a) for $s = 0$.
From $u \in H^2(\widehat{K})$, we obtain $u|_f \in H^{3/2}(f)$ for every face $f \in \mathcal{F}(\widehat{K})$ by the trace theorem, with the bound $\|u\|_{H^{3/2}(f)} \lesssim \|u\|_{H^2(\widehat{K})}$. Theorem 4.24, (ii) then shows for every face $f \in \mathcal{F}(\widehat{K})$ and $s \in [0, 1]$ that

$$\|u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u\|_{H^{1-s}(f)} \leq C p^{-(1/2+s)} \|u\|_{H^2(\widehat{K})}.$$

Now note that $u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u$ is continuous on $\partial\widehat{K}$, hence we get in particular for $s = 0$ and $s = 1$

$$\|u - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1} u\|_{H^{1-s}(\partial\widehat{K})} \leq C p^{-(1/2+s)} \|u\|_{H^2(\widehat{K})}, \tag{4.69}$$

from which the estimates for $s \in (0,1)$ follow by interpolation.
We now observe that $P^{\text{grad},3d}u - \widehat{\Pi}^{\text{grad},3d}_{p+1}u$ is discrete harmonic, i.e.

$$\Big(\nabla(P^{\text{grad},3d}u - \widehat{\Pi}^{\text{grad},3d}_{p+1}u), \nabla v\Big)_{L^2(\widehat{K})} = 0 \qquad \forall v \in \mathring{W}_{p+1}(\widehat{K}),$$

cf. (4.2d) and (4.65a). We now use a continuous, polynomial preserving lifting $\mathcal{L}^{\text{grad},3d}$: $H^{1/2}(\partial\widehat{K}) \to H^1(\widehat{K})$, cf. [52, Thm. 1] or [30, Thm. 6.1]. Imitating equation (4.37) from the 2D case, we get

$$|P^{\text{grad},3d}u - \widehat{\Pi}^{\text{grad},3d}_{p+1}u|_{H^1(\widehat{K})} \lesssim \|P^{\text{grad},3d}u - \widehat{\Pi}^{\text{grad},3d}_{p+1}u\|_{H^{1/2}(\partial\widehat{K})}, \tag{4.70}$$

thus we obtain with Lemma 4.26, (4.69) and (4.70)

$$\begin{aligned} |u - \widehat{\Pi}^{\text{grad},3d}_{p+1}u|_{H^1(\widehat{K})} &\leq |u - P^{\text{grad},3d}u|_{H^1(\widehat{K})} + |P^{\text{grad},3d}u - \widehat{\Pi}^{\text{grad},3d}_{p+1}u|_{H^1(\widehat{K})} \\ &\overset{\text{Lemma 4.26,(4.70)}}{\lesssim} p^{-1}\|u\|_{H^2(\widehat{K})} + \|P^{\text{grad},3d}u - \widehat{\Pi}^{\text{grad},3d}_{p+1}u\|_{H^{1/2}(\partial\widehat{K})} \\ &\overset{(4.69)}{\lesssim} p^{-1}\|u\|_{H^2(\widehat{K})} + \|u - P^{\text{grad},3d}u\|_{H^1(\widehat{K})} \\ &\overset{\text{Lemma 4.26}}{\lesssim} p^{-1}\|u\|_{H^2(\widehat{K})}, \end{aligned} \tag{4.71}$$

which proves (4.68a) for $s = 0$.

Step 2: We show (4.68a) for the case $\omega \leq 2\pi/3$, i.e. $s = 1$, by a duality argument.
We define $\widetilde{e} := u - \widehat{\Pi}^{\text{grad},3d}_{p+1}u$. Let then $z \in H^2(\widehat{K}) \cap H^1_0(\widehat{K})$ be given by the dual problem

$$\begin{aligned} -\Delta z &= \widetilde{e} \quad \text{on } \widehat{K}, \\ z &= 0 \quad \text{on } \partial\widehat{K}. \end{aligned}$$

Integration by parts implies

$$\|\widetilde{e}\|^2_{L^2(\widehat{K})} = \int_{\widehat{K}} \nabla z \cdot \nabla\widetilde{e} - \int_{\partial\widehat{K}} \partial_n z\widetilde{e}. \tag{4.72}$$

We now estimate the first term in (4.72), using the orthogonality properties satisfied by $\widetilde{e}$ Lemma 2.23 and (4.71), and obtain

$$|(\nabla z, \nabla\widetilde{e})_{L^2(\widehat{K})}| \leq \inf_{\pi \in \mathring{W}_{p+1}(\widehat{K})} \|z - \pi\|_{H^1(\widehat{K})}\|\nabla\widetilde{e}\|_{L^2(\widehat{K})} \overset{\text{Lemma 2.23}}{\lesssim} p^{-1}\|z\|_{H^2(\widehat{K})}\|\nabla\widetilde{e}\|_{L^2(\widehat{K})} \tag{4.73}$$

$$\lesssim p^{-1}\|\widetilde{e}\|_{L^2(\widehat{K})}\|\nabla\widetilde{e}\|_{L^2(\widehat{K})} \overset{(4.71)}{\lesssim} p^{-2}\|\widetilde{e}\|_{L^2(\widehat{K})}\|u\|_{H^2(\widehat{K})}. \tag{4.74}$$

For the second term in (4.72), we apply the 2D result Lemma 4.18 for each face $f \in \mathcal{F}(\widehat{K})$. Since $\omega_{max}(f) < 2\pi/3$ for each $f \in \mathcal{F}(\widehat{K})$ by assumption, Lemma 4.18 holds with $s = 3/2$. Hence,

$$\begin{aligned} |(\partial_n z, \widetilde{e})_{L^2(f)}| \leq \|\partial_n z\|_{H^{1/2}(f)} \|\widetilde{e}\|_{\widetilde{H}^{-1/2}(f)} &\overset{\text{Lemma 4.18}}{\lesssim} p^{-2} \|\partial_n z\|_{H^{1/2}(f)} \|u\|_{H^{3/2}(f)} \\ &\lesssim p^{-2} \|z\|_{H^2(\widehat{K})} \|u\|_{H^2(\widehat{K})} \lesssim p^{-2} \|\widetilde{e}\|_{L^2(\widehat{K})} \|u\|_{H^2(\widehat{K})}. \end{aligned} \tag{4.75}$$

The equations (4.72), (4.73) and (4.75) show estimate (4.68a) for $s = 1$.

Step 3: We show (4.68a) for the case $2\pi/3 < \omega < \pi$, i.e. $s = \pi/\omega - 1/2 < 1$, by a duality argument.

The argumentation is similar to step 2. We define $\widetilde{e} := u - \widehat{\Pi}_{p+1}^{\text{grad},3d} u$. We now need an estimate for the norm

$$\|\widetilde{e}\|_{H^{3/2-\pi/\omega}(\widehat{K})} \lesssim \|\widetilde{e}\|_{\widetilde{H}^{3/2-\pi/\omega}(\widehat{K})} = \sup_{v \in H^{\pi/\omega-3/2}(\widehat{K})} \frac{(\widetilde{e}, v)_{L^2(\widehat{K})}}{\|v\|_{H^{\pi/\omega-3/2}(\widehat{K})}}, \tag{4.76}$$

cf. [45, Thm. 3.34, Thm. 3.40], where $\widetilde{H}^t(\Omega) = H^t(\Omega)$ for a Lipschitz domain Ω and $t \in [0, 1/2)$ is shown. For $v \in H^{\pi/\omega-3/2}(\widehat{K})$, let then $z \in H^{\pi/\omega+1/2}(\widehat{K}) \cap H_0^1(\widehat{K})$ be given by the dual problem

$$\begin{aligned} -\Delta z &= v \quad \text{on } \widehat{K}, \\ z &= 0 \quad \text{on } \partial\widehat{K}. \end{aligned} \tag{4.77}$$

Note that $-1/2 < \pi/\omega - 3/2 < 0$, thus the regularity of (4.77) is obtained by interpolation between $H^{-1}(\widehat{K})$ and $L^2(\widehat{K})$ (for which convexity of $\widehat{K}$ is exploited). Integration by parts implies

$$(\widetilde{e}, v)_{L^2(\widehat{K})} = \int_{\widehat{K}} \nabla z \cdot \nabla \widetilde{e} - \int_{\partial\widehat{K}} \partial_n z \widetilde{e}. \tag{4.78}$$

We mention that $\partial_n z \in L^2(\partial\widehat{K})$ since $\omega < \pi$, hence it is possible to split the integral over $\partial\widehat{K}$ into a sum of face contributions. We now estimate the first term in (4.78), using the orthogonality properties satisfied by $\widetilde{e}$, Lemma 2.23 and (4.71), and obtain

$$\begin{aligned} |(\nabla z, \nabla \widetilde{e})_{L^2(\widehat{K})}| &\leq \inf_{\pi \in \mathring{W}_{p+1}(\widehat{K})} \|z - \pi\|_{H^1(\widehat{K})} \|\nabla \widetilde{e}\|_{L^2(\widehat{K})} \\ &\overset{\text{Lemma 2.23}}{\lesssim} p^{-(\pi/\omega - 1/2)} \|z\|_{H^{\pi/\omega+1/2}(\widehat{K})} \|\nabla \widetilde{e}\|_{L^2(\widehat{K})} \\ &\lesssim p^{-(\pi/\omega-1/2)} \|v\|_{H^{\pi/\omega-3/2}(\widehat{K})} \|\nabla \widetilde{e}\|_{L^2(\widehat{K})} \\ &\overset{(4.71)}{\lesssim} p^{-(\pi/\omega-3/2)} \|v\|_{H^{\pi/\omega+1/2}(\widehat{K})} \|u\|_{H^2(\widehat{K})}. \end{aligned} \tag{4.79}$$

For the second term in (4.78), we apply the 2D result Lemma 4.18 for each face $f \in \mathcal{F}(\widehat{K})$. Since $\omega_{max}(f) < \omega$ for each $f \in \mathcal{F}(\widehat{K})$ by assumption, Lemma 4.18 holds with $s = \pi/\omega$.

Hence,

$$|(\partial_n z, \widetilde{e})_{L^2(f)}| \leq \|\partial_n z\|_{H^{\pi/\omega-1}(f)} \|\widetilde{e}\|_{\widetilde{H}^{1-\pi/\omega}(f)} \overset{\text{Lemma 4.18}}{\lesssim} p^{-(1/2+\pi/\omega)} \|\partial_n z\|_{H^{\pi/\omega-1}(f)} \|u\|_{H^{3/2}(f)}$$
$$\lesssim p^{-(1/2+\pi/\omega)} \|z\|_{H^{\pi/\omega+1/2}(\widehat{K})} \|u\|_{H^2(\widehat{K})} \lesssim p^{-(1/2+\pi/\omega)} \|v\|_{H^{\pi/\omega-3/2}(\widehat{K})} \|u\|_{H^2(\widehat{K})}. \tag{4.80}$$

The equations (4.78), (4.79) and (4.80), inserted in (4.76), show estimate (4.68a) for $s = \pi/\omega - 1/2 < 1$.

Step 4: We show (4.68a) for $s \in (0, \min(\pi/\omega - 1/2, 1))$.
In the first three steps, we have shown the desired estimate for the cases $s = 0$ and, depending on ω, $s = \min(\pi/\omega - 1/2, 1)$. The intermediate values now follow by interpolation.

Step 5: We show (4.68b) for $s = \min(\pi/\omega - 1/2, 1)$, by a duality argument.
With the notation $\widetilde{e} := u - \widehat{\Pi}^{\text{grad},3d}_{p+1} u$ as before, we need an estimate for the norm

$$\|\nabla \widetilde{e}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K})} = \sup_{\mathbf{v} \in \mathbf{H}^s(\widehat{K})} \frac{(\nabla \widetilde{e}, \mathbf{v})_{L^2(\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^s(\widehat{K})}}. \tag{4.81}$$

According to Lemma 2.29, any $\mathbf{v} \in \mathbf{H}^s(\widehat{K})$ can be decomposed as $\mathbf{v} = \nabla \varphi + \mathbf{curl}\, \mathbf{z}$ with $\varphi \in H^{s+1}(\widehat{K}) \cap H^1_0(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{K})$, where also the estimate

$$\|\varphi\|_{H^{s+1}(\widehat{K})} + \|\mathbf{z}\|_{\mathbf{H}^{s+1}(\widehat{K})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^s(\widehat{K})}$$

holds. We proceed by integration by parts to get

$$(\nabla \widetilde{e}, \mathbf{v})_{L^2(\widehat{K})} = (\nabla \widetilde{e}, \nabla \varphi)_{L^2(\widehat{K})} + (\Pi_\tau \nabla \widetilde{e}, \gamma_\tau \mathbf{z})_{L^2(\partial \widehat{K})}.$$

The first term is handled by Lemma 2.23 and (4.71) to obtain

$$\left|(\nabla \widetilde{e}, \nabla \varphi)_{L^2(\widehat{K})}\right| \lesssim \|\nabla \widetilde{e}\|_{L^2(\widehat{K})} \inf_{\pi \in \mathring{W}_{p+1}(\widehat{K})} \|\varphi - \pi\|_{H^1(\widehat{K})} \lesssim p^{-1} \|u\|_{H^2(\widehat{K})} p^{-s} \|\varphi\|_{H^{s+1}(\widehat{K})}$$
$$\lesssim p^{-(s+1)} \|u\|_{H^2(\widehat{K})} \|\mathbf{v}\|_{\mathbf{H}^s(\widehat{K})},$$

imitating (4.79). Now note that $\mathbf{z} \in \mathbf{H}^{s+1}(\widehat{K})$ implies $\mathbf{z} \in \mathbf{H}^{s+1/2}(f)$ for each face $f \in \mathcal{F}(\widehat{K})$. Thus, we use Lemma 4.18 with $s = \min(\pi/\omega - 1/2, 1) + 1/2$ in order to treat the second term, which yields

$$\left|(\Pi_\tau \nabla \widetilde{e}, \gamma_\tau \mathbf{z})_{L^2(f)}\right| = \left|(\nabla_f \widetilde{e}, \gamma_\tau \mathbf{z})_{L^2(f)}\right| \lesssim \|\nabla_f \widetilde{e}\|_{\widetilde{\mathbf{H}}^{-(s+1/2)}(f)} \|\gamma_\tau \mathbf{z}\|_{\mathbf{H}^{s+1/2}(f)}$$
$$\overset{\text{Lemma 4.18}}{\lesssim} p^{-(1+s)} \|u\|_{H^{3/2}(f)} \|\mathbf{z}\|_{H^{s+1}(\widehat{K})} \lesssim p^{-(1+s)} \|u\|_{H^2(\widehat{K})} \|\mathbf{v}\|_{\mathbf{H}^s(\widehat{K})}.$$

Inserting the last two estimates in (4.81) gives us (4.68b) for $s = \min(\pi/\omega - 1/2, 1)$.

