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The Realizability of Revision and Contraction Operators in Epistemic Spaces

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Abstract

This paper studies the realizability of belief revision and belief contraction operators in epistemic spaces. We observe that AGM revision and AGM contraction operators for epistemic spaces are only realizable in precisely determined epistemic spaces. We define the class of linear change operators, which are a special kind of maxichoice operators. When AGM revision, respectively, AGM contraction, is realizable, linear change operators are a canonical realization.

1 Introduction

The area of belief change deals with the adaption of agents' beliefs in the light of new information from a fundamental perspective. A general novel framework for considering belief change is the framework of belief change operators for epistemic spaces (Schwind, Konieczny, and Pino Pérez 2022). In this framework, belief change happens on a specific defined set of epistemic states. The belief change operators are global mathematical objects that define how each state changes in the light of arbitrary new information. An advantage of this framework is that it recognizes the varying nature of epistemic states of individual agents, which is given, e.g., due to the individual capabilities of the agents. Many other frameworks do not allow such a differentiation, because they (implicitly) assume that all belief sets are available.

In this paper, we study the *realizability* of belief change operators: the question of whether there exists a belief change operator that satisfies a given set of postulates under a given set of assumptions. This problem been addressed for many types of belief change operators in various frameworks (Fermé and Hansson 2018; Falakh, Rudolph, and Sauerwald 2022; Ribeiro et al. 2013) and are an ongoing research subject, but has been not considered for the expressive framework of epistemic spaces.

The generality of the framework for epistemic spaces and the global nature of belief change operators in this framework has consequences for the problem of realizability. To check realizability, one must consider all belief changes on all epistemic states (available in the considered epistemic space) together and decide if all those belief changes do not interfere with each other. This is different to deliberations about the realizability of belief change operators considered in many other frameworks, where a realizability is about the changes in one particular belief set or sequences of changes.

In the following, we list the main content of this paper, which also includes the main contributions of this submission:

- [Non-Existence] Our first observation is that AGM revision and AGM contraction operators (Alchourrón, Gärdenfors, and Makinson 1985) for epistemic states (Darwiche and Pearl 1997) do not exist in all epistemic spaces. Furthermore, the corresponding epistemic spaces do not coincide.
- [Realizability of Revision and Contraction] A precise characterization of those epistemic spaces is given, where any AGM revision operators, respectively, any AGM contraction operators, exist at all.
- [Realizability of Operators] We consider then realizability of concrete belief revision and contraction operator approaches: full meet, maxichoice (Alchourron and Makinson 1982) and newly defined linear change operators. For contraction, we obtain the realizability of all these operators coincides with realizability of contraction. When considering realizability of revision operators, we observe that realizability of general revision, linear revision and maxichoice revision coincide. For full meet revision the realizibility is different.
- [Non-Interdefinability of Revision and Contraction] A consequence of our investigation is that there are epistemic space where one of AGM revision and AGM contraction is realizable and the other type of operator not. This shows that in some epistemic spaces revision and contraction are not interdefinable, which is an extension to a result obtained by Konieczny and Pérez (2017).

We have proofs for all results given in this paper, but, due to space limitations, the proofs are not presented here[1].

2 Propositional Logic, Minima and Orders

Let $\Sigma = \{a, b, c, \ldots\}$ be a finite set of propositional atoms and let $\mathcal{L}$ be a propositional language over Σ. The set of propositional interpretations is denoted by Ω. Propositional entailment is denoted by $\models$ and the set of models of α with $[\![\alpha]\!]$. For $L \subseteq \mathcal{L}$, entailment and model sets are defined as usual, i.e., $[\![L]\!] = \bigcap_{\alpha \in L} [\![\alpha]\!]$ and $L \models \beta$ if for all $\alpha \in L$ holds $\alpha \models \beta$. We define $Cn(L) = \{\beta \mid L \models \beta\}$ and we

[1]The proofs can be found in the supplementary material of the arXiv version of this paper (Sauerwald and Thimm 2024).

define $L + \alpha = \mathrm{Cn}(L \cup \{\alpha\})$. Furthermore, L is called deductively closed if $L = \mathrm{Cn}(L)$ and $\mathcal{L}^{\mathrm{Bel}}$ is the set of all deductively closed sets. Given $\Omega' \subseteq \Omega$ and a total preorder $\leq \subseteq \Omega \times \Omega$ (total and transitive relation), we denote with $\min(\Omega', \leq) = \{\omega \in \Omega' \mid \omega \leq \omega' \text{ for all } \omega' \in \Omega'\}$ the set of all minimal interpretations of Ω' with respect to $\leq$. A linear order $\ll \subseteq \Omega \times \Omega$ is a total preorder that is antisymmetric.