Step 6: The estimate (4.68b) for $s \in (0, \min(\pi/\omega - 1/2, 1))$ now follows by interpolation. □

Remark 4.30. A short look at the condition on the angles in the previous result reveals that the statements in Proposition 4.29 hold for $s \in [0,1]$ as long as the maximal interior angle of the faces of $\widehat{K}$ is less than $2\pi/3$, which is obviously true for every choice of the reference tetrahedron with only acute angles. As a consequence, there is no restriction for $s \in [0,1]$ necessary if we choose the regular tetrahedron or the tetrahedron, where the vertices have the Cartesian coordinates $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, which are both natural choices for reference tetrahedra.

4.7.2 Stability of $\widehat{\Pi}_p^{\mathrm{curl},3d}$

The main goal in this subsection is the proof of the stability properties of $\widehat{\Pi}_p^{\mathrm{curl},3d}$, in an analogous way to Lemma 4.22. In 2D, the existence of a suitable lifting operator from the boundary was crucial. This fact carries over to the three-dimensional case, where a lifting operator is required, too.
In [31, Thm. 7.2] a lifting operator has been constructed for the space $\mathbf{H}(\widehat{K}, \mathbf{curl})$, which lives in the Banach space $\mathbf{X}^{-1/2}$, cf. [31, Sec. 2].

Definition 4.31. *We define the space*

$$\mathbf{X}^{-1/2} := \Pi_\tau \mathbf{H}(\widehat{K}, \mathbf{curl}),$$

which is equipped with the quotient norm

$$\|\mathbf{z}\|_{\mathbf{X}^{-1/2}} := \inf_{\substack{\mathbf{v} \in \mathbf{H}(\widehat{K}, \mathbf{curl}) \\ \Pi_\tau \mathbf{v} = \mathbf{z}}} \|\mathbf{v}\|_{\mathbf{H}(\widehat{K}, \mathbf{curl})}.$$

For each face $f \in \mathcal{F}(\widehat{K})$ and $s > 1/2$, we define the space

$$\mathbf{H}^s_T(f) := \{\mathbf{z} \in \mathbf{H}^s(f) : \mathbf{z} \cdot \mathbf{n}_f = 0\}$$

with the usual $\mathbf{H}^s(f)$-norm (which will be sometimes denoted by $\|\cdot\|_{\mathbf{H}^s_T(f)}$ to emphasize that only tangential vector fields are considered).

In the following lemma, we take the lifting operator from [31], state the appropriate mapping properties and add an additional orthogonality that is needed later on.

Lemma 4.32. *There exist $C > 0$, which is independent of p, and, for each $p \in \mathbb{N}$, a lifting operator $\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d} : \Pi_\tau \mathbf{Q}_p(\widehat{K}) \to \mathbf{Q}_p(\widehat{K})$ satisfying the following properties:*

(i) For all $\mathbf{z} \in \mathbf{Q}_p(\widehat{K})$, there holds $\Pi_\tau \boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}(\Pi_\tau \mathbf{z}) = \Pi_\tau \mathbf{z}$.

(ii) There holds the stability estimate

$$\|\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d} \mathbf{z}\|_{\mathbf{H}(\widehat{K}, \mathbf{curl})} \le C \|\mathbf{z}\|_{\mathbf{X}^{-1/2}}.$$

(iii) There holds the orthogonality $(\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d} \mathbf{z}, \nabla v)_{L^2(\widehat{K})} = 0$ for all $v \in \mathring{W}_{p+1}(\widehat{K})$.

Proof. We define

$$\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\mathbf{z} := \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z} - \mathbf{w}_0,$$

where $\boldsymbol{\mathcal{E}}^{\mathrm{curl}} : \Pi_\tau(\mathbf{H}(\widehat{K},\mathbf{curl})) \to \mathbf{H}(\widehat{K},\mathbf{curl})$ is the lifting operator from [31] and where $\mathbf{w}_0$ is defined by the following saddle point problem:
Find $\mathbf{w}_0 \in \mathring{\mathbf{Q}}_p(\widehat{K})$ and $\varphi \in \mathring{W}_{p+1}(\widehat{K})$ such that for all $\mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K})$ and all $\mu \in \mathring{W}_{p+1}(\widehat{K})$

$$\left(\mathbf{curl}\,\mathbf{w}_0, \mathbf{curl}\,\mathbf{q}\right)_{L^2(\widehat{K})} + \left(\mathbf{q}, \nabla\varphi\right)_{L^2(\widehat{K})} = \left(\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}), \mathbf{curl}\,\mathbf{q}\right)_{L^2(\widehat{K})} \tag{4.82a}$$

$$\left(\mathbf{w}_0, \nabla\mu\right)_{L^2(\widehat{K})} = \left(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}, \nabla\mu\right)_{L^2(\widehat{K})}. \tag{4.82b}$$

In order to show that the saddle point problem (4.82) is uniquely solvable, we need to check coercivity and the inf-sup-condition:
With the bilinear forms $a(\mathbf{w},\mathbf{q}) := \left(\mathbf{curl}\,\mathbf{w}, \mathbf{curl}\,\mathbf{q}\right)_{L^2(\widehat{K})}$ and $b(\mathbf{w},\varphi) := \left(\mathbf{w}, \nabla\varphi\right)_{L^2(\widehat{K})}$ for $\mathbf{w},\mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K})$ and $\varphi \in \mathring{W}_{p+1}(\widehat{K})$, coercivity of a on

$$\ker b = \{\mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K})\colon (\mathbf{q},\nabla\mu)_{L^2(\widehat{K})} = 0\,\forall\mu \in \mathring{W}_{p+1}\} = \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$$

is a direct consequence of the Friedrichs inequality (Lemma 2.35) by

$$a(\mathbf{v},\mathbf{v}) = \|\,\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\widehat{K})} \geq \frac{1}{2C^2}\|\mathbf{v}\|^2_{L^2(\widehat{K})} + \frac{1}{2}\|\,\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\widehat{K})} \geq \min\{\frac{1}{2C^2},\frac{1}{2}\}\|\mathbf{v}\|^2_{\mathbf{H}(\widehat{K},\mathbf{curl})}$$

for all $\mathbf{v} \in \ker b$. For the validity of the inf-sup condition

$$\inf_{\varphi\in\mathring{W}_{p+1}(\widehat{K})}\;\sup_{\mathbf{w}\in\mathring{\mathbf{Q}}_p(\widehat{K})} \frac{b(\mathbf{w},\varphi)}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\|\varphi\|_{H^1(\widehat{K})}} \geq C,$$

we choose $\mathbf{w} = \nabla\varphi \in \mathring{\mathbf{Q}}_p(\widehat{K})$ for a given $\varphi \in \mathring{W}_{p+1}(\widehat{K})$. Since φ is zero on the boundary, the standard Poincaré inequality implies

$$\frac{b(\mathbf{w},\varphi)}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\|\varphi\|_{H^1(\widehat{K})}} = \frac{\|\nabla\varphi\|^2_{L^2(\widehat{K})}}{\|\nabla\varphi\|_{L^2(\widehat{K})}\|\varphi\|_{H^1(\widehat{K})}} \geq C.$$

Hence, the saddle point problem (4.82) has a unique solution

$$(\mathbf{w}_0,\varphi) \in \mathring{\mathbf{Q}}_p(\widehat{K}) \times \mathring{W}_{p+1}(\widehat{K}).$$

Choosing $\mathbf{q} = \nabla\varphi$ as test function in (4.82a) shows $\varphi = 0$.
The lifting operator $\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}$ now obviously satisfies (iii) by construction, cf. (4.82b). Statement (i) also holds since $\Pi_\tau\mathbf{w}_0 = 0$ and the operator $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ has the desired polynomial preserving property, cf. [31, Thm. 7.2].
We now show (ii). Note that the solution $\mathbf{w}_0$ satisfies the estimate

$$\|\mathbf{w}_0\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|f\| + \|g\|, \tag{4.83}$$

where $f(\mathbf{v}) = (\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}), \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}$ and $g(v) = (\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}, \nabla v)_{L^2(\widehat{K})}$, and $\|\cdot\|$ denotes the operator norm, cf. [12, Thm. 4.2.3]. Hence, we have

$$\|f\| = \sup_{\|\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\le 1} |(\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}), \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}| \le \|\,\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z})\|_{L^2(\widehat{K})} \lesssim \|\mathbf{z}\|_{\mathbf{X}^{-1/2}}, \tag{4.84}$$

since the operator $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ already satisfies the continuity property (ii). We can also estimate

$$\|g\| \lesssim \sup_{\|v\|_{H^1(\widehat{K})}\le 1} |(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}, \nabla v)_{L^2(\widehat{K})}| \le \|\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}\|_{L^2(\widehat{K})} \lesssim \|\mathbf{z}\|_{\mathbf{X}^{-1/2}} \tag{4.85}$$

analogously. Thus, in view of (4.83), (4.84) and (4.85), the triangle inequality

$$\|\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \le \|\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\mathbf{w}_0\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|\mathbf{z}\|_{\mathbf{X}^{-1/2}}$$

shows (ii). □

Lemma 4.33. *A function* $\mathbf{z} \in \mathbf{T} := \Pi_\tau \mathbf{H}^2(\widehat{K})$ *is in* $L^2(\partial\widehat{K})$ *and facewise in* $\mathbf{H}_T^{3/2}(f)$*. Moreover, there exists* $C > 0$ *such that the inequality*

$$\|\mathbf{z}\|_{\mathbf{X}^{-1/2}} \le C \sum_{f\in\mathcal{F}(\widehat{K})} \left[\|\mathbf{z}\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} + \|\,\mathrm{curl}_f\,\mathbf{z}\|_{\widetilde{H}^{-1/2}(f)}\right]$$

holds, where $\|\cdot\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)}$ *is defined as the dual norm to* $\|\cdot\|_{\mathbf{H}_T^{1/2}(f)}$.

Proof. We proceed in several steps.

Step 1: Clearly, $\mathbf{z}$ is in $L^2(\partial\widehat{K})$ and facewise in $\mathbf{H}_T^{3/2}(f)$. The surface curl of $\mathbf{z} \in \mathbf{T}$ which is denoted by $\mathrm{curl}_{\partial\widehat{K}}\,\mathbf{z}$, is by definition $\mathbf{n}\cdot\mathbf{curl}\,\widetilde{\mathbf{z}} \in H^{-1/2}(\partial\widehat{K})$ for any lifting $\widetilde{\mathbf{z}} \in \mathbf{H}(\widehat{K},\mathbf{curl})$ of $\mathbf{z}$. Note that this definition is meaningful since it is indeed independent of the lifting: The difference $\boldsymbol{\delta}$ of two liftings is namely clearly in the space $\mathbf{H}_0(\widehat{K},\mathbf{curl})$. The deRham diagram (2.30) then implies $\mathbf{curl}\,\boldsymbol{\delta} \in \mathbf{H}_0(\widehat{K},\mathrm{div})$ and thus $\mathbf{n}\cdot\mathbf{curl} = 0$.
Furthermore, since by assumption an $\mathbf{H}^2$-lifting of $\mathbf{z}$ exists, $\mathrm{curl}_{\partial\widehat{K}}\,\mathbf{z} \in H^{-1/2}(\partial\widehat{K})$ is facewise in $\mathbf{H}_T^{1/2}(f)$ and coincides facewise with $\mathrm{curl}_f\,\mathbf{z}$.

Step 2: We construct a particular lifting $\mathbf{Z} \in \mathbf{H}(\widehat{K},\mathbf{curl})$ of $\mathbf{z} \in \mathbf{X}^{-1/2}$. In order to find the lifting, we pose the following (constrained) minimization problem: Minimize $\|\,\mathbf{curl}\,\mathbf{Y}\|_{L^2(\widehat{K})}$ under the constraints $\Pi_\tau\mathbf{Y} = \mathbf{z}$ and $(\mathbf{Y},\nabla\varphi)_{L^2(\widehat{K})} = 0$ for all $\varphi \in H_0^1(\widehat{K})$. Writing $\mathbf{Y} = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z} + \mathbf{Y}_0$ with $\mathbf{Y}_0 \in \mathbf{H}_0(\widehat{K},\mathbf{curl})$, the Lagrange functional $\mathfrak{L}$ is defined by

$$\mathfrak{L}(\mathbf{Y},\psi) := \|\,\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z} + \mathbf{Y}_0)\|^2_{L^2(\widehat{K})} + (\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z} + \mathbf{Y}_0, \nabla\psi)_{L^2(\widehat{K})}.$$

Minimization is now equivalent to solving the following saddle point problem: Find $\mathbf{Y}_0 \in \mathbf{H}_0(\widehat{K},\mathbf{curl})$ and $\varphi \in H_0^1(\widehat{K})$ such that for all $\mathbf{q} \in \mathbf{H}_0(\widehat{K},\mathbf{curl})$ and $\mu \in H_0^1(\widehat{K})$,

$$\begin{aligned}(\mathbf{curl}\,\mathbf{Y}_0, \mathbf{curl}\,\mathbf{q})_{L^2(\widehat{K})} + (\mathbf{q},\nabla\varphi)_{L^2(\widehat{K})} &= -(\mathbf{curl}(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z}), \mathbf{curl}\,\mathbf{q})_{L^2(\widehat{K})}\\ (\mathbf{Y}_0,\nabla\mu)_{L^2(\widehat{K})} &= -(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\mathbf{z},\nabla\mu)_{L^2(\widehat{K})}.\end{aligned}$$

This problem can be solved in a similar way to (2.48) (or to (4.82), but with continuous functions). As was observed above, the Lagrange multiplier φ in fact vanishes so that we conclude that the minimizer $\mathbf{Z}$ solves

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{Z} = 0, \qquad \operatorname{div}\mathbf{Z} = 0, \qquad \Pi_\tau \mathbf{Z} = \mathbf{z}.$$

Step 3: We estimate $\mathbf{w} := \mathbf{curl}\,\mathbf{Z}$.
The assertions

$$\mathbf{curl}\,\mathbf{w} = 0, \qquad \operatorname{div}\mathbf{w} = 0, \qquad \mathbf{n}\cdot\mathbf{w} = \operatorname{curl}_{\partial\widehat{K}}\mathbf{z}. \tag{4.86}$$

are clear from step 2. Since $\mathbf{curl}\,\mathbf{w} = 0$ (in the distributional sense), there exists a function $\psi \in H^1(\widehat{K})$ such that $\mathbf{w} = \nabla\psi$. The second and third conditions in (4.86) then imply that ψ solves the Neumann problem

$$\begin{aligned} -\Delta\psi &= 0, \\ \partial_n\psi &= \mathbf{n}\cdot\mathbf{w} = \operatorname{curl}_{\partial\widehat{K}}\mathbf{z} \quad \text{on } \partial\widehat{K}. \end{aligned}$$

The integrability condition is satisfied since $(\mathbf{n}\cdot\mathbf{w},1)_{L^2(\partial\widehat{K})} = (\operatorname{div}\mathbf{w},1)_{L^2(\widehat{K})} = 0$. Hence, standard estimates for the Laplace problem give us

$$\|\mathbf{curl}\,\mathbf{Z}\|_{L^2(\widehat{K})} = \|\mathbf{w}\|_{L^2(\widehat{K})} = \|\nabla\psi\|_{L^2(\widehat{K})} \lesssim \|\operatorname{curl}_{\partial\widehat{K}}\mathbf{z}\|_{H^{-1/2}(\partial\widehat{K})}. \tag{4.87}$$

Step 4: I order to bound $\mathbf{Z}$, we use the right inverses introduced in Lemma 2.27. By Lemma 2.28, there exists $\widetilde{\mathbf{z}} \in \mathbf{H}^1(\widehat{K})$ and $\phi \in H^1(\widehat{K})$ such that $\mathbf{Z} = \nabla\phi + \widetilde{\mathbf{z}}$ with the estimate

$$\|\widetilde{\mathbf{z}}\|_{H^1(\widehat{K})} \lesssim \|\mathbf{curl}\,\mathbf{Z}\|_{L^2(\widehat{K})} \lesssim \|\operatorname{curl}_{\partial\widehat{K}}\mathbf{z}\|_{H^{-1/2}(\partial\widehat{K})}. \tag{4.88}$$

In order to find estimates for ϕ, we use integration by parts. Since $\operatorname{div}\mathbf{Z} = 0$, we obtain

$$\nabla\phi + \widetilde{\mathbf{z}} = \mathbf{Z} = \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\mathbf{Z}) = \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\nabla\phi) + \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\widetilde{\mathbf{z}}), \tag{4.89}$$

hence

$$(\mathbf{curl}\,\mathbf{Z},\mathbf{v})_{L^2(\widehat{K})} = (\mathbf{Z},\mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} - (\mathbf{z},\gamma_\tau\mathbf{v})_{L^2(\partial\widehat{K})}.$$

follows. Choosing $\mathbf{v} = \mathbf{R}^{\mathrm{curl}}(\nabla\phi) \in \mathbf{H}^1(\widehat{K})$ and using (4.89) yields

$$\begin{aligned} (\mathbf{curl}\,\mathbf{Z},\mathbf{R}^{\mathrm{curl}}(\nabla\phi))_{L^2(\widehat{K})} &= (\nabla\phi + \widetilde{\mathbf{z}},\mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\widetilde{\mathbf{z}}))_{L^2(\widehat{K})} \\ &\quad - (\mathbf{z},\gamma_\tau\mathbf{R}^{\mathrm{curl}}(\nabla\phi))_{L^2(\partial\widehat{K})}. \end{aligned}$$

In view of the mapping property $\mathbf{R}^{\mathrm{curl}} : L^2(\widehat{K}) \to \mathbf{H}^1(\widehat{K})$, we now obtain

$$\begin{aligned} \|\nabla\phi\|^2_{L^2(\widehat{K})} \lesssim \|\mathbf{curl}\,\mathbf{Z}\|_{L^2(\widehat{K})}\|\nabla\phi\|_{L^2(\widehat{K})} &+ \|\widetilde{\mathbf{z}}\|_{L^2(\widehat{K})}\|\widetilde{\mathbf{z}} - \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\widetilde{\mathbf{z}})\|_{L^2(\widehat{K})} \\ &+ \|\widetilde{\mathbf{z}} - \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\widetilde{\mathbf{z}})\|_{L^2(\widehat{K})}\|\nabla\phi\|_{L^2(\widehat{K})} \\ &+ \|\widetilde{\mathbf{z}}\|_{L^2(\widehat{K})}\|\nabla\phi\|_{L^2(\widehat{K})} + \left|(\mathbf{z},\gamma_\tau\mathbf{R}^{\mathrm{curl}}(\nabla\phi))_{L^2(\partial\widehat{K})}\right|. \end{aligned} \tag{4.90}$$

Combining the estimates (4.88) and (4.89) implies

$$\begin{aligned}\|\mathbf{Z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} &\lesssim \|\widetilde{\mathbf{z}}\|_{L^2(\widehat{K})} + \|\nabla\phi\|_{L^2(\widehat{K})} + \|\,\mathbf{curl}\,\mathbf{Z}\|_{L^2(\widehat{K})} \\ &\lesssim \sup_{\mathbf{v}\in\mathbf{H}^1(\widehat{K})} \frac{(\mathbf{z},\gamma_\tau\mathbf{v})_{L^2(\partial\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}} + \|\operatorname{curl}_{\partial\widehat{K}}\mathbf{z}\|_{H^{-1/2}(\partial\widehat{K})}.\end{aligned} \tag{4.91}$$

Step 5: Note that both functions $\mathbf{z}$ and $\operatorname{curl}_{\partial\widehat{K}}\mathbf{z}$ lie in the space $L^2(\partial\widehat{K})$. Thus, the norm $\|\cdot\|_{\mathbf{X}^{-1/2}}$ can be estimated in a localized way by the continuity of the inclusions $H^{1/2}(\partial\widehat{K}) \subset \prod_{f\in\mathcal{F}(\widehat{K})} H^{1/2}(f)$ and $\gamma_\tau\mathbf{H}^1(\widehat{K}) \subset \prod_{f\in\mathcal{F}(\widehat{K})} \mathbf{H}_T^{1/2}(f)$. The estimates for the dual spaces

$$\|\operatorname{curl}_{\partial\widehat{K}}\mathbf{z}\|_{H^{-1/2}(\partial\widehat{K})} \lesssim \sum_{f\in\mathcal{F}(\widehat{K})} \|\operatorname{curl}_f\mathbf{z}\|_{\widetilde{H}^{-1/2}(f)}, \tag{4.92a}$$

$$\sup_{\mathbf{v}\in\mathbf{H}^1(\widehat{K})} \frac{(\mathbf{z},\gamma_\tau\mathbf{v})_{L^2(\partial\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}} \lesssim \sum_{f\in\mathcal{F}(\widehat{K})} \|\mathbf{z}\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} \tag{4.92b}$$

follow, and the bound

$$\|\mathbf{z}\|_{\mathbf{X}^{-1/2}} \lesssim \|\mathbf{Z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \overset{(4.91),\,(4.92)}{\lesssim} \sum_{f\in\mathcal{F}(\widehat{K})} \|\mathbf{z}\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} + \|\operatorname{curl}_f\mathbf{z}\|_{\widetilde{H}^{-1/2}(f)}$$

finishes the proof. □

This lifting operator will now be used for estimating the interpolation error.

Proposition 4.34. *Let* $\mathbf{u}\in\mathbf{H}^1(\widehat{K},\mathbf{curl})$*. Then there exists* $C>0$ *independent of* $p\in\mathbb{N}$ *such that*

$$\|\mathbf{u}-\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \le Cp^{-1} \inf_{\mathbf{v}\in\mathbf{Q}_p(\widehat{K})} \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \tag{4.93}$$

holds.

Proof. Step 1: By the projection property of $\widehat{\Pi}_p^{\mathrm{curl},3d}$, it is again sufficient to show the estimate for $\mathbf{v}=0$.

Step 2: We decompose the function $\mathbf{u}\in\mathbf{H}^1(\widehat{K},\mathbf{curl})$ as $\mathbf{u}=\nabla\varphi+\mathbf{v}$, where $\varphi\in H^2(\widehat{K})$ and $\mathbf{v}\in\mathbf{H}^2(\widehat{K})$ satisfy the estimates

$$\|\varphi\|_{H^2(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \quad\text{and}\quad \|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})} \lesssim \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^1(\widehat{K})}, \tag{4.94}$$

cf. Lemma 2.28. The commuting diagram and the interpolation result Proposition 4.29 then imply

$$\begin{aligned}\|\nabla\varphi-\widehat{\Pi}_p^{\mathrm{curl},3d}\nabla\varphi\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} = \|\nabla(\varphi-\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi)\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} = |\varphi-\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi|_{H^1(\widehat{K})} \\ \lesssim p^{-1}\|\varphi\|_{H^2(\widehat{K})}.\end{aligned} \tag{4.95}$$

Step 3: For $\mathbf{v} \in \mathbf{H}^2(\widehat{K})$, Lemma 4.33 gives us

$$\|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\mathbf{X}^{-1/2}} \lesssim \tag{4.96}$$

$$\sum_{f \in \mathcal{F}(\widehat{K})} \|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} + \|\operatorname{curl}_f(\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v}))\|_{\widetilde{H}^{-1/2}(f)}. \tag{4.97}$$

For each face $f \in \mathcal{F}(\widehat{K})$, we can now apply the 2D results Lemma 4.19 and Lemma 4.22 to obtain

$$\begin{aligned}\|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\widetilde{\mathbf{H}}_T^{-1/2}(f)} &\overset{\text{Lem. 4.19}}{\lesssim} p^{-1/2}\|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\mathbf{H}(f,\mathrm{curl})} \\ &\overset{\text{Lem. 4.22}}{\lesssim} p^{-1/2-1/2}\|\Pi_\tau\mathbf{v}\|_{\mathbf{H}^{1/2}(f,\mathrm{curl})}\end{aligned}$$

by the continuity of the trace operator $\Pi_\tau : \mathbf{H}^2(\widehat{K}) \to \mathbf{H}_T^{3/2}(f) \subset \mathbf{H}^{1/2}(f, \mathrm{curl})$. Similarly, Lemma 4.20 yields

$$\begin{aligned}\|\operatorname{curl}(\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v}))\|_{\widetilde{H}^{-1/2}(f)} &\overset{\text{Lem. 4.20}}{\lesssim} p^{-1/2}\|\operatorname{curl}(\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v}))\|_{L^2(f)} \\ &\lesssim p^{-1/2-1/2}\|\Pi_\tau\mathbf{v}\|_{\mathbf{H}^{1/2}(f,\mathrm{curl})} \lesssim p^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})}.\end{aligned}$$

Inserting the last two estimates in (4.96) then implies

$$\|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\mathbf{X}^{-1/2}} \leq Cp^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})}. \tag{4.98}$$