3 Epistemic Spaces, Revision and Contraction

We consider the background on epistemic spaces, AGM revision and AGM contraction. In this work, we model agents by the means of logic. Deductive closed sets of formulas, which we denote from now as *belief set*, represent deductive capabilities; agents are assumed to be perfect reasoners. The interpretations represent worlds that the agent is capable to imagine. The following notion describes the space of epistemic possibilities of an agent's mind in a general way.

Definition 1 (Schwind, Konieczny, and Pino Pérez 2022; adapted)**.** *A tuple* $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ *is called an* epistemic space *if* $\mathcal{E}$ *is a non-empty set and* $\mathrm{Bel} : \mathcal{E} \to \mathcal{L}^{\mathrm{Bel}}$.

We call the elements of $\mathcal{E}$ *epistemic states* and use $[\![\Psi]\!]$ as shorthand for $[\![\mathrm{Bel}(\Psi)]\!]$. Definition 1 differs from the definition given by Schwind, Konieczny, and Pino Pérez (2022) insofar that we do *not* exclude inconsistent belief sets and forbid emptiness of $\mathcal{E}$. Belief change operators for an epistemic space $\mathbb{E}$ are global objects, functions on all epistemic states in the mathematical sense.

Definition 2. *Let* $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ *be an epistemic space. A* belief change operator for $\mathbb{E}$ *is a function* $\circ : \mathcal{E} \times \mathcal{L} \to \mathcal{E}$.

In general, operators from Definition 2 could behave arbitrarily, and, of course, for considering revision and contraction, we have to add additional constraints, which we will consider in the following.

AGM Belief Revision for Epistemic Spaces. Revision operators incorporating new beliefs into an agent's belief set, consistently, whenever this is possible. We use an adaption of the AGM postulates for revision (Alchourrón, Gärdenfors, and Makinson 1985) for epistemic states (Darwiche and Pearl 1997), which is inspired by the approach of Katsuno and Mendelzon (1992).

Definition 3. *Let* $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ *be an epistemic space. A belief change operator* $\star$ *for* $\mathbb{E}$ *is called an* (AGM) revision operator for $\mathbb{E}$ *if the following postulates are satisfied (Darwiche and Pearl 1997):*

(R1) $\alpha \in \mathrm{Bel}(\Psi \star \alpha)$

(R2) $\mathrm{Bel}(\Psi \star \alpha) = \mathrm{Bel}(\Psi) + \alpha$ *if* $\mathrm{Bel}(\Psi) + \alpha$ *is consistent*

(R3) *If* α *is consistent, then* $\mathrm{Bel}(\Psi \star \alpha)$ *is consistent*

(R4) *If* $\alpha \equiv \beta$*, then* $\mathrm{Bel}(\Psi \star \alpha) = \mathrm{Bel}(\Psi \star \beta)$

(R5) $\mathrm{Bel}(\Psi \star (\alpha \wedge \beta)) \subseteq \mathrm{Bel}(\Psi \star \alpha) + \beta$

(R6) *If* $\mathrm{Bel}(\Psi \star \alpha) + \beta$ *is consistent,*
then $\mathrm{Bel}(\Psi \star \alpha) + \beta \subseteq \mathrm{Bel}(\Psi \star (\alpha \wedge \beta))$

AGM revision is well-known for aiming at establishing a minimal change of the prior beliefs when revising. This is carried mainly by (R2) with respect to the beliefs of an agent. Moreover, (R5) and (R6) are postulates about rational choice that ensure relational minimal change (Alchourrón, Gärdenfors, and Makinson 1985; Katsuno and Mendelzon 1992; Fermé and Hansson 2018); an interpretation that is also used in other areas (Sen 1971). For a detailed discussion of the postulates (R1)–(R6) we refer to Gärdenfors (1988). In the remaining parts of this paper, we sometimes write *revision operator* instead of *AGM revision operator*.