Step 4: With the best approximation operator $P^{\mathrm{curl},3d}$ from Lemma 4.27, we have

$$\begin{aligned}\|\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} &\leq \|\mathbf{v} - P^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v} - P^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \\ &\leq p^{-1}\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} + \|\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v} - P^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \\ &\leq p^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})} + \|\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v} - P^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})},\end{aligned}$$

since $\mathbf{v} \in \mathbf{H}^2(\widehat{K})$. Writing $\mathbf{E} := \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v} - P^{\mathrm{curl},3d}\mathbf{v} \in \mathbf{Q}_p(\widehat{K})$ for simplicity, the orthogonality conditions (4.3e), (4.3f) satisfied by $\widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v}$ and the orthogonality conditions (4.66a), (4.66b) satisfied by $P^{\mathrm{curl},3d}\mathbf{v}$ imply

$$(\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{w})_{L^2(\widehat{K})} = 0 \quad \forall \mathbf{w} \in \mathring{\mathbf{Q}}_p(\widehat{K}), \tag{4.99a}$$

$$(\mathbf{E}, \nabla w)_{L^2(\widehat{K})} = 0 \quad \forall w \in \mathring{W}_{p+1}(\widehat{K}). \tag{4.99b}$$

Since we also have

$$(\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E}, \nabla w)_{L^2(\widehat{K})} = 0 \quad \forall w \in \mathring{W}_{p+1}(\widehat{K})$$

from Lemma 4.32, (iii), it follows $\mathbf{E} - \boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$. Thus, we obtain by the discrete Friedrichs inequality (Lemma 2.35)

$$\begin{aligned}\|\mathbf{E}\|_{L^2(\widehat{K})} &\leq \|\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E}\|_{L^2(\widehat{K})} + \|\mathbf{E} - \boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E}\|_{L^2(\widehat{K})} \\ &\lesssim \|\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E}\|_{L^2(\widehat{K})} + \|\mathbf{curl}(\mathbf{E} - \boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E})\|_{L^2(\widehat{K})} \\ &\lesssim \|\boldsymbol{\mathcal{L}}^{\mathrm{curl},3d}\Pi_\tau\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\mathbf{curl}\,\mathbf{E}\|_{L^2(\widehat{K})} \\ &\lesssim \|\Pi_\tau\mathbf{E}\|_{\mathbf{X}^{-1/2}} + \|\mathbf{curl}\,\mathbf{E}\|_{L^2(\widehat{K})}.\end{aligned} \tag{4.100}$$

Since condition (4.99a) leads to

$$\begin{aligned}\|\operatorname{\mathbf{curl}}\mathbf{E}\|^2_{L^2(\widehat{K})} &= (\operatorname{\mathbf{curl}}\mathbf{E}, \operatorname{\mathbf{curl}}(\mathbf{E} - \boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E} + \boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E}))_{L^2(\widehat{K})}\\ &= (\operatorname{\mathbf{curl}}\mathbf{E}, \operatorname{\mathbf{curl}}\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E})_{L^2(\widehat{K})}\\ &\leq \|\operatorname{\mathbf{curl}}\mathbf{E}\|_{L^2(\widehat{K})}\|\operatorname{\mathbf{curl}}\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E}\|_{L^2(\widehat{K})},\end{aligned}$$

we obtain with Lemma 4.32, (ii)

$$\|\operatorname{\mathbf{curl}}\mathbf{E}\|_{L^2(\widehat{K})} \leq \|\operatorname{\mathbf{curl}}\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau\mathbf{E}\|_{L^2(\widehat{K})} \lesssim \|\Pi_\tau\mathbf{E}\|_{\mathbf{X}^{-1/2}}, \tag{4.101}$$

from which

$$\begin{aligned}\|\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} &\leq \|\mathbf{v} - P^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\\ &\overset{(4.100),(4.101)}{\lesssim} \|\mathbf{v} - P^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\Pi_\tau\mathbf{E}\|_{\mathbf{X}^{-1/2}}\\ &\lesssim \|\mathbf{v} - P^{\mathrm{curl},3d}\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\Pi_\tau(\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v})\|_{\mathbf{X}^{-1/2}} \overset{(4.98)}{\lesssim} p^{-1}\|\mathbf{v}\|_{\mathbf{H}^2(\widehat{K})}\end{aligned} \tag{4.102}$$

follows. Note that the third inequality directly follows by the triangle inequality and the definition of the $\mathbf{X}^{-1/2}$-norm as an infimum. The proof is now complete in view of the equations (4.94), (4.95) and (4.102). □

We can also control the interpolation error in negative Sobolev norms. Note that we have to pose similar conditions on the maximal possible negative norm dependent on the angles of $\widehat{K}$ as in Proposition 4.29. As a consequence, there won't be any additional restrictions for reference tetrahedra with all interior angles smaller than $2\pi/3$, see also Remark 4.30.

Proposition 4.35. *Let $\omega \in (0, \pi)$, and assume that all interior angles of the 4 faces of $\widehat{K}$ are smaller than ω. Then, for every $s \in [0, \min(\pi/\omega - 1/2, 1)]$, there exists $C_s > 0$ such that for $\mathbf{u} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$, the estimate*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \leq C_s p^{-(1+s)} \inf_{\mathbf{v}\in\mathbf{Q}_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \tag{4.103}$$

holds.

Proof. We proceed in several steps.

Step 1: Since $\widehat{\Pi}_p^{\mathrm{curl},3d}$ is a projection, it is again sufficient to prove the result for $\mathbf{v} = 0$ in the infimum.

Step 2: We show (4.103) for $s^\star = \min(\pi/\omega - 1/2, 1)$. In order to do this, we first write $\mathbf{E} := \mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}$ for simplicity and then have

$$\begin{aligned}\|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-s^\star}(\widehat{K},\mathbf{curl})} &\lesssim \|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-s^\star}(\widehat{K})} + \|\operatorname{\mathbf{curl}}\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-s^\star}(\widehat{K})}\\ &= \sup_{\mathbf{v}\in\mathbf{H}^{s^\star}(\widehat{K})} \frac{(\mathbf{E},\mathbf{v})_{L^2(\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^{s^\star}(\widehat{K})}} + \sup_{\mathbf{v}\in\mathbf{H}^{s^\star}(\widehat{K})} \frac{(\operatorname{\mathbf{curl}}\mathbf{E},\mathbf{v})_{L^2(\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^{s^\star}(\widehat{K})}}.\end{aligned} \tag{4.104}$$

Step 3: We estimate the first supremum in (4.104).
By Lemma 2.31, we have for $\mathbf{v} \in \mathbf{H}^{s^\star}(\widehat{K})$ the decomposition

$$\mathbf{v} = \nabla\varphi + \mathbf{curl}\,\mathbf{curl}\,\mathbf{z},$$

where $\varphi \in H^{s^\star+1}(\widehat{K}) \cap H_0^1(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^{s^\star}(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ satisfy the estimate

$$\|\varphi\|_{H^{s^\star+1}(\widehat{K})} + \|\mathbf{z}\|_{\mathbf{H}^{s^\star}(\widehat{K},\mathbf{curl})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^{s^\star}(\widehat{K})}.$$

Since even $\mathbf{curl}\,\mathbf{z} \in \mathbf{H}^{s^\star}(\widehat{K}, \mathbf{curl})$, Lemma 2.28 implies the decomposition

$$\mathbf{curl}\,\mathbf{z} = \nabla\varphi_2 + \mathbf{z}_2 \tag{4.105}$$

with $\varphi_2 \in H^{s^\star+1}(\widehat{K})$ and $\mathbf{z}_2 \in \mathbf{H}^{s^\star+1}(\widehat{K})$. Inserting the suitable decomposition leads to

$$(\mathbf{E}, \mathbf{v})_{L^2(\widehat{K})} = (\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})} + (\mathbf{E}, \mathbf{curl}\,\mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})},$$

where the first term is estimated by using the orthogonality condition (4.3e), Lemma 2.23 and Proposition 4.34 by

$$\begin{aligned}\left|(\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})}\right| &= \left|\inf_{w\in \mathring{W}_{p+1}(\widehat{K})} (\mathbf{E}, \nabla(\varphi - w))_{L^2(\widehat{K})}\right| \lesssim p^{-s^\star}\|\varphi\|_{H^{s^\star+1}(\widehat{K})}\|\mathbf{E}\|_{L^2(\widehat{K})} \\ &\lesssim p^{-s^\star}\|\mathbf{v}\|_{H^{s^\star}(\widehat{K})}\|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim p^{-(s^\star+1)}\|\mathbf{v}\|_{H^{s^\star}(\widehat{K})}\|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})},\end{aligned} \tag{4.106}$$

and where we obtain for the second term by integration by parts and representation (4.105)

$$\begin{aligned}(\mathbf{E}, \mathbf{curl}\,\mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} &= (\mathbf{E}, \mathbf{curl}\,\mathbf{z}_2)_{L^2(\widehat{K})} \\ &= (\mathbf{curl}\,\mathbf{E}, \mathbf{z}_2)_{L^2(\widehat{K})} + (\Pi_\tau\mathbf{E}, \gamma_\tau\mathbf{z}_2)_{L^2(\partial\widehat{K})} \\ &= (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} - (\mathbf{curl}\,\mathbf{E}, \nabla\varphi_2)_{L^2(\widehat{K})} + (\Pi_\tau\mathbf{E}, \gamma_\tau\mathbf{z}_2)_{L^2(\partial\widehat{K})} \\ &= (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} - (\mathbf{n}\cdot\mathbf{curl}\,\mathbf{E}, \varphi_2)_{L^2(\partial\widehat{K})} + (\Pi_\tau\mathbf{E}, \gamma_\tau\mathbf{z}_2)_{L^2(\partial\widehat{K})}.\end{aligned} \tag{4.107}$$

We estimate the three terms in (4.107) separately. For the first term, we use the orthogonality condition (4.3f) to obtain

$$\left|(\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})}\right| = \left|\inf_{\mathbf{w}\in\mathring{\mathbf{Q}}_p(\widehat{K})} (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}(\mathbf{z} - \mathbf{w}))_{L^2(\widehat{K})}\right|. \tag{4.108}$$

We now want to apply Lemma 4.27 in order to get an estimate for the infimum, however, in general $P^{\mathrm{curl},3d}\mathbf{z} \notin \mathring{\mathbf{Q}}_p(\widehat{K})$. Hence, we proceed as in Lemma 4.19 and use the lifting

$$\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d} : \Pi_\tau\mathbf{Q}_p(\widehat{K}) \to \mathbf{Q}_p(\widehat{K})$$

from Lemma 4.32. Since

$$P^{\mathrm{curl},3d}\mathbf{z} - \boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau P^{\mathrm{curl},3d}\mathbf{z} \in \mathring{\mathbf{Q}}_p(\widehat{K}),$$

this function can be used as $\mathbf{w}$ in the infimum in (4.108). Since

$$\|\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau P^{\mathrm{curl},3d}\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \lesssim \|\Pi_\tau P^{\mathrm{curl},3d}\mathbf{z}\|_{\mathbf{X}^{-1/2}} = \|\Pi_\tau(\mathbf{z} - P^{\mathrm{curl},3d}\mathbf{z})\|_{\mathbf{X}^{-1/2}} \\ \lesssim \|\mathbf{z} - P^{\mathrm{curl},3d}\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})},$$

we get with Lemma 2.23 and Proposition 4.34

$$\begin{aligned} \left|(\mathbf{curl}\,\mathbf{E},\mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})}\right| &\le \|\,\mathbf{curl}\,\mathbf{E}\|_{L^2(\widehat{K})}\|\mathbf{z} - (P^{\mathrm{curl},3d}\mathbf{z} - \boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau P^{\mathrm{curl},3d}\mathbf{z})\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} \\ &\lesssim p^{-1}\|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}\left(\|\mathbf{z} - P^{\mathrm{curl},3d}\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})} + \|\boldsymbol{\mathcal{L}}_p^{\mathrm{curl},3d}\Pi_\tau P^{\mathrm{curl},3d}\mathbf{z}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\right) \\ &\lesssim p^{-(s^\star+1)}\|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}\|\mathbf{z}\|_{\mathbf{H}^{s^\star}(\widehat{K},\mathbf{curl})} \\ &\lesssim p^{-(s^\star+1)}\|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}\|\mathbf{v}\|_{\mathbf{H}^{s^\star}(\widehat{K})}. \end{aligned} \tag{4.109}$$

For the second term in (4.107), we first note that $\mathbf{curl}\,\mathbf{E} \in \mathbf{H}^1(\widehat{K})$. Hence, the integral over $\partial\widehat{K}$ can be split into a sum of face contributions, and it also holds

$$(\mathbf{n}\cdot\mathbf{curl}\,\mathbf{E})|_f = \underset{f}{\mathrm{curl}}\,\Pi_\tau\mathbf{E}. \tag{4.110}$$

We observe that our assumptions allow us to choose $s = \min(\pi/\omega, 3/2) = s^\star + 1/2$ in Lemma 4.20 and Lemma 4.22, since $\pi/\omega_{max} > s$. We then get for each face, using Lemma 4.22,

$$\begin{aligned} \left|(\mathrm{curl}_f\,\Pi_\tau\mathbf{E},\varphi_2)_{L^2(f)}\right| &\overset{\text{Lem. 4.20}}{\lesssim} p^{-(s^\star+1/2)}\|\,\mathrm{curl}_f\,\Pi_\tau\mathbf{E}\|_{L^2(f)}\|\varphi_2\|_{H^{s^\star+1/2}(f)} \\ &\overset{\text{Lem. 4.22}}{\lesssim} p^{-(s^\star+1)}\|\Pi_\tau\mathbf{u}\|_{\mathbf{H}^{1/2}(f,\mathrm{curl})}\|\varphi_2\|_{H^{s^\star+1}(\widehat{K})} \\ &\lesssim p^{-(s^\star+1)}\|\mathbf{u}\|_{\mathbf{H}^1(\mathbf{curl},\widehat{K})}\|\mathbf{v}\|_{\mathbf{H}^{s^\star}(\widehat{K})}. \end{aligned} \tag{4.111}$$

The third term in (4.107) now follows with Lemma 4.19 and Lemma 4.22

$$\begin{aligned} \left|(\Pi_\tau\mathbf{E},\gamma_\tau\mathbf{z}_2)_{L^2(f)}\right| &\overset{\text{Lem. 4.19}}{\lesssim} p^{-(s^\star+1/2)}\|\Pi_\tau\mathbf{E}\|_{\mathbf{H}(f,\mathrm{curl})}\|\gamma_\tau\mathbf{z}_2\|_{\mathbf{H}^{s^\star+1/2}(f)} \\ &\overset{\text{Lem. 4.22}}{\lesssim} p^{-(s^\star+1)}\|\Pi_\tau\mathbf{u}\|_{\mathbf{H}^{1/2}(f,\mathrm{curl})}\|\mathbf{z}_2\|_{\mathbf{H}^{s^\star+1}(\widehat{K})} \\ &\lesssim p^{-(s^\star+1)}\|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}\|\mathbf{v}\|_{\mathbf{H}^{s^\star}(\widehat{K})}. \end{aligned} \tag{4.112}$$

Adding (4.111) and (4.112) over all faces $f \in \mathcal{F}(\widehat{K})$ and inserting these estimates together with (4.109) in (4.106) and (4.107) finishes this step.

Step 4: We estimate the second supremum in (4.104).
For the second supremum in (4.104), we decompose $\mathbf{v} \in \mathbf{H}^{s^\star}(\widehat{K})$ as

$$\mathbf{v} = \nabla\varphi + \mathbf{curl}\,\mathbf{z}$$

with $\varphi \in H^{s^\star+1}(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^{s^\star}(\widehat{K}, \mathbf{curl}) \cap \mathbf{H}_0(\widehat{K}, \mathbf{curl})$ according to Lemma 2.31. This leads to

$$(\mathbf{curl}\,\mathbf{E}, \mathbf{v})_{L^2(\widehat{K})} = (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} + (\mathbf{curl}\,\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})}, \tag{4.113}$$

where we now have to bound both expressions on the right-hand side. For the first term we use the orthogonality (4.3f) and Proposition 4.34 to obtain

$$\begin{aligned}\left|(\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})}\right| &= \left|\inf_{\mathbf{w}\in\mathring{\mathbf{Q}}_p(\widehat{K})} (\mathbf{curl}\,\mathbf{E}, \mathbf{curl}(\mathbf{z}-\mathbf{w}))_{L^2(\widehat{K})}\right| \\ &\lesssim p^{-s^\star}\|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\|\mathbf{z}\|_{\mathbf{H}^{s^\star}(\widehat{K},\mathbf{curl})} \lesssim p^{-(s^\star+1)}\|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}\|\mathbf{v}\|_{\mathbf{H}^{s^\star}(\widehat{K},\mathbf{curl})},\end{aligned}$$

cf. the arguments in step 3 for handling the infimum. For the second term of (4.113), we use integration by parts and obtain in view of (4.110)

$$(\mathbf{curl}\,\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})} = \sum_{f\in\mathcal{F}(\widehat{K})} (\operatorname{curl}_f \Pi_\tau \mathbf{E}, \varphi)_{L^2(f)},$$

where the decomposition into face contributions is again possible since both $\mathbf{curl}\,\mathbf{E}$ and $\nabla\varphi$ are at least in L^2 on the boundary. We obtain

$$\begin{aligned}\left|(\operatorname{curl}_f \Pi_\tau \mathbf{E}, \varphi)_{L^2(f)}\right| &\lesssim p^{-(s^\star+1/2)}\|\Pi_\tau\mathbf{E}\|_{\mathbf{H}(f,\operatorname{curl})}\|\varphi\|_{H^{s^\star+1/2}(f)} \\ &\lesssim p^{-(s^\star+1)}\|\mathbf{u}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}\|\mathbf{v}\|_{\mathbf{H}^{s^\star}(\widehat{K})}\end{aligned}$$

by Lemmas 4.20 and 4.22, which finishes this step.

Step 5: The general result now follows by interpolation between the cases $s = 0$, cf. Proposition 4.34, and $s = s^\star$. □

For functions $\mathbf{u}$ with discrete **curl**, we have the following result.

Lemma 4.36. *Let $\omega \in (0, \pi)$, and assume that all interior angles of the 4 faces of $\widehat{K}$ are smaller than ω. Then, for all $k \geq 1$ and for all $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ with $\mathbf{curl}\,\mathbf{u} \in \mathbf{V}_p(\widehat{K})$, there holds*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \leq C_{s,k} p^{-(k+s)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}, \qquad s \in [0, \min(\pi/\omega - 1/2, 1)]. \tag{4.114}$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

Proof. We imitate the proof of Lemma 4.23. Using the right inverses R^{grad} and $\mathbf{R}^{\mathrm{curl}}$, Lemma 2.28 again yields

$$\mathbf{u} = \nabla R^{\mathrm{grad}}(\mathbf{u} - \mathbf{R}^{\mathrm{curl}}\,\mathbf{curl}\,\mathbf{u}) + \mathbf{R}^{\mathrm{curl}}\,\mathbf{curl}\,\mathbf{u} =: \nabla\varphi + \mathbf{v}$$

with $\varphi \in H^{k+1}(\widehat{K})$ and $\mathbf{v} \in \mathbf{H}^k(\widehat{K})$, together with the estimate

$$\|\varphi\|_{H^{k+1}(\widehat{K})} + \|\mathbf{v}\|_{\mathbf{H}^k(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})} + \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^{k-1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}. \tag{4.115}$$

Since $\mathbf{curl}\,\mathbf{u} \in \mathbf{V}_p(\widehat{K})$ has been assumed, Lemma 2.27, (v) implies $\mathbf{v} = \mathbf{R}^{\mathrm{curl}}\,\mathbf{curl}\,\mathbf{u} \in \mathbf{Q}_p(\widehat{K})$. Since $\widehat{\Pi}_p^{\mathrm{curl},3d}$ is a projection, it follows $\mathbf{v} - \widehat{\Pi}_p^{\mathrm{curl},3d}\mathbf{v} = 0$, and the commuting diagram property $\nabla\widehat{\Pi}_{p+1}^{\mathrm{grad},3d} = \widehat{\Pi}_p^{\mathrm{curl},3d}\nabla$ and Proposition 4.29 give

$$\begin{aligned}\|(\mathrm{I}-\widehat{\Pi}_p^{\mathrm{curl},3d})\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} &= \|(\mathrm{I}-\widehat{\Pi}_p^{\mathrm{curl},3d})\nabla\varphi + \underbrace{(\mathrm{I}-\widehat{\Pi}_p^{\mathrm{curl},3d})\mathbf{v}}_{=0}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \\ &= \|\nabla(\mathrm{I}-\widehat{\Pi}_{p+1}^{\mathrm{grad},3d})\varphi\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K})} \lesssim p^{-(k+s)}\|\varphi\|_{H^{k+1}(\widehat{K})}.\end{aligned}$$

Note that the restriction on s posed in (4.114) is necessary due to the assumptions from Proposition 4.29. Equation (4.114) now follows immediately with (4.115).
The additional claim that we can replace the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$ is clear since the operator $\widehat{\Pi}_p^{\mathrm{curl},3d}$ reproduces polynomials of degree p. □

4.7.3 Stability of $\widehat{\Pi}_p^{\mathrm{div},3d}$

Unlike the two-dimensional case where the interpolation operators of interest are $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$ and $\widehat{\Pi}_p^{\mathrm{curl},2d}$, we also have $\widehat{\Pi}_p^{\mathrm{div},3d}$ as third interpolation operator in 3D. This subsection, which is devoted to the error estimates of $\widehat{\Pi}_p^{\mathrm{div},3d}$, is structured in a similar way to Subsection 4.7.2, since the concepts of proof are similar, however, there are less technical problems to deal with.
Similar to Lemma 4.21 in 2D, where we proffered the interpolation error of $\widehat{\Pi}_p^{\mathrm{curl},2d}$ on edges, we start with the error of $\widehat{\Pi}_p^{\mathrm{div},3d}$ on faces, as they are the lowest-order manifold of interest here.

Lemma 4.37. *Let $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$. Then there holds for each face $f \in \mathcal{F}(\widehat{K})$ and $s \geq 0$*

$$\|(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}) \cdot \mathbf{n}_f\|_{\widetilde{H}^{-s}(f)} \leq C_s p^{-s} \inf_{v \in V_p(f)} \|\mathbf{u}\cdot\mathbf{n}_f - v\|_{L^2(f)}. \tag{4.116}$$

Proof. Note that Lemma 4.11 gives us $\mathbf{u}\cdot\mathbf{n}_f \in L^2(f)$, thus all expression in (4.116) are indeed meaningful.
Let $\tilde{e} := \Big(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\Big)\cdot\mathbf{n}_f$ be the error. We already know that the operator $\widehat{\Pi}_p^{\mathrm{div},3d}$ is the L^2-projection on faces $f \in \mathcal{F}(\widehat{K})$, i.e.

$$(\mathbf{n}_f \cdot (\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}), w)_{L^2(f)} = 0 \qquad \forall w \in V_p(f), \tag{4.117}$$

cf. (4.11). Hence, using $w = \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\cdot\mathbf{n}_f$ as test function implies (4.116) for $s = 0$.
The case $s > 0$ now follows by a standard duality argument. We have to estimate the norm

$$\|\tilde{e}\|_{\widetilde{H}^{-s}(f)} = \sup_{v\in H^s(f)} \frac{(\tilde{e}, v)_{L^2(f)}}{\|v\|_{H^s(f)}}.$$

Since $(\tilde{e}, v)_{L^2(f)} = 0$ for all $v \in \mathcal{P}_p(f)$ by (4.117), we obtain

$$\begin{aligned}\big|(\tilde{e}, v)_{L^2(f)}\big| = \left|\inf_{w\in\mathcal{P}_p(f)} (\tilde{e}, v - w)_{L^2(f)}\right| &\leq \|\tilde{e}\|_{L^2(f)} \inf_{w\in\mathcal{P}_p(f)} \|v - w\|_{L^2(f)} \\ &\lesssim p^{-s}\|\tilde{e}\|_{L^2(f)}\|v\|_{H^s(f)}\end{aligned}$$

by Lemma 2.23. □

The analysis of the error of $\widehat{\Pi}_p^{\mathrm{curl},3d}$ was based on the existence of a lifting operator from $\partial\widehat{K}$ on $\widehat{K}$ with suitable properties. There, we used the lifting constructed in [31] and modified it slightly in order to get an additional orthogonality. We now follow this concept by taking the lifting operator from [32] and adjusting it to our needs.

Lemma 4.38. *Denote the (normal) trace space of* $\mathbf{V}_p(\widehat{K})$ *by*

$$V_p(\partial\widehat{K}) := \{v \in L^2(\partial\widehat{K}) \,|\, \exists \mathbf{v} \in \mathbf{V}_p(\widehat{K}) \text{ such that } \mathbf{n}_f \cdot \mathbf{v}|_f = v|_f \quad \forall f \in \mathcal{F}(\widehat{K})\}.$$

There exist $C > 0$*, which is independent of* p*, and, for each* $p \in \mathbb{N}_0$ *a lifting operator* $\boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d} : V_p(\partial\widehat{K}) \to \mathbf{V}_p(\widehat{K})$ *satisfying the following properties:*

(i) For each $f \in \mathcal{F}(\widehat{K})$ *and* $z \in V_p(\partial\widehat{K})$*, there holds* $\mathbf{n}_f \cdot \boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d} z = z|_f$.

(ii) There holds the stability estimate

$$\|\boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d} z\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \leq C \|z\|_{\widetilde{H}^{-1/2}(\partial\widehat{K})}.$$

(iii) There holds the orthogonality $(\boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d} z, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0$ *for all* $\mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K})$.