AGM Belief Contraction for Epistemic Spaces. Contraction is the process of withdrawing beliefs, without adding new beliefs. Postulates for AGM contraction (Alchourrón, Gärdenfors, and Makinson 1985) where adapted by Caridroit, Konieczny, and Marquis (2015) to the setting of propositional logic. An adapted version of these postulates for AGM contraction operators for epistemic spaces from Konieczny and Pérez (2017) is given in the following.

Definition 4 (Adapted, Konieczny and Pérez, 2017)**.** *Let* $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ *be an epistemic space. A belief change operator* $\div$ *for* $\mathbb{E}$ *is called an* (AGM) contraction operator for $\mathbb{E}$ *if the following postulates are satisfied:*

(C1) $\mathrm{Bel}(\Psi \div \alpha) \subseteq \mathrm{Bel}(\Psi)$

(C2) *If* $\alpha \notin \mathrm{Bel}(\Psi)$*, then* $\mathrm{Bel}(\Psi) \subseteq \mathrm{Bel}(\Psi \div \alpha)$

(C3) *If* $\alpha \not\equiv \top$*, then* $\alpha \notin \mathrm{Bel}(\Psi \div \alpha)$

(C4) $\mathrm{Bel}(\Psi) \subseteq \mathrm{Bel}(\Psi \div \alpha) + \alpha$

(C5) *If* $\alpha \equiv \beta$*, then* $\mathrm{Bel}(\Psi \div \alpha) = \mathrm{Bel}(\Psi \div \beta)$

(C6) $\mathrm{Bel}(\Psi \div \alpha) \cap \mathrm{Bel}(\Psi \div \beta) \subseteq \mathrm{Bel}(\Psi \div (\alpha \wedge \beta))$

(C7) *If* $\beta \notin \mathrm{Bel}(\Psi \div (\alpha \wedge \beta))$*,*
then $\mathrm{Bel}(\Psi \div (\alpha \wedge \beta)) \subseteq \mathrm{Bel}(\Psi \div \beta)$

As in the case of revision, AGM contraction is well-known for aiming at establishing a minimal change of the prior beliefs when contracting, which is carried mainly by (C1), (C2), (C6) and (C7). Again, we refer to Gärdenfors (1988) for a detailed discussion of the postulates (C1)–(C7). In the remaining parts of this paper we sometimes write *contraction operator* instead of *AGM contraction operator*.

4 Realizability of Contraction Operators

We define realizability of AGM contraction as notion that captures whether there is any contraction operator.

Definition 5 (Contraction Realizability)**.** *We say that* AGM contraction is realizable *in an epistemic space* $\mathbb{E}$*, if there exists an AGM contraction operator for* $\mathbb{E}$.

The following proposition fully captures those epistemic spaces $\mathbb{E}$ for which an AGM contraction operator for $\mathbb{E}$ exists.

Theorem 6. *Let* $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ *be an epistemic space. AGM contraction is realizable in* $\mathbb{E}$ *if and only if* $\mathbb{E}$ *satisfies:*

(ZC) *For each* $\Psi \in \mathcal{E}$ *and for each* $M \subseteq \Omega$ *with* $[\![\Psi]\!] \subseteq M$
there exists $\Psi_M \in \mathcal{E}$ *such that* $[\![\Psi_M]\!] = M$.

The postulate (ZC) describes that agents are always capable of being more undecided about the state of the world. Clearly, Theorem 6 implies that contraction operators do not

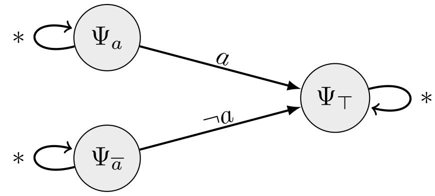

Figure 1: Graphical representation of the contraction operator $\div$ for $\mathbb{E}$ given in Example 9. Nodes are the epistemic states of $\mathbb{E}$. Edges represent the behaviour of $\div$; there is an edge from Ψ_1 to Ψ_2 with label x if $x \equiv \alpha$ implies $\Psi_2 = \Psi_1 \div \alpha$ hold. We use $*$ as placeholder label that stand for all x that are not explicitly mentioned.

exist in every epistemic space. For specific types of contraction operators, the realizability could be even more strict. To that end, we now define some basic types of contraction operators for epistemic states that are inspired by classical approaches to contraction (Hansson 1999).