Proof. We mention that we need the space

$$\mathring{\mathbf{Q}}_{p,\perp}(\widehat{K}) = \{\mathbf{q} \in \mathring{\mathbf{Q}}_p(\widehat{K}) \colon (\mathbf{q}, \nabla\psi)_{L^2(\widehat{K})} = 0 \,\forall \psi \in \mathring{W}_{p+1}(\widehat{K})\}$$

that was first defined in Lemma 2.35.
Let now $z \in \widetilde{H}^{-1/2}(\partial\widehat{K})$ be a function with the property $z|_f \in V_p(f)$ for all faces $f \in \mathcal{F}(\widehat{K})$.
We define the lifting operator

$$\boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d} z := \boldsymbol{\mathcal{E}}^{\mathrm{div}} z - \mathbf{w}_0,$$

where $\boldsymbol{\mathcal{E}}^{\mathrm{div}} : H^{-1/2}(\partial\widehat{K}) \to \mathbf{H}(\widehat{K}, \mathrm{div})$ is the lifting operator from [32] and where $\mathbf{w}_0$ is defined by the following saddle point problem:
Find $\mathbf{w}_0 \in \mathring{\mathbf{V}}_p(\widehat{K})$ and $\boldsymbol{\varphi} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$ such that

$$(\mathrm{div}\,\mathbf{w}_0, \mathrm{div}\,\mathbf{v})_{L^2(\widehat{K})} + (\mathbf{v}, \mathbf{curl}\,\boldsymbol{\varphi})_{L^2(\widehat{K})} = (\mathrm{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z), \mathrm{div}\,\mathbf{v})_{L^2(\widehat{K})} \quad \forall \mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}) \tag{4.118a}$$

$$(\mathbf{w}_0, \mathbf{curl}\,\boldsymbol{\mu})_{L^2(\widehat{K})} = (\boldsymbol{\mathcal{E}}^{\mathrm{div}} z, \mathbf{curl}\,\boldsymbol{\mu})_{L^2(\widehat{K})} \quad \forall \boldsymbol{\mu} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K}). \tag{4.118b}$$

In order to show that the saddle point problem (4.118) is uniquely solvable, we again have to check coercivity and the inf-sup-condition:
With the bilinear forms $a(\mathbf{w}, \mathbf{q}) := (\mathrm{div}\,\mathbf{w}, \mathrm{div}\,\mathbf{q})_{L^2(\widehat{K})}$ and $b(\mathbf{w}, \boldsymbol{\varphi}) := (\mathbf{w}, \mathbf{curl}\,\boldsymbol{\varphi})_{L^2(\widehat{K})}$ for $\mathbf{w}, \mathbf{q} \in \mathring{\mathbf{V}}_p(\widehat{K})$ and $\boldsymbol{\varphi} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$, coercivity of a on

$$\ker b = \{\mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}) : (\mathbf{v}, \mathbf{curl}\,\boldsymbol{\mu})_{L^2(\widehat{K})} = 0 \,\forall \boldsymbol{\mu} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})\}$$

is a direct consequence of the Friedrichs inequality for the divergence operator (Lemma 2.36) by

$$a(\mathbf{v},\mathbf{v}) = \|\operatorname{div}\mathbf{v}\|^2_{L^2(\widehat{K})} \geq \frac{1}{2C^2}\|\mathbf{v}\|^2_{L^2(\widehat{K})} + \frac{1}{2}\|\operatorname{div}\mathbf{v}\|^2_{L^2(\widehat{K})} \geq \min\{\frac{1}{2C^2},\frac{1}{2}\}\|\mathbf{v}\|^2_{\mathbf{H}(\widehat{K},\mathrm{div})}$$

for all $\mathbf{v} \in \ker b$. For the validity of the inf-sup-condition

$$\inf_{\boldsymbol{\varphi}\in\mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})} \sup_{\mathbf{w}\in\mathring{\mathbf{V}}_p(\widehat{K})} \frac{b(\mathbf{w},\boldsymbol{\varphi})}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}} \geq C,$$

we choose $\mathbf{w} = \mathbf{curl}\,\boldsymbol{\varphi} \in \mathring{\mathbf{V}}_p(\widehat{K})$ for a given $\boldsymbol{\varphi} \in \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K})$. The Friedrichs inequality for the **curl**-operator (Lemma 2.35) then implies

$$\frac{b(\mathbf{w},\boldsymbol{\varphi})}{\|\mathbf{w}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}} = \frac{\|\mathbf{curl}\,\boldsymbol{\varphi}\|^2_{L^2(\widehat{K})}}{\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{L^2(\widehat{K})}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}} \overset{\text{Lem. 2.35}}{\geq} C.$$

Hence, the saddle point problem (4.118) has a unique solution

$$(\mathbf{w}_0,\boldsymbol{\varphi}) \in \mathring{\mathbf{V}}_p(\widehat{K}) \times \mathring{\mathbf{Q}}_{p,\perp}(\widehat{K}).$$

Choosing $\mathbf{v} = \mathbf{curl}\,\boldsymbol{\varphi}$ as test function in (4.118a) shows $\boldsymbol{\varphi} = 0$.
The lifting operator $\boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d}$ now obviously satisfies (iii) by construction, cf. (4.118b). Statement (i) also holds, since $\mathbf{w}_0 \cdot \mathbf{n}_f$ for each face $f \in \mathcal{F}(\widehat{K})$ and the operator $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$ has the desired polynomial preserving property, cf. [32, Theorem 7.1].
We now show (ii). Note that the solution $\mathbf{w}_0$ satisfies the estimate

$$\|\mathbf{w}_0\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \lesssim \|f\| + \|g\|, \tag{4.119}$$

where $f(\mathbf{v}) = (\operatorname{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z), \operatorname{div}\mathbf{v})_{L^2(\widehat{K})}$ and $g(v) = (\boldsymbol{\mathcal{E}}^{\mathrm{div}} z, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}$, and $\|\cdot\|$ denotes the operator norm. Thus, we have

$$\|f\| = \sup_{\|\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}\leq 1} |(\operatorname{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z), \operatorname{div}\mathbf{v})_{L^2(\widehat{K})}| \leq \|\operatorname{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z)\|_{L^2(\widehat{K})} \lesssim \|z\|_{\widetilde{H}^{-1/2}(\partial\widehat{K})}, \tag{4.120}$$

since the operator $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$ already satisfies the continuity property (ii). The estimate

$$\|g\| = \sup_{\|\mathbf{v}\|_{\mathbf{H}(\widehat{K},\mathbf{curl})}\leq 1} |(\boldsymbol{\mathcal{E}}^{\mathrm{div}} z, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})}| \leq \|\boldsymbol{\mathcal{E}}^{\mathrm{div}} z\|_{L^2(\widehat{K})} \lesssim \|z\|_{\widetilde{H}^{-1/2}(\partial\widehat{K})} \tag{4.121}$$

follows analogously. Hence, (ii) now follows with (4.119), (4.120) and (4.121) from

$$\|\boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d} z\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \leq \|\boldsymbol{\mathcal{E}}^{\mathrm{div}} z\|_{\mathbf{H}(\widehat{K},\mathrm{div})} + \|\mathbf{w}_0\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \lesssim \|z\|_{\widetilde{H}^{-1/2}(\partial\widehat{K})}.$$

□

This lifting operator now enables us to estimate the interpolation error in the $\mathbf{H}(\widehat{K},\mathrm{div})$-norm and afterwards in negative Sobolev norms.

Proposition 4.39. *Let* $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$*. Then there exists* $C > 0$ *independent of* $p \in \mathbb{N}$ *such that*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \leq Cp^{-1/2} \inf_{\mathbf{v}\in\mathbf{V}_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})} \tag{4.122}$$

holds.

Proof. We proceed in several steps.

Step 1: By the projection property of $\widehat{\Pi}_p^{\mathrm{div},3d}$, it is again sufficient to show the estimate for $\mathbf{v} = 0$.

Step 2: Since $\mathbf{u} \cdot \mathbf{n}_f \in L^2(f)$ on each face $f \in \mathcal{F}(\widehat{K})$, cf. Lemma 4.11, we get from Lemma 4.37

$$\|(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}) \cdot \mathbf{n}_f\|_{\widetilde{H}^{-1/2}(f)} \lesssim p^{-1/2}\|\mathbf{u} \cdot \mathbf{n}_f\|_{L^2(f)} \lesssim p^{-1/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}. \tag{4.123}$$

Step 3: We now estimate the error $\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}$.
With the best approximation operator $P^{\mathrm{div},3d}$ from Lemma 4.28, we define $\mathbf{E} := \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u} \in \mathbf{V}_p(\widehat{K})$. Note that $\widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}$ satisfies the orthogonality conditions (4.4c) and (4.4d), and $P^{\mathrm{div},3d}\mathbf{u}$ satisfies (4.67a) and (4.67b), thus we obtain the conditions

$$(\mathrm{div}\,\mathbf{E}, \mathrm{div}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathring{\mathbf{V}}_p(\widehat{K}), \tag{4.124a}$$

$$(\mathbf{E}, \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}). \tag{4.124b}$$

Since $\mathbf{E} - \boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n}) \in \mathring{\mathbf{V}}_p(\widehat{K})$ with the orthogonality condition

$$(\mathbf{E} - \boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n}), \mathbf{curl}\,\mathbf{v})_{L^2(\widehat{K})} = 0 \qquad \forall \mathbf{v} \in \mathring{\mathbf{Q}}_p(\widehat{K}),$$

cf. (4.124b) and Lemma 4.38, (iii), we obtain with the discrete Friedrichs inequality (Lemma 2.36)

$$\begin{aligned}\|\mathbf{E}\|_{L^2(\widehat{K})} &\leq \|\boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n})\|_{L^2(\widehat{K})} + \|\mathbf{E} - \boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n})\|_{L^2(\widehat{K})} \\ &\lesssim \|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})} + \|\,\mathrm{div}(\mathbf{E} - \boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n}))\|_{L^2(\widehat{K})} \\ &\lesssim \|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})} + \|\,\mathrm{div}\,\mathbf{E}\|_{L^2(\widehat{K})}.\end{aligned} \tag{4.125}$$

Equation (4.124a) and the stability property of the lifting (Lemma 4.38, (ii)) imply

$$\|\,\mathrm{div}\,\mathbf{E}\|^2_{L^2(\widehat{K})} = (\mathrm{div}\,\mathbf{E}, \mathrm{div}\,\boldsymbol{\mathcal{L}}_p^{\mathrm{div},3d}(\mathbf{E}\cdot\mathbf{n}))_{L^2(\widehat{K})} \lesssim \|\,\mathrm{div}\,\mathbf{E}\|_{L^2(\widehat{K})}\|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})}. \tag{4.126}$$

Hence, combining the estimates (4.125) and (4.126) yields

$$\|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} \lesssim \|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})}. \tag{4.127}$$

Step 4: We estimate the interpolation error in the $\mathbf{H}(\widehat{K}, \mathrm{div})$-norm.

This is now achieved by the triangle inequality, equation (4.127) together with the 2D-result (4.123), and the best approximation property of Lemma 4.28,

$$\begin{aligned}\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} &\leq \|\mathbf{u} - P^{\mathrm{div},3\mathrm{d}}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} + \|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}\\ &\overset{(4.127)}{\lesssim} \|\mathbf{u} - P^{\mathrm{div},3\mathrm{d}}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} + \|\mathbf{E}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\widehat{K})}\\ &\lesssim \|\mathbf{u} - P^{\mathrm{div},3d}\mathbf{u}\|_{\mathbf{H}(\widehat{K},\mathrm{div})} + \sum_{f\in\mathcal{F}(\widehat{K})} \|(\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u})\cdot\mathbf{n}_f\|_{\widetilde{H}^{-1/2}(f)}\\ &\overset{(4.123),\mathrm{Lem.}\ 4.28}{\lesssim} p^{-1/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}.\end{aligned}$$

□

Proposition 4.40. *Let* $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K},\mathrm{div})$*. Then, for* $s \in [0,1]$*, there exists* $C_s > 0$ *such that the estimate*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \leq C_s p^{-1/2-s} \inf_{\mathbf{v}\in\mathbf{V}_p(\widehat{K})} \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}$$

holds.

Proof. We proceed in several steps.

Step 1: Since $\widehat{\Pi}_p^{\mathrm{div},3d}$ is a projection, it again suffices to prove the result with $\mathbf{v} = 0$ in the infimum.