Definition 7. *Let $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ be an epistemic space and let $\div$ be a contraction operator for $\mathbb{E}$. We say $\div$ is a* full meet contraction operator for $\mathbb{E}$ *if for all $\Psi \in \mathcal{E}$ and $\alpha \in \mathcal{L}$ holds:*

$$[\![\Psi \div \alpha]\!] = \begin{cases} [\![\Psi]\!] & \text{if } [\![\Psi]\!] \cap [\![\neg\alpha]\!] \neq \emptyset \\ [\![\Psi]\!] \cup [\![\neg\alpha]\!] & \text{otherwise} \end{cases}$$

We say $\div$ is a maxichoice contraction operator for $\mathbb{E}$ *if for each $\Psi \in \mathcal{E}$ exists a linear order $\ll_\Psi$ such that for all $\alpha \in \mathcal{L}$ holds:*

$$[\![\Psi \div \alpha]\!] = \begin{cases} [\![\Psi]\!] & \text{if } [\![\Psi]\!] \cap [\![\neg\alpha]\!] \neq \emptyset \\ [\![\Psi]\!] \cup \min([\![\neg\alpha]\!], \ll_\Psi) & \text{otherwise} \end{cases}$$

Note that presupposing existence of $\div$ in the beginning of Definition 7, guarantees implicitly the existence of the epistemic states used by $\div$, as otherwise, $\div$ would not exist. Hence, in Definition 7, we do not have to deal with the question of realizability.

For a maxichoice contraction operators for epistemic states, for each epistemic state Ψ, the linear order $\ll_\Psi$ could be different. We denote maxichoice contraction operators that are based, globally, on a single linear order as *linear contraction operators*:

Definition 8. *Let $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ be an epistemic space. A $\div$ contraction operator for $\mathbb{E}$ is called a* linear contraction operator for $\mathbb{E}$ *if $\div$ is a maxichoice contraction and there is a linear order $\ll$ on Ω such that for all $\Psi \in \mathcal{E}$ holds $\ll_\Psi = \ll$ for $\ll_\Psi$ from Definition 7.*

Next, we consider an example.

Example 9. *Let $\Sigma = \{a\}$ and thus $\Omega = \{a, \overline{a}\}$. Consider the following epistemic space $\mathbb{E}^{\mathrm{E9}} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ with $\mathcal{E} = \{\Psi_a, \Psi_{\overline{a}}, \Psi_\top\}$ and Bel is given by:*

$$[\![\Psi_a]\!] = \{a\} \qquad [\![\Psi_{\overline{a}}]\!] = \{\overline{a}\} \qquad [\![\Psi_\top]\!] = \Omega$$

The set $\mathcal{E}$ contains only epistemic states for which the beliefs are consistent, i.e., $[\![\Psi]\!] \neq \emptyset$ for all $\Psi \in \mathcal{E}$. Let $\div$ be a belief change operator for $\mathbb{E}^{\mathrm{E9}}$ given as follows for all $\Psi \in \mathcal{E}$ and $\alpha \in \mathcal{L}$:

$$\Psi \div \alpha = \begin{cases} \Psi_\top & \text{if } \alpha \in \mathrm{Bel}(\Psi) \\ \Psi & \text{otherwise} \end{cases}$$

For a graphical representation of $\div$ consider Figure 1. One can check that $\div$ is a linear contraction operator, because the linear order $\ll$ given by $a \ll \overline{a}$ gives rise to this operator.

The following proposition points out that, when AGM contraction is realizable, i.e., there is an AGM contraction operators, then there is also an instance of all the concrete contraction operators considered here.

Proposition 10. *Let $\mathbb{E}$ be an epistemic space. The following statements are equivalent:*

(I) AGM contraction is realizable in $\mathbb{E}$.
(II) There exists a linear contraction operator $\div$ for $\mathbb{E}$.
(III) There exists a maxichoice contraction operator $\div$ for $\mathbb{E}$.
(IV) There exists a full meet contraction operator $\div$ for $\mathbb{E}$.

In the next section, we will see that for revision the situation is much more complex, as realizablity does not carry over between different types of revision operators.

5 Realizability of Revision Operators

Now, we will determine the conditions of realizability of revision operators and specific types of revision operators. Realizability of AGM revision is defined analogously to Definition 5.