Step 2: In order to show the desired estimate for $s = 1$, we introduce the abbreviation $\mathbf{E} := \mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}$ and now have to estimate the norm

$$\|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-1}(\widehat{K},\mathrm{div})} \lesssim \|\mathbf{E}\|_{\widetilde{\mathbf{H}}^{-1}(\widehat{K})} + \|\operatorname{div}\mathbf{E}\|_{\widetilde{H}^{-1}(\widehat{K})} \tag{4.128}$$

$$= \sup_{\mathbf{v}\in\mathbf{H}^1(\widehat{K})} \frac{(\mathbf{E},\mathbf{v})_{L^2(\widehat{K})}}{\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}} + \sup_{v\in H^1(\widehat{K})} \frac{(\operatorname{div}\mathbf{E}, v)_{L^2(\widehat{K})}}{\|v\|_{H^1(\widehat{K})}}. \tag{4.129}$$

Step 3: We estimate the first supremum in (4.128).
According to Lemma 2.31, we write $\mathbf{v} \in \mathbf{H}^1(\widehat{K})$ as

$$\mathbf{v} = \nabla\varphi + \mathbf{curl}\,\mathbf{z}$$

with $\varphi \in H^2(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^1(\widehat{K},\mathbf{curl}) \cap \mathbf{H}_0(\widehat{K},\mathbf{curl})$ that satisfy the bound

$$\|\varphi\|_{H^2(\widehat{K})} + \|\mathbf{z}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \lesssim \|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.$$

We now have to handle the two terms in

$$(\mathbf{E},\mathbf{v})_{L^2(\widehat{K})} = (\mathbf{E},\mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})} + (\mathbf{E},\nabla\varphi)_{L^2(\widehat{K})} \tag{4.130}$$

separately. For the first term of (4.130), the orthogonality (4.4c) and Proposition 4.39 give

$$\begin{aligned}\left|(\mathbf{E}, \mathbf{curl}\,\mathbf{z})_{L^2(\widehat{K})}\right| &= \left|\inf_{\mathbf{w}\in\mathring{\mathbf{Q}}_p(\widehat{K})} (\mathbf{E}, \mathbf{curl}(\mathbf{z}-\mathbf{w}))_{L^2(\widehat{K})}\right| \lesssim p^{-1}\|\mathbf{E}\|_{L^2(\widehat{K})}\|\mathbf{z}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \\ &\lesssim p^{-3/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})},\end{aligned}$$

where the infimum is estimated as in Proposition 4.35, without repeating the arguments here. For the second term of (4.130), we use integration by parts to obtain

$$(\mathbf{E}, \nabla\varphi)_{L^2(\widehat{K})} = -(\operatorname{div}\mathbf{E}, \varphi)_{L^2(\widehat{K})} + \sum_{f\in\mathcal{F}(\widehat{K})} (\mathbf{E}\cdot\mathbf{n}_f, \varphi)_{L^2(f)}, \tag{4.131}$$

where the decomposition in the sum over the faces is allowed by sufficient regularity of $\mathbf{E}$ and φ, since $\mathbf{E}\cdot\mathbf{n}_f \in L^2(f)$ for each face $f \in \mathcal{F}(\widehat{K})$ by Lemma 4.11. If we denote by $\overline{\varphi} := (\int_{\widehat{K}} \varphi)/|\widehat{K}|$ the average of φ, integration by parts together with condition (4.4a) yields

$$(\operatorname{div}\mathbf{E}, \varphi)_{L^2(\widehat{K})} = (\operatorname{div}\mathbf{E}, \varphi-\overline{\varphi})_{L^2(\widehat{K})} + \overline{\varphi}(\mathbf{E}\cdot\mathbf{n}, 1)_{L^2(\partial\widehat{K})} = (\operatorname{div}\mathbf{E}, \varphi-\overline{\varphi})_{L^2(\widehat{K})}. \tag{4.132}$$

We then define the function $\psi \in H^2(\widehat{K})$ as the solution of the Neumann problem

$$\begin{aligned}\Delta\psi &= \varphi - \overline{\varphi}, \\ \partial_n\psi &= 0 \quad \text{on } \partial\widehat{K}\end{aligned}$$

and set $\mathbf{\Phi} := \nabla\psi \in \mathbf{H}^1(\widehat{K})$. Since $\operatorname{div}\mathbf{\Phi} = \Delta\psi = \varphi - \overline{\varphi} \in H^1(\widehat{K})$, it follows $\mathbf{\Phi} \in \mathbf{H}^1(\widehat{K}, \mathrm{div})$, and we get

$$\begin{aligned}\left|(\operatorname{div}\mathbf{E}, \varphi-\overline{\varphi})_{L^2(\widehat{K})}\right| &= \left|(\operatorname{div}\mathbf{E}, \operatorname{div}\mathbf{\Phi})_{L^2(\widehat{K})}\right| \\ &\overset{(4.4d)}{=} \left|\inf_{\mathbf{w}\in\mathring{\mathbf{V}}_p(\widehat{K})} (\operatorname{div}\mathbf{E}, \operatorname{div}(\mathbf{\Phi}-\mathbf{w}))_{L^2(\widehat{K})}\right| \lesssim p^{-1}\|\mathbf{E}\|_{\mathbf{H}(\widehat{K},\mathrm{div})}\|\mathbf{\Phi}\|_{\mathbf{H}^1(\widehat{K},\mathrm{div})} \\ &\lesssim p^{-3/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}\|\varphi\|_{H^1(\widehat{K})} \lesssim p^{-3/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.\end{aligned} \tag{4.133}$$

Note that the estimate of the infimum in (4.133) follows the same lines of argumentation as in Proposition 4.35. One simply has to take the $\mathbf{H}(\widehat{K}, \mathrm{div})$-lifting from Lemma 4.38 here. Thus, the estimates for the volume term on the right-hand side of (4.131) are complete. For estimates for the boundary terms, we use the orthogonality properties (4.4a) and (4.4b) as well as Lemma 2.23 and Lemma 4.37 to obtain

$$\begin{aligned}\left|(\mathbf{E}\cdot\mathbf{n}, \varphi)_{L^2(f)}\right| &= \left|\inf_{w\in V_p(f)} (\mathbf{E}\cdot\mathbf{n}, \varphi - w)_{L^2(f)}\right| \lesssim p^{-1}\|\mathbf{E}\cdot\mathbf{n}\|_{\widetilde{H}^{-1/2}(f)}\|\varphi\|_{H^{3/2}(f)} \\ &\lesssim p^{-3/2}\|\mathbf{u}\cdot\mathbf{n}\|_{L^2(f)}\|\varphi\|_{H^2(\widehat{K})} \lesssim p^{-3/2}\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}\|\mathbf{v}\|_{\mathbf{H}^1(\widehat{K})}.\end{aligned}$$

This finishes the estimate of the first term of (4.128).

Step 4: We estimate the second supremum in (4.128).

This estimate is completely analogous to (4.132) and (4.133). Note that the $H^2(\widehat{K})$-regularity of the function φ was not necessary in these equations, while $H^1(\widehat{K})$-regularity was indeed sufficient.

Step 5: The cases $s \in (0,1)$ now follow by interpolation between $s = 0$ (cf. Proposition 4.39) and $s = 1$. ☐

For functions $\mathbf{u}$ with discrete divergence is a polynomial, we have the following result similar to Lemma 4.36.

Lemma 4.41. *Let $\omega \in (0,\pi)$, and assume that all interior angles of the 4 faces of $\widehat{K}$ are smaller than ω. Then, for all $k \geq 1$ and for all $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ with $\operatorname{div}\mathbf{u} \in \mathcal{P}_p(\widehat{K})$, there holds*

$$\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \leq C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}, \qquad s \in [0, \min(\pi/\omega - 1/2, 1)]. \tag{4.134}$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

Proof. We imitate the proofs of Lemma 4.23 and Lemma 4.36. Using the right inverses $\mathbf{R}^{\mathrm{curl}}$ and $\mathbf{R}^{\mathrm{div}}$, Lemma 2.33 yields

$$\mathbf{u} = \mathbf{curl}\,\mathbf{R}^{\mathrm{curl}}(\mathbf{u} - \mathbf{R}^{\mathrm{div}}\operatorname{div}\mathbf{u}) + \mathbf{R}^{\mathrm{div}}\operatorname{div}\mathbf{u} =: \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z}$$

with $\boldsymbol{\varphi} \in \mathbf{H}^{k+1}(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^k(\widehat{K})$, together with the estimate

$$\|\boldsymbol{\varphi}\|_{\mathbf{H}^{k+1}(\widehat{K})} + \|\mathbf{z}\|_{\mathbf{H}^k(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})} + \|\operatorname{div}\mathbf{u}\|_{H^{k-1}(\widehat{K})} \leq C\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}. \tag{4.135}$$

Since we assumed $\operatorname{div}\mathbf{u} \in \mathcal{P}_p(\widehat{K})$, Lemma 2.27, (vi) implies $\mathbf{z} = \mathbf{R}^{\mathrm{div}}\operatorname{div}\mathbf{u} \in \mathbf{V}_p(\widehat{K})$. By the projection property of $\widehat{\Pi}_p^{\mathrm{div},3d}$, it follows $\mathbf{z} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{z} = 0$, hence, we get from the commuting diagram property $\mathbf{curl}\,\widehat{\Pi}_p^{\mathrm{curl},3d} = \widehat{\Pi}_p^{\mathrm{div},3d}\,\mathbf{curl}$, Lemma 2.23, Proposition 4.35 and (4.135)

$$\begin{aligned}
\|(\mathrm{I} - \widehat{\Pi}_p^{\mathrm{div},3d})\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} &= \|(\mathrm{I} - \widehat{\Pi}_p^{\mathrm{div},3d})\,\mathbf{curl}\,\boldsymbol{\varphi} + \underbrace{(\mathrm{I} - \widehat{\Pi}_p^{\mathrm{div},3d})\mathbf{z}}_{=0}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \\
&= \|\mathbf{curl}(\mathrm{I} - \widehat{\Pi}_p^{\mathrm{curl},3d})\boldsymbol{\varphi}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \leq \|(\mathrm{I} - \widehat{\Pi}_p^{\mathrm{curl},3d})\boldsymbol{\varphi}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \\
&\overset{\text{Prop. 4.35}}{\lesssim} p^{-(1+s)} \inf_{\mathbf{v} \in \mathbf{Q}_p(\widehat{K})} \|\boldsymbol{\varphi} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \lesssim p^{-(1+s)} \inf_{\mathbf{v} \in (\mathcal{P}_p(\widehat{K}))^3} \|\boldsymbol{\varphi} - \mathbf{v}\|_{\mathbf{H}^2(\widehat{K})} \\
&\overset{\text{Lemma 2.23}}{\lesssim} p^{-(k+s)}\|\boldsymbol{\varphi}\|_{\mathbf{H}^{k+1}(\widehat{K})} \overset{(4.135)}{\lesssim} p^{-(k+s)}\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}.
\end{aligned}$$

Note that the restrictions on s posed in (4.134) are necessary because of the application of Proposition 4.35.
The additional claim that we can replace the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$ is clear since the operator $\widehat{\Pi}_p^{\mathrm{div},3d}$ reproduces polynomials of degree p. ☐

4.7.4 The main results in 3D

The following theorem is the main result about the interpolation operators in three dimensions and has been fully proved in the previous subsections.

Theorem 4.42. *Let $\omega \in (0, \pi)$, and assume that all interior angles of the 4 faces of the reference tetrahedron $\widehat{K} \subseteq \mathbb{R}^3$ are smaller than ω. Define the value $s^\star := \min(\pi/\omega - 1/2, 1)$. Then there are constants C_s and $C_{s,k}$, depending only on s, k, and $\widehat{K}$, such that the following assertions hold:*

(i) The operators $\widehat{\Pi}^{\mathrm{grad},3d}_{p+1}$, $\widehat{\Pi}^{\mathrm{curl},3d}_{p}$, $\widehat{\Pi}^{\mathrm{div},3d}_{p}$, $\widehat{\Pi}^{L^2}_{p}$ are well-defined, projections, and the diagram (4.12) commutes.

(ii) For all $\varphi \in H^2(\widehat{K})$ there holds

$$\|\varphi - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1}\varphi\|_{H^{1-s}(\widehat{K})} \leq C_s p^{-(1+s)} \inf_{v \in W_{p+1}(\widehat{K})} \|\varphi - v\|_{H^2(\widehat{K})}, \qquad s \in [0, s^\star],$$

$$\|\nabla(\varphi - \widehat{\Pi}^{\mathrm{grad},3d}_{p+1}\varphi)\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K})} \leq C_s p^{-(1+s)} \inf_{v \in W_{p+1}(\widehat{K})} \|\varphi - v\|_{H^2(\widehat{K})}, \qquad s \in [0, s^\star].$$

(iii) For all $\mathbf{u} \in \mathbf{H}^1(\widehat{K}, \mathbf{curl})$ there holds

$$\|\mathbf{u} - \widehat{\Pi}^{\mathrm{curl},3d}_{p}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \leq C_s p^{-(1+s)} \inf_{\mathbf{v} \in \mathbf{Q}_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})}, \qquad s \in [0, s^\star].$$

(iv) For all $k \geq 1$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ with $\mathbf{curl}\,\mathbf{u} \in \mathbf{V}_p(\widehat{K}) \supseteq (\mathcal{P}_p(\widehat{K}))^3$ there holds

$$\|\mathbf{u} - \widehat{\Pi}^{\mathrm{curl},3d}_{p}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \leq C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}, \qquad s \in [0, s^\star].$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

(v) For all $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K}, \mathrm{div})$ there holds

$$\|\mathbf{u} - \widehat{\Pi}^{\mathrm{div},3d}_{p}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \leq C_s p^{-(1/2+s)} \inf_{\mathbf{v} \in \boldsymbol{V}_p(\widehat{K})} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})}, \qquad s \in [0, 1].$$

(vi) For all $k \geq 1$ and all $\mathbf{u} \in \mathbf{H}^k(\widehat{K})$ with $\mathrm{div}\,\mathbf{u} \in \mathcal{P}_p(\widehat{K})$ there holds

$$\|\mathbf{u} - \widehat{\Pi}^{\mathrm{div},3d}_{p}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \leq C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}, \qquad s \in [0, s^\star].$$

If $p \geq k-1$, then the full norm $\|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$ can be replaced with the seminorm $|\mathbf{u}|_{\mathbf{H}^k(\widehat{K})}$.

Proof. For the subjects stated in (i), see Lemma 4.9, Lemma 4.10, Lemma 4.11 and Theorem 4.14. Item (ii) is exactly Proposition 4.29, and (iii) is shown in Proposition 4.35. Assertion (iv) is seen in Lemma 4.36 and statement (v) in Proposition 4.40. Finally, (vi) is proven in Lemma 4.41. □

Remark 4.43. For a reference tetrahedron $\widehat{K}$, all statements in Theorem 4.42 hold for $s \in [0,1]$ which is the case for the most natural choices of reference tetrahedra, cf. Remark 4.30. We also mention that the case $s = 0$ is always admissible, completely independent of the angles of $\widehat{K}$.

If we start with a more regular function, then we obtain better approximation properties. The following result, which is the 3D-analog to Corollary 4.25, deals with this observation.