Definition 11 (Revision Realizability)**.** *We say that* AGM revision is realizable *in an epistemic space $\mathbb{E}$, if there exists an AGM contraction operator for $\mathbb{E}$.*

The next theorem characterizes exactly those epistemic spaces that permit existence of revision operators.

Theorem 12. *Let $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ be an epistemic space. AGM revision is realizable in $\mathbb{E}$ if and only if $\mathbb{E}$ satisfies:*

(ZR1) *For all $\omega \in \Omega$ there is some $\Psi_\omega \in \mathcal{E}$ with $[\![\Psi_\omega]\!] = \{\omega\}$.*
(ZR2) *For all $\Psi \in \mathcal{E}$ and for all $M \subseteq [\![\Psi]\!]$ there exists $\Psi_M \in \mathcal{E}$ with $[\![\Psi_M]\!] = M$.*

The postulates (ZR1) and (ZR2) characterize epistemic spaces which have revision operators. Figuratively, (ZR1) describes that the agent is capable of imagining that the "real world" is as described by exactly one interpretation; (ZR2) describes that if an agent can consider a set of interpretations as possible, then this is true for every subset thereof. A consequence of Theorem 12 is that there are epistemic spaces where no revision operator exist at all.

As for contraction, we define types of revision operators, for which we will also consider the realizability.

Definition 13. *Let $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ be an epistemic space and let $\star$ be a revision operator for $\mathbb{E}$. We say $\star$ is a* full meet revision operator for $\mathbb{E}$ *if for each $\Psi \in \mathcal{E}$ and $\alpha \in \mathcal{L}$ holds:*

$$[\![\Psi \star \alpha]\!] = \begin{cases} [\![\Psi]\!] \cap [\![\alpha]\!] & \text{if } [\![\Psi]\!] \cap [\![\alpha]\!] \neq \emptyset \\ [\![\alpha]\!] & \text{otherwise} \end{cases}$$

We say $\star$ is a maxichoice revision operator for $\mathbb{E}$ *if for each $\Psi \in \mathcal{E}$ exists a linear order $\ll_\Psi$ such that for all $\alpha \in \mathcal{L}$ holds:*

$$[\![\Psi \star \alpha]\!] = \begin{cases} [\![\Psi]\!] \cap [\![\alpha]\!] & \text{if } [\![\Psi]\!] \cap [\![\alpha]\!] \neq \emptyset \\ \min([\![\alpha]\!], \ll_\Psi) & \text{otherwise} \end{cases}$$

We denote maxichoice revision operators that are based, globally, on a single linear order as *linear revision operators*.

Definition 14. *Let $\mathbb{E} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ be an epistemic space. A $\star$ revision operator for $\mathbb{E}$ is called a* linear revision operator for $\mathbb{E}$ *if $\star$ is a maxichoice revision and there is a linear order $\ll$ on Ω such that for all $\Psi \in \mathcal{E}$ holds $\ll_\Psi = \ll$ for $\ll_\Psi$ from Definition 13.*

We consider an example epistemic space with a maxichoice revision operator, yet no full meet revision operators.

Example 15. *Let $\Sigma = \{a, b\}$ and thus $\Omega = \{ab, \overline{a}b, a\overline{b}, \overline{a}\overline{b}\}$. Consider the following epistemic space $\mathbb{E}^{\mathrm{E15}} = \langle \mathcal{E}, \mathrm{Bel} \rangle$ with $\mathcal{E} = \{\Psi_\omega \mid \omega \in \Omega\} \cup \{\Psi_\perp\}$ and Bel is given by:*

$$[\![\Psi_\omega]\!] = \{\omega\} \qquad\qquad [\![\Psi_\perp]\!] = \emptyset$$

Let $\ll_1$ and $\ll_2$ be two linear orders on Ω given by:

$$ab \ll_1 a\overline{b} \ll_1 \overline{a}b \ll_1 \overline{a}\overline{b} \qquad ab \ll_2 \overline{a}b \ll_2 a\overline{b} \ll_2 \overline{a}\overline{b}$$

We employ $\ll_1$ and $\ll_2$ to define a maxichoice revision operator $\star$ for $\mathbb{E}$ given as follows for all $\omega \in \Omega$ and $\alpha \in \mathcal{L}$:

$$\Psi_\omega \star \alpha = \begin{cases} \Psi_\perp & \text{if } [\![\alpha]\!] = \emptyset \\ \Psi_\omega & \text{if } \omega \in [\![\alpha]\!] \\ \Psi_{\min([\![\alpha]\!], \ll_1)} & \text{otherwise} \end{cases}$$

$$\Psi_\perp \star \alpha = \begin{cases} \Psi_\perp & \text{if } [\![\alpha]\!] = \emptyset \\ \Psi_{\min([\![\alpha]\!], \ll_2)} & \text{otherwise} \end{cases}$$

For a graphical representation of $\star$ consider Figure 2. To see that there is no full meet revision operator for $\mathbb{E}^{\mathrm{E15}}$, we consider the epistemic state $\Psi_\perp$ and the formula $\alpha = b$. If $\star_{\mathrm{fm}}$ were a full meet revision operator for $\mathbb{E}^{\mathrm{E15}}$, then we would obtain $[\![\Psi_\perp \star_{\mathrm{fm}} \alpha]\!] = \{ab, \overline{a}b\}$. However, this is impossible as there is no epistemic state $\Psi \in \mathcal{E}$ with $[\![\Psi]\!] = \{ab, \overline{a}b\}$.

Example 15 is a witness for the following observation.

Proposition 16. *AGM revision is realizable in $\mathbb{E}^{\mathrm{E15}}$, yet there exist no full meet revision for $\mathbb{E}^{\mathrm{E15}}$.*

The next proposition exactly determines the realizability of linear revision operators, maxichoice revision operators, and of full meet revision operators.

Proposition 17. *Let $\mathbb{E}$ be an epistemic space. The following statements hold:*

(I) *AGM revision is realizable in $\mathbb{E}$ if and only if there exists a linear revision operator $\star$ for $\mathbb{E}$.*

(II) *AGM revision is realizable in $\mathbb{E}$ if and only if there exists a maxichoice revision operator $\star$ for $\mathbb{E}$.*

(III) *There exists a full meet revision operator $\star$ for $\mathbb{E}$ if and only if the following postulate is satisfied by $\mathbb{E}$:*

(Unbiased) *For all belief sets $B \subseteq \mathcal{L}$ there exists an epistemic state $\Psi_B \in \mathcal{E}$ with* $\mathrm{Bel}(\Psi_B) = B$.

Due to Proposition 17 (I), linear revision operators are a canonical type of revision operators. Clearly, because linear revision operators are maxichoice revision operators, we obtain Proposition 17 (II) from Proposition 17 (I). Moreover, Proposition 17 (III) shows that full meet revision operators exist only when the agent is able to believe in any belief sets.

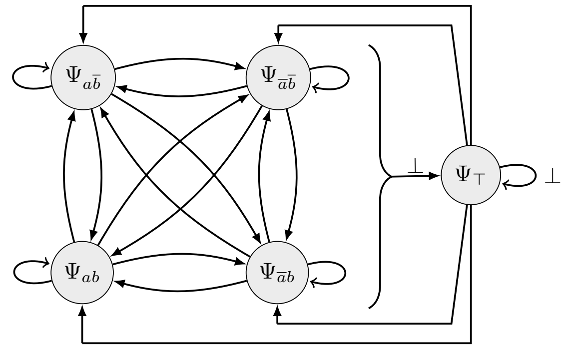

Figure 2: Graphical representation of the revision operator $\star$ for $\mathbb{E}$ given in Example 15. Due to space reasons we omitted most labels.

6 Consequences for the Interdefinability of Revision and Contraction

In the classical AGM belief change framework, revision and contraction are interdefinable (Fermé and Hansson 2018), via the Levi-identity (Levi 1977) and Harper-identity (Harper 1976). As observed at first by Konieczny and Pérez (2017), revision and contraction operators over epistemic spaces are not interdefinable in general.

Proposition 18 (Konieczny and Pérez, 2017)**.** *There is an epistemic space $\mathbb{E}$ such that there exist more*[2] *revision operators for $\mathbb{E}$ than contraction operators for $\mathbb{E}$.*

The results of this paper allow further remarks to the matter of interdefinability of revision and contraction. First, observe that $\mathbb{E}^{\mathrm{E15}}$ from Example 15 contains the epistemic state Ψ_{ab} with $[\![\Psi_{ab}]\!] = \{ab\}$, yet there is *no* epistemic state $\Psi \in \mathcal{E}$ such that $[\![\Psi]\!] = \{ab, a\overline{b}\}$ holds. Hence, due to Theorem 6, there is no contraction operator for $\mathbb{E}^{\mathrm{E15}}$; showing that the situation can be more drastic than Proposition 18 suggests.