Corollary 4.44. *Using the notation of Theorem 4.42, the following statements hold for $k \geq 1$:*

$$\|\varphi - \widehat{\Pi}_{p+1}^{\mathrm{grad},3d}\varphi\|_{H^{1-s}(\widehat{K})} \leq C_{s,k} p^{-(k+s)} \|\varphi\|_{H^{k+1}(\widehat{K})}, \qquad s \in [0, \min(\pi/\omega - 1/2, 1)], \tag{4.136}$$

$$\|\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{curl},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} \leq C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^{k}(\widehat{K},\mathbf{curl})}, \qquad s \in [0, \min(\pi/\omega - 1/2, 1)], \tag{4.137}$$

$$\|\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{div},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \leq C_{s,k} p^{-(k+s)} \|\mathbf{u}\|_{\mathbf{H}^{k}(\widehat{K},\mathrm{div})}, \qquad s \in [0, 1]. \tag{4.138}$$

Proof. The estimate (4.136) follows directly from Theorem 4.42, (ii), together with the best approximation property of Lemma 2.23.
In order to show (4.137), we write $\mathbf{u} \in \mathbf{H}^k(\widehat{K}, \mathbf{curl})$ as

$$\mathbf{u} = \nabla\varphi + \mathbf{z}$$

with $\varphi \in H^{k+1}(\widehat{K})$, $\mathbf{z} \in \mathbf{H}^{k+1}(\widehat{K})$, and the bounds $\|\varphi\|_{H^{k+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathbf{curl})}$ and $\|\mathbf{z}\|_{\mathbf{H}^{k+1}(\widehat{K})} \lesssim \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^k(\widehat{K})}$, cf. Lemma 2.28. Thus, Theorem 4.42, (iii) and Lemma 2.23 yield

$$\begin{aligned}
\|\mathbf{u} - \widehat{\Pi}_{p}^{\mathrm{curl},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathbf{curl})} &\lesssim p^{-(1+s)} \inf_{\substack{v \in W_{p+1}(\widehat{K}),\\ \mathbf{q} \in \mathbf{Q}_p(\widehat{K})}} \|\nabla\varphi + \mathbf{z} - (\nabla v + \mathbf{q})\|_{\mathbf{H}^1(\widehat{K},\mathbf{curl})} \\
&\lesssim p^{-(1+s)} \left[\inf_{v \in W_{p+1}(\widehat{K})} \|\varphi - v\|_{H^2(\widehat{K})} + \inf_{\mathbf{q} \in \mathbf{Q}_p(\widehat{K})} \|\mathbf{z} - \mathbf{q}\|_{\mathbf{H}^2(\widehat{K})} \right] \\
&\overset{\text{Lemma 2.23}}{\lesssim} p^{-(1+s)-(k+1-2)} \left[\|\varphi\|_{H^{k+1}(\widehat{K})} + \|\mathbf{z}\|_{\mathbf{H}^{k+1}(\widehat{K})} \right] \lesssim p^{-(s+k)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathbf{curl})}.
\end{aligned}$$

For the estimate (4.138) we use the decomposition of $\mathbf{u} \in \mathbf{H}^k(\widehat{K}, \mathrm{div})$ in

$$\mathbf{u} = \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z},$$

where $\boldsymbol{\varphi} \in \mathbf{H}^{k+1}(\widehat{K})$ and $\mathbf{z} \in \mathbf{H}^{k+1}(\widehat{K})$ with the estimates $\|\boldsymbol{\varphi}\|_{\mathbf{H}^{k+1}(\widehat{K})} \lesssim \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathrm{div})}$ and $\|\mathbf{z}\|_{\mathbf{H}^{k+1}(\widehat{K})} \lesssim \|\,\mathrm{div}\,\mathbf{u}\|_{H^k(\widehat{K})}$ according to Lemma 2.33. Theorem 4.42, (v) and Lemma 2.23

then give us

$$
\begin{aligned}
&\|\mathbf{u} - \widehat{\Pi}_p^{\mathrm{div},3d}\mathbf{u}\|_{\widetilde{\mathbf{H}}^{-s}(\widehat{K},\mathrm{div})} \\
&\quad\lesssim p^{-(1/2+s)} \inf_{\mathbf{q}\in\mathbf{Q}_p(\widehat{K}),\mathbf{v}\in\mathbf{V}_p(\widehat{K})} \|\,\mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{z} - (\mathbf{curl}\,\mathbf{q} + \mathbf{v})\|_{\mathbf{H}^{1/2}(\widehat{K},\mathrm{div})} \\
&\quad\lesssim p^{-(1/2+s)} \left[\inf_{\mathbf{q}\in\mathbf{Q}_p(\widehat{K})} \|\boldsymbol{\varphi} - \mathbf{q}\|_{\mathbf{H}^{3/2}(\widehat{K})} + \inf_{\mathbf{v}\in\mathbf{V}_p(\widehat{K})} \|\mathbf{z} - \mathbf{v}\|_{\mathbf{H}^{3/2}(\widehat{K})} \right] \\
&\quad\overset{\text{Lemma 2.23}}{\lesssim} p^{-(s+k)} \|\mathbf{u}\|_{\mathbf{H}^k(\widehat{K},\mathrm{div})},
\end{aligned}
$$

which completes the proof. □

4.8 Extensions to finite elements of the second kind

In addition to the classical finite elements introduced in (2.31) which are called elements of the first kind, one can also use elements of the second kind, defined by full polynomial spaces as spaces of functions together with suitable degrees of freedom, cf. [26, Sec. 2], [55, Sec. 3] or [62, Sec. 4]. This gives (adapting the notation in order to support the notation in all previous chapters) the discrete spaces

$$
\begin{aligned}
W_{p+1}(\widehat{K}) &:= \mathcal{P}_{p+3}(\widehat{K}), \\
\mathbf{Q}_p(\widehat{K}) &:= (\mathcal{P}_{p+2}(\widehat{K}))^3, \\
\mathbf{V}_p(\widehat{K}) &:= (\mathcal{P}_{p+1}(\widehat{K}))^3
\end{aligned}
\tag{4.139}
$$

with the following exact sequence

$$
\mathbb{R} \xrightarrow{\mathrm{id}} W_{p+1}(\widehat{K}) \xrightarrow{\nabla} \mathbf{Q}_p(\widehat{K}) \xrightarrow{\mathbf{curl}} \mathbf{V}_p(\widehat{K}) \xrightarrow{\mathrm{div}} \mathcal{P}_p(\widehat{K}) \xrightarrow{0} \{0\},
\tag{4.140}
$$

cf. [26, eq. (45)]. We then define the interpolation operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, $\widehat{\Pi}_p^{\mathrm{curl},3d}$, $\widehat{\Pi}_p^{\mathrm{div},3d}$ and $\widehat{\Pi}_p^{L^2}$ analogously to Definitions 4.1 - 4.4. The same arguments as in Sections 4.2 - 4.4 show that they are well-defined and that the diagram

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\mathrm{id}} & H^2(\widehat{K}) & \xrightarrow{\nabla} & \mathbf{H}^1(\widehat{K},\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \mathbf{H}^1(\widehat{K},\mathrm{div}) & \xrightarrow{\mathrm{div}} & H^1(\widehat{K}) & \xrightarrow{0} & \{0\} \\
 & & \big\downarrow \widehat{\Pi}_{p+1}^{\mathrm{grad},3d} & & \big\downarrow \widehat{\Pi}_p^{\mathrm{curl},3d} & & \big\downarrow \widehat{\Pi}_p^{\mathrm{div},3d} & & \big\downarrow \widehat{\Pi}_p^{L^2} & & \\
\mathbb{R} & \xrightarrow{\mathrm{id}} & W_{p+1}(\widehat{K}) & \xrightarrow{\nabla} & \mathbf{Q}_p(\widehat{K}) & \xrightarrow{\mathbf{curl}} & \mathbf{V}_p(\widehat{K}) & \xrightarrow{\mathrm{div}} & \mathcal{P}_p(\widehat{K}) & \xrightarrow{0} & \{0\}
\end{array}
\tag{4.141}
$$

commutes. To get a bit more into details, the dimension arguments from Section 4.3 still hold by counting, and only the exact sequence property is relevant for the commuting diagram property.
Anyway, all results about the interpolation operators shown for the elements of the first kind, can be completely repeated for the elements of the second kind. This is based on the fact that the concrete form of the discrete spaces is mostly irrelevant:

- Many arguments are based on the commuting diagram property or the discrete spaces building an exact sequence. This holds true for the elements of the first kind as well as for the elements of the second kind.
- We used the best approximation operators $P^{\mathrm{grad},3d}$, $P^{\mathrm{curl},3d}$ and $P^{\mathrm{div},3d}$, introduced in Lemmas 4.26, 4.27 and 4.28, for the interpolation error estimates. They map in the corresponding discrete spaces and satisfy orthogonality conditions, which still holds for elements of the second kind, cf. [26, Thm. 5.2] and the lines at the beginning of [26, Sec. 5.1].
- Another main ingredient of our studies are the right-inverse operators R^{grad}, $\mathbf{R}^{\mathrm{curl}}$ and $\mathbf{R}^{\mathrm{div}}$ from [19], presented in our work in Lemma 2.27. As is seen in [19, Sec. 4.2], the properties stated in Lemma 2.27, (iv), (v), (vi) only rely on the polynomial spaces forming an exact sequence and thus hold for elements of the second kind.
- We sometimes made use of continuous, polynomial-preserving liftings from the boundary. Since $W_{p+1}(\widehat{K}) = \mathcal{P}_{p+1}(\widehat{K})$ for elements of both first and second kind, and since the liftings $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ and $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$ are even polynomial-preserving for full polynomial spaces, cf. [31, Thm. 7.2], [32, Thm. 7.1], the arguments concerning the liftings also work for elements of the second kind.
- Since the discrete Friedrichs inequalities presented in Section 2.6 rely more or less on the right-inverse operators, they still hold in the current setting, cf. the proofs in [26, Sec. 5].

Thus, our main result Theorem 4.42 holds for elements of the second kind.

Theorem 4.45. *Define the interpolation operators* $\widehat{\Pi}_{p+1}^{\mathrm{grad},3d}$, $\widehat{\Pi}_{p}^{\mathrm{curl},3d}$ *and* $\widehat{\Pi}_{p}^{\mathrm{div},3d}$ *analogously to* (4.2) - (4.4), *corresponding to the discrete spaces* $W_{p+1}(\widehat{K})$, $\mathbf{Q}_p(\widehat{K})$ *and* $\mathbf{V}_p(\widehat{K})$ *defined in* (4.139). *Then, under the hypotheses of Theorem 4.42, all assertions of Theorem 4.42 are valid in the setting of elements of the second kind.*

In 2D, the elements of the second kind are also defined by full polynomial spaces as the function spaces, i.e.

$$\begin{aligned} W_{p+1}(\widehat{f}) &:= \mathcal{P}_{p+2}(\widehat{f}), \\ \mathbf{Q}_p(\widehat{f}) &:= (\mathcal{P}_{p+1}(\widehat{f}))^2 \end{aligned} \tag{4.142}$$

with the exact sequence

$$\mathbb{R} \xrightarrow{\mathrm{id}} W_{p+1}(\widehat{f}) \xrightarrow{\nabla} \mathbf{Q}_p(\widehat{f}) \xrightarrow{\mathbf{curl}} \mathcal{P}_p(\widehat{f}) \xrightarrow{0} \{0\}, \tag{4.143}$$

cf. [26, eq. (22)]. If we define the interpolation operators $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$, $\widehat{\Pi}_{p}^{\mathrm{curl},2d}$ and $\widehat{\Pi}_{p}^{L^2}$ analogously to Definitions 4.5 - 4.7, then the diagram

$$\begin{array}{ccccccccc} \mathbb{R} & \xrightarrow{\mathrm{id}} & H^{3/2}(\widehat{f}) & \xrightarrow{\nabla} & \mathbf{H}^{1/2}(\widehat{f},\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H^{1/2}(\widehat{f}) & \xrightarrow{0} & \{0\} \\ & & \big\downarrow \widehat{\Pi}_{p+1}^{\mathrm{grad},2d} & & \big\downarrow \widehat{\Pi}_{p}^{\mathrm{curl},2d} & & \big\downarrow \widehat{\Pi}_{p}^{L^2} & & \\ \mathbb{R} & \xrightarrow{\mathrm{id}} & W_{p+1}(\widehat{f}) & \xrightarrow{\nabla} & \mathbf{Q}_p(\widehat{f}) & \xrightarrow{\mathrm{curl}} & \mathcal{P}_p(\widehat{f}) & \xrightarrow{0} & \{0\} \end{array} \tag{4.144}$$

commutes.
Similar lines of argumentation as in 3D now work for the case of two spatial dimensions. Hence, Theorem 4.24 holds true for 2D-elements of the second kind.

Theorem 4.46. *Define the interpolation operators* $\widehat{\Pi}_{p+1}^{\mathrm{grad},2d}$ *and* $\widehat{\Pi}_{p}^{\mathrm{curl},2d}$ *analogously to* (4.6) - (4.7), *corresponding to the discrete spaces* $W_{p+1}(\widehat{K})$ *and* $\mathbf{Q}_p(\widehat{K})$ *defined in* (4.142). *Then, under the hypotheses of Theorem 4.24, all assertions of Theorem 4.24 are valid in the setting of elements of the second kind.*

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