Proposition 19. *AGM contraction is not realizable in $\mathbb{E}^{\mathrm{E15}}$, but AGM revision is realizable in $\mathbb{E}^{\mathrm{E15}}$.*

Moreover, in specific epistemic spaces a situation that is dual to Proposition 18 appears, as witnessed by Example 9.

Proposition 20. *AGM revision is not realizable in $\mathbb{E}^{\mathrm{E9}}$, but AGM contraction is realizable in $\mathbb{E}^{\mathrm{E9}}$.*

We obtain the following dual to Proposition 18.

Corollary 21. *There is an epistemic space $\mathbb{E}$ such that there exist more contraction operators for $\mathbb{E}$ than revision operators for $\mathbb{E}$.*

7 Conclusion and Discussion

In this paper, we considered the problem of realizability for AGM revision and AGM contraction in epistemic spaces. Next, we highlight and discuss some of this paper's results, outline a plan for future work, and identify open problems.

Realizability and Canonical Operators. The main results, Theorem 6 and Theorem 12, show that AGM revision and

[2]By cardinality; the set of all revision operators for $\mathbb{E}$ has strictly more elements than the set of all contraction operators for $\mathbb{E}$.

AGM contraction, are only realizable in specific epistemic spaces. Our results support that for epistemic spaces, linear contraction and linear revision are conical instances, as they are guaranteed to exist when the respective kind of belief change is realizable at all (Proposition 10 and Proposition 17). We want to highlight that the observation that linear revisions are a canonical form of revision for epistemic spaces is converse to the observations made when generalizing AGM revision to arbitrary Tarskian logics, where full meet revision operators always exist and, thus, are canonical instances for AGM revision (Falakh, Rudolph, and Sauerwald 2022). Not that the setting considered here is more general than in (Falakh, Rudolph, and Sauerwald 2022), as, e.g., a closure under conjunction over the available epistemic states is not assumed. Our results on realizability are also different from those of Ribeiro et al. (2013), who investigated which logics AGM revision and AGM contraction operators exist. Still, they did not consider the restrictions given by considering epistemic spaces, which leads to different results.

Potential Application. We predict that our realizability results can be used, for instance, to checking beforehand, whether AGM revision or AGM contraction can be applied out of the box in a specific domain. Another application are algorithms that compute, for a given epistemic space $\mathbb{E}$, whether a revision or contraction operator for $\mathbb{E}$ exists.

Linear Contraction and Linear Revision. Note that linear contraction and linear revision are novel types of operators, which make sense when considering the setting of epistemic spaces. Before us, Herzig, Konieczny, and Perrussel (2003) already employed linear orders to define belief change operators. However, the difference is that they did not employ linear orders to specify the global behaviour of operators as we did here in Section 4 and Section 5; their definition corresponds to what is here called maxichoice.

Future Work and Open Problems. In future work, we extend our investigations of revision and contraction constrained by iteration postulates (Sauerwald, Kern-Isberner, and Beierle 2020; Schwind, Konieczny, and Pino Pérez 2022; Darwiche and Pearl 1997). We are also planning to consider the exact relation of belief change in epistemic spaces to hyper-intensional belief revision (Souza and Wassermann 2022; Berto 2019; Özgün and Berto 2021) and to belief change in subclassical logics, for which abstract logic (Lewitzka and Brunner 2009) will be useful, as well as the work by Flouris (Flouris 2006; Flouris and Plexousakis 2006) and results in the understanding of revision in Horn logic (Delgrande and Peppas 2015). As highlighted in this paper, another interesting problem to study is the interrelation of revision and contraction in epistemic spaces. The interested reader may consult Konieczny and Pérez (2017) and Booth and Chandler (2019) as a starting point. An open problem for the area of (iterated) belief change is to consider more expressive logics and explore what are reasonable belief changes in these settings. The notion of epistemic spaces might be useful for such endeavours. We expect that our realizability results carry over to more expressive settings, and we plan to check whether this conjecture holds.

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